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Abstract Plancherel (Trace) Formulas over Homogeneous Spaces of Compact Groups

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Abstract. This paper introduces a unified operator theory approach to the abstract Plancherel (trace) formulas over homogeneous spaces of compact groups. Let G be a compact group and let H be a closed subgroup of G. Let G/H be the left coset space of H in G and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. Then we present a generalized abstract notion of Plancherel (trace) formula for the Hilbert space $L^2(G/H,\mu)$.

1 Introduction

The abstract aspects of harmonic analysis over homogeneous spaces of compact non-Abelian groups or, precisely, left coset (resp. right coset) spaces of non-normal subgroups of compact non-Abelian groups are placed as building blocks for coherent states analysis [2–4, 11], theoretical and particle physics [1, 9, 10, 12]. Over the last decades, abstract and computational aspects of Plancherel formulas over symmetric spaces have achieved significant popularity in geometric analysis, mathematical physics, and scientific computing (computational engineering); see [6, 7, 12–17] and references therein.

Let G be a compact group, let H be a closed subgroup of G, and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. The left coset space G/H is considered as a compact homogeneous space on which G acts via the left action. This paper, which contains 4 sections, is organized as follows. Section 2 is devoted to fixing notations and preliminaries, including a brief summary of Hilbert-Schmidt operators, non-Abelian Fourier analysis over compact groups, and classical results on abstract harmonic analysis over locally compact homogeneous spaces. We present some abstract harmonic analysis aspects of the Hilbert function space $L^2(G/H,\mu)$ in Section 3. Then we define the abstract notion of dual space $\widehat{G/H}$ for the homogeneous space G/H, and we will show that this definition is precisely the standard dual space for the compact quotient group G/H when H is a closed normal subgroup of G. We then introduce the definition of abstract operator-valued Fourier transform over the Banach function space $L^1(G/H,\mu)$. The paper closes by a presentation of a generalized version of the abstract Plancherel (trace) formula for the Hilbert function space $L^2(G/H,\mu)$.

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2 Preliminaries and Notations

Let $\mathcal{H}$ be a separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *Hilbert–Schmidt operator* if for one, hence for any, orthonormal basis $\{e_k\}$ of $\mathcal{H}$ we have $\sum_k \|Te_k\|^2 < \infty$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathrm{HS}(\mathcal{H})$, and for $T \in \mathrm{HS}(\mathcal{H})$ the Hilbert–Schmidt norm of T is $\|T\|_{\mathrm{HS}}^2 = \sum_k \|Te_k\|^2$. The set $\mathrm{HS}(\mathcal{H})$ is a self adjoint two sided ideal in $\mathcal{B}(\mathcal{H})$, and if $\mathcal{H}$ is finite-dimensional we have $\mathrm{HS}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is trace-class whenever $\|T\|_{\mathrm{tr}} = \mathrm{tr}[|T|] < \infty$, if $\mathrm{tr}[T] = \sum_k \langle Te_k, e_k\rangle$ and $|T| = (TT^*)^{1/2}$ [19].

Let G be a compact group with the probability Haar measure dx. Then each irreducible representation of G is finite dimensional and every unitary representation of G is a direct sum of irreducible representations; see [1,9]. The set of of all unitary equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$. This definition of $\widehat{G}$ is in essential agreement with the classical definition when G is Abelian, since each character of an Abelian group is a one dimensional representation of G. If π is any unitary representation of G, for $\zeta, \xi \in \mathcal{H}_\pi$ the functions $\pi_{\zeta,\xi}(x) = \langle \pi(x)\zeta, \xi\rangle$ are called the *matrix elements* of π. If $\{e_j\}$ is an orthonormal basis for $\mathcal{H}_\pi$, then π_{ij} means π_{e_i,e_j}. The notation $\mathcal{E}_\pi$ is used for the linear span of the matrix elements of π and the notation $\mathcal{E}$ is used for the linear span of $\bigcup_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$. Then the Peter–Weyl Theorem [1,9] guarantees that if G is a compact group, $\mathcal{E}$ is uniformly dense in $\mathcal{C}(G)$, $L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi$, and $\{d_\pi^{-1/2}\pi_{ij} : i, j = 1, \dots, d_\pi, [\pi] \in \widehat{G}\}$ is an orthonormal basis for $L^2(G)$. For $f \in L^1(G)$ and $[\pi] \in \widehat{G}$, the Fourier transform of f at π is defined in the weak sense as an operator in $\mathcal{B}(\mathcal{H}_\pi)$ by

$$\widehat{f}(\pi) = \int_G f(x)\pi(x)^* dx. \tag{2.1}$$

If $\pi(x)$ is represented by the matrix $(\pi_{ij}(x)) \in \mathbb{C}^{d_\pi \times d_\pi}$, then $\widehat{f}(\pi) \in \mathbb{C}^{d_\pi\times d_\pi}$ is the matrix with entries given by $\widehat{f}(\pi)_{ij} = d_\pi^{-1} c_{ji}^\pi(f)$, which satisfies

$$\sum_{i,j=1}^{d_\pi} c_{ij}^\pi(f)\pi_{ij}(x) = d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ji}\pi_{ij}(x) = d_\pi \mathrm{tr}\big[\widehat{f}(\pi)\pi(x)\big],$$

where $c_{i,j}^\pi(f) = d_\pi\langle f, \pi_{ij}\rangle_{L^2(G)}$. Then as a consequence of the Peter–Weyl Theorem, we get [18,21]

$$\|f\|_{L^2(G)}^2 = \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{f}(\pi)\|_{\mathrm{HS}}^2. \tag{2.2}$$

Let H be a closed subgroup of G with the probability Haar measure dh. The left coset space G/H is considered as a compact homogeneous space that G acts on it from the left, and $q\colon G \to G/H$, given by $x \mapsto q(x) := xH$, is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces have been quite well studied by several authors; see [5,8–10,20] and references therein. If G is compact, each transitive G-space can be considered as a left coset space G/H for some closed subgroup H of G. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f \in \mathcal{C}(G)$ and

$$T_H(f)(xH) = \int_H f(xh)dh.$$

Let μ be a Radon measure on G/H and $x \in G$. The translation μ_x of μ is defined by $\mu_x(E) = \mu(xE)$, for all Borel subsets E of G/H. The measure μ is called G-*invariant* if $\mu_x = \mu$, for all $x \in G$. The homogeneous space G/H has a normalized G-invariant measure μ, which satisfies the following Weil formula [1, 20]:

$$\int_{G/H} T_H(f)(xH)d\mu(xH) = \int_G f(x)dx \quad \text{for all } f \in L^1(G), \tag{2.3}$$

and also the following norm-decreasing formula

$$\|T_H(f)\|_{L^1(G/H,\mu)} \le \|f\|_{L^1(G)} \quad \text{for all } f \in L^1(G).$$

3 Abstract Harmonic Analysis of Hilbert Function Spaces over Homogeneous Spaces of Compact Groups

Throughout this paper we assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure dh, and also μ is the normalized G-invariant measure on the homogeneous space G/H that satisfies (2.3).

In this section, we present some properties of the Hilbert function space $L^2(G/H,\mu)$ in the framework of abstract harmonic analysis.

First we shall show that the linear map T_H has a unique extension to a bounded linear map from $L^2(G)$ onto $L^2(G/H,\mu)$.

Theorem 3.1 *Let H be a closed subgroup of a compact group G and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. The linear map $T_H\colon \mathcal{C}(G) \to \mathcal{C}(G/H)$ has a unique extension to a bounded linear map from $L^2(G)$ onto $L^2(G/H,\mu)$.*

Proof Let μ be the normalized G-invariant measure on the homogeneous space G/H that satisfies (2.3) and $f \in \mathcal{C}(G)$. Then we claim that

$$\|T_H(f)\|_{L^2(G/H,\mu)} \le \|f\|_{L^2(G)}. \tag{3.1}$$

To this end, using compactness of H, we have

$$\begin{aligned}\|T_H(f)\|^p_{L^2(G/H,\mu)} &= \int_{G/H} |T_H(f)(xH)|^2 d\mu(xH) = \int_{G/H}\Big|\int_H f(xh)dh\Big|^2 d\mu(xH)\\ &\le \int_{G/H}\Big(\int_H |f(xh)|dh\Big)^2 d\mu(xH)\\ &\le \int_{G/H}\int_H |f(xh)|^2 dh d\mu(xH).\end{aligned}$$

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Then, by Weil's formula, we get

$$\begin{aligned}\int_{G/H}\int_H |f(xh)|^2 dh d\mu(xH) &= \int_{G/H}\int_H |f|^2(xh) dh d\mu(xH)\\ &= \int_{G/H} T_H(|f|^2)(xH) d\mu(xH)\\ &= \int_G |f(x)|^2 dx = \|f\|^2_{L^2(G)},\end{aligned}$$

which implies (3.1). Thus, we can extend T_H to a bounded linear operator from $L^2(G)$ onto $L^2(G/H,\mu)$, which we still denote by T_H, and that satisfies

$$\|T_H(f)\|_{L^2(G/H,\mu)} \le \|f\|_{L^2(G)} \quad \text{for all } f \in L^2(G). \qquad \blacksquare$$

Let $\mathcal{J}^2(G,H) := \{f \in L^2(G) : T_H(f) = 0\}$ and $\mathcal{J}^2(G,H)^\perp$ be the orthogonal completion of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

As an immediate consequence of Theorem 3.1, we deduce the following result.

Proposition 3.2 *Let H be a closed subgroup of a compact group G and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. Then $T_H: L^2(G) \to L^2(G/H,\mu)$ is a partial isometric linear map.*

Proof Let $\varphi \in L^2(G/H,\mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^2(G)$ with

$$\|\varphi_q\|_{L^2(G)} = \|\varphi\|_{L^2(G/H,\mu)}. \tag{3.2}$$

Indeed, using Weil's formula, we can write

$$\begin{aligned}\|\varphi_q\|^2_{L^2(G)} &= \int_G |\varphi_q(x)|^2 dx = \int_{G/H} T_H(|\varphi_q|^2)(xH) d\mu(xH)\\ &= \int_{G/H}\Big(\int_H |\varphi_q(xh)|^2 dh\Big) d\mu(xH),\end{aligned}$$

and since H is compact and dh is a probability measure, we get

$$\begin{aligned}\int_{G/H}\Big(\int_H |\varphi_q(xh)|^2 dh\Big) d\mu(xH) &= \int_{G/H}\Big(\int_H |\varphi(xhH)|^2 dh\Big) d\mu(xH)\\ &= \int_{G/H}\Big(\int_H |\varphi(xH)|^2 dh\Big) d\mu(xH)\\ &= \int_{G/H} |\varphi(xH)|^2 \Big(\int_H dh\Big) d\mu(xH)\\ &= \int_{G/H} |\varphi(xH)|^2 d\mu(xH) = \|\varphi\|^2_{L^2(G/H,\mu)},\end{aligned}$$

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which implies (3.2). Then $T_H^*(\varphi) = \varphi_q$ and $T_H T_H^*(\varphi) = \varphi$. Because using Weil's formula we have

$$\begin{aligned}\langle T_H^*(\varphi), f\rangle_{L^2(G)} &= \langle \varphi, T_H(f)\rangle_{L^2(G/H,\mu)} \\ &= \int_{G/H} \varphi(xH)\overline{T_H(f)(xH)}d\mu(xH) \\ &= \int_{G/H} \varphi(xH)T_H(\overline{f})(xH)d\mu(xH) \\ &= \int_{G/H} T_H(\varphi_q.\overline{f})(xH)d\mu(xH) \\ &= \int_G \varphi_q(x)\overline{f(x)}dx = \langle \varphi_q, f\rangle_{L^2(G)}\end{aligned}$$

for all $f \in L^2(G)$, which implies that $T_H^*(\varphi) = \varphi_q$. Now a straightforward calculation shows that $T_H = T_H T_H^* T_H$. Then by [19, Theorem 2.3.3], T_H is a partial isometric operator. ■

We can then conclude the following corollaries as well.

Corollary 3.3 *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$, respectively. Then, for each $f \in L^2(G)$ and a.e. $x \in G$, we have*

(i) $P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH)$.
(ii) $P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH)$.

Corollary 3.4 *Let H be a compact subgroup of a compact group G and μ be the normalized G-invariant measure on G/H associated with Weil's formula. Then*

(i) $\mathcal{J}^2(G,H)^\perp = \{\psi_q : \psi \in L^2(G/H,\mu)\}$.
(ii) *For $f \in \mathcal{J}^2(G,H)^\perp$ and $h \in H$, we have $R_h f = f$.*
(iii) *For $\psi \in L^2(G/H,\mu)$ we have $\|\psi_q\|_{L^2(G)} = \|\psi\|_{L^2(G/H,\mu)}$.*
(iv) *For $f, g \in \mathcal{J}^2(G,H)^\perp$, we have*

$$\Big\langle T_H(f), T_H(g)\Big\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}.$$

We finish this section with the following remark.

Remark 3.5 Invoking Corollary 3.4, one can regard the Hilbert function space $L^2(G/H,\mu)$ as a closed linear subspace of the Hilbert function space $L^2(G)$; that is, the closed linear subspace consists of all $f \in L^2(G)$ that satisfy $R_h f = f$ for all $h \in H$. Then Theorem 3.1 and Proposition 3.2 guarantees that the bounded linear map

$$T_H: L^2(G) \longrightarrow L^2(G/H,\mu) \subset L^2(G)$$

is an orthogonal projection.

4 Abstract Trace Formulas over Homogeneous Spaces of Compact Groups

In this section, we present the abstract notions of dual spaces and Plancherel (trace) formulas over homogeneous spaces of compact groups.

For a closed subgroup H of G, let

$$H^{\perp} = \Big\{ [\pi] \in \widehat{G} : \pi(h) = I \text{ for all } h \in H \Big\}.$$

Then, by definition, we have $H^{\perp} \subseteq \widehat{G}$. If G is Abelian, each closed subgroup H of G is normal and the compact group G/H is Abelian and so $\widehat{G/H}$ is precisely the set of all characters (one dimensional irreducible representations) of G that are constant on H, that is precisely $H^{\perp}$. If G is a non-Abelian group and H is a closed normal subgroup of G, then the dual space $\widehat{G/H}$ which is the set of all unitary equivalence classes of unitary representations of the quotient group G/H, has meaning and is well defined. Indeed, G/H is a non-Abelian group. In this case, the map $\Phi\colon \widehat{G/H} \to H^{\perp}$ defined by $\sigma \mapsto \Phi(\sigma) := \sigma \circ q$ is a Borel isomorphism and $\widehat{G/H} = H^{\perp}$; see [1, 18, 21]. Thus, if H is normal, $H^{\perp}$ coincides with the classic definitions of the dual space either when G is Abelian or non-Abelian.

For a given closed subgroup H of G and also a continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of G, define

$$T_H^{\pi} := \int_H \pi(h)\,dh, \tag{4.1}$$

where the operator valued integral (4.1) is considered in the weak sense. In other words,

$$\langle T_H^{\pi}\zeta, \xi\rangle = \int_H \langle \pi(h)\zeta, \xi\rangle\,dh \quad \text{for } \zeta, \xi \in \mathcal{H}_\pi.$$

The function $h \mapsto \langle \pi(h)\zeta, \xi\rangle$ is bounded and continuous on H. Since H is compact, the right integral is the ordinary integral of a function in $L^1(H)$. Hence, T_H^{π} is a bounded linear operator on $\mathcal{H}_\pi$ with $\|T_H^{\pi}\| \leq 1$.

Definition 4.1 Let H be a compact subgroup of a compact group G. The dual space $\widehat{G/H}$ of the left coset space G/H is defined as the subset of $\widehat{G}$ given by

$$\widehat{G/H} := \Big\{ [\pi] \in \widehat{G} : T_H^{\pi} \neq 0 \Big\} = \Big\{ [\pi] \in \widehat{G} : \int_H \pi(h)\,dh \neq 0 \Big\}. \tag{4.2}$$

Then evidently we have

$$H^{\perp} \subseteq \widehat{G/H}. \tag{4.3}$$

First we present an interesting property of (4.2) when the left coset space G/H has the canonical quotient group structure.

The next theorem shows that the reverse inclusion of (4.3) holds if H is a normal subgroup of G.

Theorem 4.2 *Let H be a closed normal subgroup of a compact group G. Then $\widehat{G/H} = H^{\perp}$.*

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Proof Let H be a closed normal subgroup of a compact group G. Invoking the inclusion (4.3), it is sufficient to show that $\widehat{G/H} \subseteq H^\perp$. Let $[\pi] \in \widehat{G/H}$ be given. Due to normality of H in G, the map $\tau_x : H \to H$ given by $h \mapsto \tau_x(h) := x^{-1}hx$ belongs to $\mathrm{Aut}(H)$, and we also have $x^{-1}Hx = H$, for all $x \in G$. Let $x \in G$. Then by compactness of G we have $d(\tau_x(h)) = dh$, and hence we can write

$$\begin{aligned}\int_H \pi(h)dh &= \int_{xHx^{-1}} \pi(\tau_x(h))d(\tau_x(h)) = \int_H \pi(\tau_x(h))dh \\ &= \int_H \pi(x)^*\pi(h)\pi(x)dh = \pi(x)^*\Big(\int_H \pi(h)dh\Big)\pi(x) = \pi(x)^* T_H^\pi \pi(x),\end{aligned}$$

which implies that $\pi(x)T_H^\pi = T_H^\pi\pi(x)$. Since $x \in G$ was arbitrary we deduce that $T_H^\pi \in \mathcal{C}(\pi)$. Irreducibility of π guarantees that $T_H^\pi = \alpha I$ for some constant $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. By definition of $\widehat{G/H}$ we have $T_H^\pi \neq 0$, and hence we get $\alpha \neq 0$. Now let $t \in H$ be arbitrary. Then we can write

$$\pi(t) = \alpha^{-1}\pi(t)\alpha I = \alpha^{-1}\pi(t)T_H^\pi = \alpha^{-1}\int_H \pi(th)dh = \alpha^{-1}\int_H \pi(h)dh = \alpha^{-1}T_H^\pi = I,$$

which implies $[\pi] \in H^\perp$. ■

Let $\mathcal{K}_\pi^H := \{\zeta \in \mathcal{H}_\pi : \pi(h)\zeta = \zeta \ \forall h \in H\}$. Then $\mathcal{K}_\pi^H$ is a closed linear subspace of $\mathcal{H}_\pi$ and $\mathcal{R}(T_H^\pi) = \mathcal{K}_\pi^H$, where $\mathcal{R}(T_H^\pi) = \{T_H^\pi\zeta : \zeta \in \mathcal{H}_\pi\}$. It is easy to see that $[\pi] \in H^\perp$ if and only if $\mathcal{K}_\pi^H = \mathcal{H}_\pi$.

Then we can also present the following results.

Proposition 4.3 *Let H be a closed subgroup of a compact group G and let $(\pi, \mathcal{H}_\pi)$ be a continuous unitary representation of G.*

(i) *The operator T_H^π is an orthogonal projection of $\mathcal{H}_\pi$ onto $\mathcal{K}_\pi^H$.*
(ii) *The operator T_H^π is unitary if and only if $[\pi] \in H^\perp$.*

Proof (i) Using compactness of H, we have

$$(T_H^\pi)^* = \Big(\int_H \pi(h)dh\Big)^* = \int_H \pi(h)^*dh = \int_H \pi(h^{-1})dh = T_H^\pi.$$

As well, we can write

$$\begin{aligned}T_H^\pi T_H^\pi &= \Big(\int_H \pi(h)dh\Big)\Big(\int_H \pi(t)dt\Big) = \int_H \pi(h)\Big(\int_H \pi(t)dt\Big)dh \\ &= \int_H\Big(\int_H \pi(h)\pi(t)dt\Big)dh = \int_H\Big(\int_H \pi(ht)dt\Big)dh = \int_H T_H^\pi dt = T_H^\pi.\end{aligned}$$

(ii) The operator T_H^π is unitary if and only if $T_H^\pi = I$. The operator T_H is the identity if and only if $\pi(h) = I$ for all $h \in H$. Thus, T_H^π is unitary if and only if $[\pi] \in H^\perp$. ■

Let $\varphi \in L^1(G/H, \mu)$ and $[\pi] \in \widehat{G/H}$. The Fourier transform of φ at $[\pi]$ is defined as the linear operator

$$\mathcal{F}(\varphi)(\pi) = \widehat{\varphi}(\pi) := \int_{G/H} \varphi(xH)\Gamma_\pi(xH)^* d\mu(xH), \tag{4.4}$$

on the Hilbert space $\mathcal{H}_\pi$, where for each $xH \in G/H$ the notation $\Gamma_\pi(xH)$ stands for the bounded linear operator defined on the Hilbert space $\mathcal{H}_\pi$ by $\Gamma_\pi(xH) = \pi(x)T_H^\pi$;

that is,

$$\left\langle \Gamma_\pi(xH)\zeta,\xi\right\rangle = \left\langle \pi(x)T_H^\pi\zeta,\xi\right\rangle \quad \text{for } \zeta,\xi\in\mathcal{H}_\pi.$$

Then we have

$$\left\langle \Gamma_\pi(xH)\zeta,\xi\right\rangle = T_H(\pi_{\zeta,\xi})(xH),$$

for all $\zeta,\xi\in\mathcal{H}_\pi$. Indeed,

$$\begin{aligned}\left\langle \Gamma_\pi(xH)\zeta,\xi\right\rangle &= \left\langle \pi(x)T_H^\pi\zeta,\xi\right\rangle = \left\langle \pi(x)\Big(\int_H \pi(h)dh\Big)\zeta,\xi\right\rangle\\ &= \left\langle \Big(\int_H \pi(x)\pi(h)dh\Big)\zeta,\xi\right\rangle = \left\langle \Big(\int_H \pi(xh)dh\Big)\zeta,\xi\right\rangle\\ &= \int_H \left\langle \pi(xh)\zeta,\xi\right\rangle dh = \int_H \pi_{\zeta,\xi}(xh)dh = T_H(\pi_{\zeta,\xi})(xH).\end{aligned}$$

Remark 4.4 Let H be a closed normal subgroup of a compact group G and let μ be the normalized G-invariant measure over the left coset space G/H associated with Weil's formula. Then it is easy to check that μ is a Haar measure of the compact quotient group G/H, and by Theorem 4.2 we have $\widehat{G/H} = H^\perp$. Also, for each $\varphi\in L^1(G/H,\mu)$ and $[\pi]\in H^\perp$, we have

$$\widehat{\varphi}(\pi) = \int_{G/H}\varphi(xH)\pi(x)^* d\mu(xH).$$

Thus, we deduce that the abstract Fourier transform defined by (4.4) coincides with the classical Fourier transform over the compact quotient group G/H if H is normal in G.

The operator-valued integral (4.4) is considered in the weak sense; that is,

$$\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle = \int_{G/H}\varphi(xH)\langle\zeta,\Gamma_\pi(xH)^*\xi\rangle d\mu(xH) \quad \text{for } \zeta,\xi\in\mathcal{H}_\pi.$$

In other words, for $[\pi]\in\widehat{G/H}$ and $\zeta,\xi\in\mathcal{H}_\pi$ we have

$$\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle = \int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH), \tag{4.5}$$

because we can write

$$\begin{aligned}\langle\zeta,\widehat{\varphi}(\pi)\xi\rangle &= \int_{G/H}\varphi(xH)\langle\zeta,\Gamma_\pi(xH)^*\xi\rangle d\mu(xH)\\ &= \int_{G/H}\varphi(xH)\langle\Gamma_\pi(xH)\zeta,\xi\rangle d\mu(xH)\\ &= \int_{G/H}\varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH).\end{aligned}$$

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If $\zeta, \xi \in \mathcal{H}_\pi$, then we have

$$\begin{aligned}
\big|\langle \zeta, \widehat{\varphi}(\pi)\xi\rangle\big| &= \Big|\int_{G/H} \varphi(xH) T_H(\pi_{\zeta,\xi})(xH) d\mu(xH)\Big| \\
&\leq \int_{G/H} |\varphi(xH)| \big| T_H(\pi_{\zeta,\xi})(xH)\big| d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)| \Big| \int_H \pi_{\zeta,\xi}(xh) dh \Big| d\mu(xH) \\
&\leq \int_{G/H} |\varphi(xH)| \Big(\int_H |\pi_{\zeta,\xi}(xh)| dh\Big) d\mu(xH) \\
&\leq \int_{G/H} |\varphi(xH)| \Big(\int_H \|\pi(xh)\zeta\| \cdot \|\xi\| dh\Big) d\mu(xH) \\
&= \int_{G/H} |\varphi(xH)| \Big(\int_H \|\zeta\| \cdot \|\xi\| dh\Big) d\mu(xH) \\
&= \|\zeta\| \cdot \|\xi\| \cdot \Big(\int_{G/H} |\varphi(xH)| \Big(\int_H dh\Big) d\mu(xH)\Big) \\
&= \|\zeta\| \cdot \|\xi\| \cdot \|\varphi\|_{L^1(G/H,\mu)},
\end{aligned}$$

so we deduce that $\widehat{\varphi}(\pi)$ is a bounded linear operator on $\mathcal{H}_\pi$ with

$$\|\widehat{\varphi}(\pi)\| \leq \|\varphi\|_{L^1(G/H,\mu)}.$$

The following proposition presents the canonical connection of the abstract Fourier transform defined in (4.4) with the classical Fourier transform (2.1).

Proposition 4.5 *Let H be a closed subgroup of a compact group G and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. Then for $\varphi \in L^1(G/H,\mu)$ and $[\pi] \in \widehat{G/H}$, we have*

$$\widehat{\varphi}(\pi) = \widehat{\varphi_q}(\pi). \tag{4.6}$$

Proof Using Weil's formula and also (4.5), for $\zeta, \xi \in \mathcal{H}_\pi$, we can write

$$\begin{aligned}
\langle \zeta, \widehat{\varphi}(\pi)\xi\rangle &= \int_{G/H} \varphi(xH) T_H(\pi_{\zeta,\xi})(xH) d\mu(xH) \\
&= \int_{G/H} T_H(\varphi_q.\pi_{\zeta,\xi})(xH) d\mu(xH) \\
&= \int_G \varphi_q(x)\pi_{\zeta,\xi}(x) dx \\
&= \int_G \varphi_q(x)\langle \pi(x)\zeta, \xi\rangle dx \\
&= \int_G \varphi_q(x)\langle \zeta, \pi(x)^*\xi\rangle dx = \langle \zeta, \widehat{\varphi_q}(\pi)\xi\rangle,
\end{aligned}$$

which implies (4.6). ∎

In the next theorem we show that the abstract Fourier transform defined in (4.4) satisfies a generalized version of the Plancherel (trace) formula.

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Theorem 4.6 *Let H be a closed subgroup of a compact group G and let μ be the normalized G-invariant measure on G/H associated with Weil's formula. Then each $\varphi \in L^2(G/H,\mu)$ satisfies the following Plancherel formula;*

$$\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_{\mathrm{HS}}^2 = \|\varphi\|_{L^2(G/H,\mu)}^2. \tag{4.7}$$

Proof Let $\varphi \in L^2(G/H,\mu)$ be given. If $[\pi] \in \widehat{G}$ with $[\pi] \notin \widehat{G/H}$, then we have $T_H^\pi = 0$. Hence, for $\zeta, \xi \in \mathcal{H}_\pi$, we have $T_H(\pi_{\zeta,\xi}) = 0$. Therefore, we get

$$\widehat{\varphi_q}(\pi) = 0. \tag{4.8}$$

Indeed, using Weil's formula, for $\zeta, \xi \in \mathcal{H}_\pi$ we can write

$$\begin{aligned}\langle \zeta, \widehat{\varphi_q}(\pi)\xi\rangle &= \int_G \varphi_q(x)\langle \zeta, \pi(x)^*\xi\rangle dx = \int_G \varphi_q(x)\langle \pi(x)\zeta, \xi\rangle dx\\ &= \int_G \varphi(x)\pi_{\zeta,\xi}(x)dx = \int_{G/H} T_H(\varphi_q.\pi_{\zeta,\xi})(xH)d\mu(xH)\\ &= \int_{G/H} \varphi(xH)T_H(\pi_{\zeta,\xi})(xH)d\mu(xH) = 0.\end{aligned}$$

Using equations (4.6) and (4.8), the Plancherel formula (2.2), and Corollary 3.4 we achieve

$$\begin{aligned}\sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi}(\pi)\|_{\mathrm{HS}}^2 &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{\mathrm{HS}}^2\\ &= \sum_{[\pi]\in\widehat{G/H}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{\mathrm{HS}}^2 + \sum_{\{[\pi]\in\widehat{G}:[\pi]\notin\widehat{G/H}\}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{\mathrm{HS}}^2\\ &= \sum_{[\pi]\in\widehat{G}} d_\pi \|\widehat{\varphi_q}(\pi)\|_{\mathrm{HS}}^2 = \|\varphi_q\|_{L^2(G)}^2 = \|\varphi\|_{L^2(G/H,\mu)}^2,\end{aligned}$$

which implies (4.7). ∎

Remark 4.7 Let H be a closed normal subgroup of a compact group G and let μ be the normalized G-invariant measure over the left coset space G/H associated with Weil's formula. Then Theorem 4.2 implies that $\widehat{G/H} = H^\perp$, and hence the Plancherel (trace) formula (4.7) reads as follows:

$$\sum_{[\pi]\in H^\perp} d_\pi \|\widehat{\varphi}(\pi)\|_{\mathrm{HS}}^2 = \|\varphi\|_{L^2(G/H,\mu)}^2$$

for all $\varphi \in L^2(G/H,\mu)$, where

$$\widehat{\varphi}(\pi) = \int_{G/H} \varphi(xH)\pi(x)^* d\mu(xH)$$

for all $[\pi] \in H^\perp$; see Remark 4.4.

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