Canad. Math. Bull. Vol. **27** (4), 1984

A COMPACTIFICATION FOR CONVERGENCE ORDERED SPACES

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ABSTRACT. Compactifications are constructed for convergence ordered spaces and topological ordered spaces with extension properties that resemble those of the Stone-Čech compactification.

0. **Introduction.** One of the authors [4] introduced a convergence space compactification with an extension property similar to that of the topological Stone-Čech compactification. We later showed in [2] that the compactification of [4] gives rise to a topological compactification with an interesting lifting property.

This work is concerned with convergence ordered spaces, a natural generalization of the topological ordered spaces of Nachbin [3]. By "convergence ordered space" we mean a partially ordered set with a convergence structure generated by filters which have bases of convex sets. A preliminary section gives a brief introduction to such spaces.

In Section 2, a convergence ordered compactification is constructed for an arbitrary convergence ordered space by defining an appropriate partial order on a class of filters and using a "Wallman-type" construction similar to that of [4]. The extension properties of this compactification are examined in Section 3; in addition to generalizing the extension results of [4], conditions are found subject to which ours is the largest convergence ordered compactification. The last section applies the results of the preceding sections to obtain a topological ordered compactification with similar lifting properties.

Choe and Park [1] have constructed a Wallman ordered compactification for the topological setting. It is shown, under certain assumptions, that our topological ordered compactification is larger than that by Choe and Park.

1. **Preliminaries.** Let $(X, \leq)$ be a partially ordered set (or poset) equipped with a convergence structure. A convergence structure on X is a relation $\rightarrow$ between the set $F(X)$ of all filters on X and X which satisfies the following

Received by the editors November 15, 1983.
AMS Subject Classification (1980): 54A20, 54D35, 54F05

https://doi.org/10.4153/CMB-1984-081-6 Published online by Cambridge University Press

conditions:

(C_1) For each $x \in X$, $\dot{x} \to x$, where $\dot{x}$ denotes the fixed ultrafilter generated by $\{x\}$.

(C_2) If $\mathfrak{F} \to x$ and $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{G} \to x$.

(C_3) If $\mathfrak{F} \to x$, then $\mathfrak{F} \cap \dot{x} \to x$.

Starting with $A \subseteq X$, let $i(A) = \{x \in X$: for some $a \in A$, $a \leq x\}$, and $d(A) = \{x \in X$: for some $a \in A$, $x \leq a\}$. If $A = \{a\}$, we shall write $i(a)$ and $d(a)$ in place of $i(\{a\})$ and $d(\{a\})$. If $A = i(A)$ (respectively, $A = d(A)$), then A is called an *increasing* (respectively, *decreasing*) set. For any $A \subseteq X$, let $A^\wedge = i(A) \cap d(A)$; if $A = A^\wedge$, then A is said to be a *convex set.* For $\mathfrak{F} \in F(X)$, $\mathfrak{F}^\wedge$ denotes the filter generated by $\{F^\wedge : F \in \mathfrak{F}\}$; if $\mathfrak{F} = \mathfrak{F}^\wedge$, then $\mathfrak{F}$ is called a *convex filter.*

By a *convergence ordered space* (abbreviated c.o.s.), we shall mean a poset $(X, \leq)$ along with a convergence structure $\to$ on X that satisfies the condition: $\mathfrak{F}^\wedge \to x$ whenever $\mathfrak{F} \to x$. We shall commonly refer to a convergence ordered space $(X, \leq, \to)$ simply as X.

In working with a c.o.s. X, we shall make use of two "order relations" on $F(X)$. The first is set inclusion: $\mathfrak{F} \subseteq \mathfrak{G}$ means "$\mathfrak{G}$ is finer than $\mathfrak{F}$" or "$\mathfrak{G}$ is coarser than $\mathfrak{G}$." By $\mathfrak{F} \vee \mathfrak{G}$, we shall always mean the least upper bound (if it exists) of $\mathfrak{F}$ and $\mathfrak{G}$ relative to inclusion; in other words, $\mathfrak{F} \vee \mathfrak{G}$ is the filter generated by $\{F \cap G : F \in \mathfrak{F}, G \in \mathfrak{G}\}$, assuming all such intersections are non-empty. A second order relation (actually a preorder relation) on $F(X)$ is defined as follows: $\mathfrak{F} \lesssim \mathfrak{G}$ iff $i(\mathfrak{F}) \subseteq \mathfrak{G}$ and $d(\mathfrak{G}) \subseteq \mathfrak{F}$, where $i(\mathfrak{F})$ is the filter generated by $\{i(F) : F \in \mathfrak{F}\}$ and $d(\mathfrak{F})$ is defined dually. The relation "$\lesssim$" is always reflexive and transitive, and is antisymmetric when restricted to convex filters.

Let $F^\wedge(X)$ be the set of all convex filters on X. Both of the relations $\subseteq$ and $\lesssim$ are partial orders of $F^\wedge(X)$. The maximal elements of $F^\wedge(X)$ relative to the relation $\subseteq$ will be called *maximal convex filters*; these obviously include the fixed ultrafilters. The set of all non-convergent, maximal convex filters on a c.o.s. X will be denoted by X'. A useful characterization of maximal convex filters is given by the first proposition.

PROPOSITION 1.1. *A filter $\mathfrak{F} \in F^\wedge(X)$ is maximal iff, whenever A and B are convex sets and $A \cup B \in \mathfrak{F}$, either $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.*

Proof. If $\mathfrak{F}$ is a maximal convex filter, the condition is easily proved by considering the traces of $\mathfrak{F}$ on A and B. Conversely, if $\mathfrak{F}$ is not maximal, then there is a convex set G not belonging to $\mathfrak{F}$ such that $\mathfrak{F}$ has a trace on G. Since $G = i(G) \cap d(G)$, either $i(G)$ or $d(G)$ is not in $\mathfrak{F}$. If $i(G) \notin \mathfrak{F}$, then it is also true that $X - i(G) \notin \mathfrak{F}$ (since $\mathfrak{F}$ has a trace on $i(G)$). If $A = i(G)$ and $B = X - i(G)$, then A and B are convex sets and $A \cup B \in \mathfrak{F}$, but neither A nor B is in $\mathfrak{F}$. $\square$

We next consider some separation axioms. A convergence space X is said to

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be T_1 if each singleton set is closed, and T_2 if each convergent filter has a unique limit. X is regular if $cl_X\mathfrak{F} \to x$ whenever $\mathfrak{F} \to x$ (where cl_X denotes the closure operator); a regular T_2 space is called a T_3 *space*. If X is a convergence ordered space, then X is said to be T_1-*ordered* if $i(x)$ and $d(x)$ are closed sets for each $x \in X$. If $x \leq y$ whenever $\mathfrak{F} \to x$, $\mathfrak{G} \to y$, and $\mathfrak{F} \lesssim \mathfrak{G}$, then X is defined to be T_2-ordered. Clearly a T_1-ordered c.o.s. is T_1 and a T_2-ordered c.o.s. is T_2. Alternate characterizations of a T_2-ordered c.o.s. are given in the next proposition.

PROPOSITION 1.2. *For a c.o.s. X, the following statements are equivalent.*

(1) *X is T_2-ordered.*

(2) *The set $R = \{(x, y) : x \leq y\}$ is a closed subset of the product space $X \times X$.*

(3) *If $\mathfrak{F} \to x$, $\mathfrak{G} \to y$, and $(F \times G) \cap R \neq \emptyset$ for all $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, then $x \leq y$.*

An additional separation property will be needed for what follows. A c.o.s. X satisfies *condition S* if the following hold:

(S_1) If $\mathfrak{F} \to x$, $\mathfrak{G} \in X'$, and $\mathfrak{F} \lesssim \mathfrak{G}$, then $d(\mathfrak{G}) \subseteq \dot{x}$.

(S_2) If $\mathfrak{F} \to x$, $\mathfrak{G} \in X'$, and $\mathfrak{G} \lesssim \mathfrak{F}$, then $i(\mathfrak{G}) \subseteq \dot{x}$.

a T_1 (respectively, T_2) c.o.s. X which satisfies condition S is said to be strongly T_1-ordered (respectively, strongly T_2-ordered).

A function f from a poset X into a poset Y is *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$. If X and Y are convergence ordered spaces, f an order isomorphism and homeomorphic embedding, Y compact, and $f(X)$ dense in Y, then (Y, f) will be called a *convergence ordered compactification* of X.

2. **The compactification.** Throughout this section, X is assumed to be a convergence ordered space. Let $X^* = \{\dot{x} : x \in X\} \cup X'$, and define a partial order $\leq^*$ on X^* as follows: $\mathfrak{F} \leq^* \mathfrak{G}$ iff $\mathfrak{F} \lesssim \mathfrak{G}$, where the relation "$\lesssim$" between filters is defined in Section 1. Since the elements of X^* are all maximal convex filters, the relation $\leq^*$ can be described in several equivalent ways.

PROPOSITION 2.1. *for $\mathfrak{F}$, $\mathfrak{G}$ in X^*, the following statements are equivalent:* (1) $\mathfrak{F} \lesssim \mathfrak{G}$; (2) $i(\mathfrak{F}) \subseteq \mathfrak{G}$; (3) $d(\mathfrak{G}) \subseteq \mathfrak{F}$; (4) $i(\mathfrak{F}) \vee d(\mathfrak{G})$ *exists.*

The natural map $\phi : X \to X^*$, defined by $\phi(x) = \dot{x}$ for all $x \in X$, is one-to-one and increasing.

If $A \subseteq X$, define $A^* = \{\mathfrak{F} \in X^* : A \in \mathfrak{F}\}$; if $\mathfrak{F} \in F(X)$, let $\mathfrak{F}^*$ be the filter on X^* generated by $\{F^* : F \in \mathfrak{F}\}$. Observe that $A^* \mid \phi(X) = \phi(A)$ and $\phi^{-1}(A^*) = A$. It is easy to see that $A^* = B^*$ iff $A = B$.

In the next proposition, the first inequality is trivial, and the second follows directly from Proposition 1.1.

PROPOSITION 2.2. *If A and B are convex subsets of X, then $(A \cap B)^* = A^* \cap B^*$ and $(A \cup B)^* = A^* \cup B^*$.*

In X^*, the increasing and decreasing set operators will be denoted by $\imath^*$ and d^*, respectively. We omit the straightforward proof of the next proposition.

PROPOSITION 2.3. (a) *For any subset A of X*, $\imath^*(A^*) \subseteq \imath(A)^*$ *and* $d^*(A^*) \subseteq (d(A))^*$. (b) *If A is a convex subset of X, then* A^* *is a convex subset of* X^*.

Having established some basic properties of the poset $(X^*, \leq^*)$, we next define a convergence structure $\xrightarrow{*}$ on X^* as follows:

$\mathcal{A} \xrightarrow{*} \dot{x} \in \phi(X)$ iff there is $\mathfrak{F} \to x$ such that $\mathfrak{F}^* \subseteq \mathcal{A}$,

$\mathcal{A} \xrightarrow{*} \mathfrak{G} \in X'$ iff $\mathfrak{G}^* \subseteq \mathcal{A}$.

The verification that $\xrightarrow{*}$ is a convergence structure is easy and will be omitted. Proposition 2.3(b) can be applied to show that X^* is c.o.s.

If $\mathcal{A}$ is an ultrafilter on X^*, let $\mathfrak{F}_{\mathcal{A}}$ be the filter on X generated by all convex sets A such that $A^* \in \mathcal{A}$. If $\mathcal{A}$, $\mathcal{B}$ are two arbitrary filters on X^*, we shall use the notation $\mathcal{A} \lesssim^* \mathcal{B}$ to mean $\imath^*(\mathcal{A}) \subseteq \mathcal{B}$ and $d^*(\mathcal{B}) \subseteq \mathcal{A}$.

LEMMA 2.4. (a) *If* $\mathcal{A}$ *is an ultrafilter on* X^*, *then* $\mathfrak{F}_{\mathcal{A}}$ *is a maximal convex filter on X.*

(b) *If* $\mathcal{A} \lesssim^* \mathcal{B}$ *in* $F(X^*)$, *then* $\mathfrak{F}_{\mathcal{A}} \lesssim \mathfrak{F}_{\mathcal{B}}$ *in* $F(X)$.

Proof. (a) If A and B are convex sets such that $A \cup B \in \mathfrak{F}_{\mathcal{A}}$, then by Proposition 2.2, $A^* \cup B^* \in \mathcal{A}$, and so one of these is in $\mathcal{A}$, which implies that A or B is in $\mathfrak{F}_{\mathcal{A}}$. Thus $\mathfrak{F}_{\mathcal{A}}$ is maximal convex by Proposition 1.1.

(b) If $A \in \mathfrak{F}_{\mathcal{A}}$, then $A^* \in \mathcal{A}$ and $\imath^*(A^*) \in \mathcal{B}$. By Proposition 2.3(a), $(\imath(A))^* \in \mathcal{B}$, which implies $\imath(A) \in \mathfrak{F}_{\mathcal{B}}$. The proof that $d(\mathfrak{F}_{\mathcal{B}}) \subseteq \mathfrak{F}_{\mathcal{A}}$ is similar. □

THEOREM 2.5. *If X is any convergence ordered space, then* (X^*, ϕ) *is a convergence ordered compactification of X.*

Proof. We have already observed that X^* is a c.o.s. and that ϕ is an order isomorphism. From the definition of $\xrightarrow{*}$ and the fact that $\phi^{-1}(\mathfrak{F}^*) = \mathfrak{F}$ for any $\mathfrak{F} \in F(X)$, it follows easily that $\phi: X \to X^*$ is a homeomorphism. Each filter of the form $\mathfrak{F}^*$, where $\mathfrak{F} \in F(X)$, has a trace on $\phi(X)$, and this implies that $\phi(X)$ is dense in X^*.

To show that X^* is compact, consider an ultrafilter $\mathcal{A}$ on X^*. Then $\mathfrak{F}^*_{\mathcal{A}} \subseteq \mathcal{A}$, and $\mathfrak{F}_{\mathcal{A}}$ is maximal convex by Lemma 2.4. If $\mathfrak{F}_{\mathcal{A}} \to x$ in X, then $\mathcal{A} \xrightarrow{*} \dot{x}$; if $\mathfrak{F}_{\mathcal{A}} \in X'$, then $\mathcal{A} \xrightarrow{*} \mathfrak{F}_{\mathcal{A}}$. □

The next three propositions concern separation properties of the compactification space; we omit the straightforward proofs of the first two.

PROPOSITION 2.6. X^* *is* T_1 (*respectively*, T_2) *iff X is* T_1 (*respectively*, T_2).

PROPOSITION 2.7. *If X is strongly* T_1*-ordered, then* X^* *is* T_1*-ordered.*

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PROPOSITION 2.8. *X^* is T_2-ordered iff X is strongly T_2-ordered.*

Proof. Assume X is strongly T_2-ordered, and let $\mathcal{A} \xrightarrow{*} \mathfrak{F}$, $\mathcal{B} \xrightarrow{*} \mathfrak{G}$, and $\mathcal{A} \lesssim^* \mathcal{B}$. If $\mathfrak{F}$ and $\mathfrak{G}$ are both in X', then $\mathfrak{F}^* \subseteq \mathcal{A}$, $\mathfrak{G}^* \subseteq \mathcal{B}$, and if $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$, then $i^*(F^*) \cap G^* \neq \emptyset$. Using Proposition 2.3(a), we deduce that $\mathfrak{F} \lesssim \mathfrak{G}$ in $F(X)$ or, equivalently, $\mathfrak{F} \leq^* \mathfrak{G}$ in X^*. If $\mathfrak{F} = \dot{x}$ and $\mathfrak{G} = \dot{y}$, then $\mathfrak{F}_{\mathcal{A}} \to x$ and $\mathfrak{F}_{\mathcal{B}} \to y$ in X; by Lemma 2.4(b), $\mathfrak{F}_{\mathcal{A}} \lesssim \mathfrak{F}_{\mathcal{B}}$, and since X is T_2-ordered, $x \leq y$, which implies $\dot{x} \leq^* \dot{y}$. For the last case, assume $\mathfrak{F} = \dot{x}$ and $\mathfrak{G} \in X'$. Then $\mathfrak{F}_{\mathcal{A}} \to x$ and $\mathfrak{F}_{\mathcal{B}} = \mathfrak{G}$, which implies $\mathfrak{F}_{\mathcal{A}} \lesssim \mathfrak{G}$; the desired conclusion, $\dot{x} \leq^* \mathfrak{G}$, follows by condition S.

Conversely, let X^* be T_2-ordered. Since this property is hereditary, X is also T_2-ordered, and so it remains only to verify that X satisfies condition S. Let $\mathfrak{F} \to x$ and $\mathfrak{F} \lesssim \mathfrak{G}$, where $\mathfrak{G} \in X'$. Since $\mathfrak{F}^* \xrightarrow{*} \dot{x}$, $\mathfrak{G}^* \xrightarrow{*} \mathfrak{G}$, and X^* is T_2-ordered, we must have $\dot{x} \leq^* \mathfrak{G}$, which implies $d(\mathfrak{G}) \subseteq \dot{x}$. □

3. **Lifting properties.**

PROPOSITION 3.1. *Let X and Y be posets, and let $f: X \to Y$ be an increasing function.*

(a) *If $\mathfrak{F} \lesssim \mathfrak{G}$ in $F(X)$, then $f(\mathfrak{F}) \lesssim f(\mathfrak{G})$ in $F(Y)$.*

(b) *If $\mathfrak{F}$ is a maximal convex filter on X, then $f(\mathfrak{F})^\wedge$ is a maximal convex filter on Y.*

Let $f: X \to Y$ be a continuous, increasing function, where X and Y are convergence ordered spaces and Y is compact and T_2-ordered. If $\mathcal{H}$ is a maximal convex filter on Y, then $\mathcal{H}$ necessarily converges to a unique limit; $\mathcal{H}$ converges because it can be expressed in the form $\mathcal{H} = \mathcal{K}^\wedge$ for some ultrafilter $\mathcal{K}$ on Y, and the uniqueness of the limit is a consequence of Y being T_2. It follows by Proposition 3.1(b) that, for each $\mathfrak{F} \in X^*$, $f(\mathfrak{F})^\wedge$ converges to a unique element of Y which we denote by $y_{\mathfrak{F}}$. Let $f_*: X^* \to Y$ be defined by $f_*(\mathfrak{F}) = y_{\mathfrak{F}}$ for all $\mathfrak{F} \in X^*$. In case $\mathfrak{F} = \dot{x}$, note that $f_*(\dot{x}) = f(x)$, and so f_* is an extension of f to X^*.

PROPOSITION 3.2. *Let X, Y, and f conform to the assumptions of the preceding paragraph. Then $f_*: X^* \to Y$ is an increasing function. Furthermore, if A is a subset of X, then $f_*(A^*) \subseteq cl_Y f(A)$.*

Proof. Let $\mathfrak{F} \leq^* \mathfrak{G}$ in X^*. By Proposition 3.1, $f(\mathfrak{F}) \lesssim f(\mathfrak{G})$ in Y, and $y_{\mathfrak{F}} \leq y_{\mathfrak{G}}$ since Y is T_2-ordered. Thus f is increasing.

Let $y \in f_*(A^*)$. If $y = f_*(\dot{x})$, then $f(x) = y$, and $y \in f(A)$. If $y = f_*(\mathfrak{F})$ for $\mathfrak{F} \in X'$, then $A \in \mathfrak{F}$ and $f(A) \in f(\mathfrak{F})$. But $f(\mathfrak{F}) \to y$ in Y, and so $y \in cl_Y f(A)$. □

THEOREM 3.3. *If X is a c.o.s., Y a compact, regular, T_2-ordered c.o.s., and $f: X \to Y$ a continuous, increasing function, then there is a unique, continuous, increasing extension $f_*: X^* \to Y$.*

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Proof. It remains only to verify that f_* is continuous. Let $\mathcal{A} \xrightarrow{*} \mathfrak{F}$ in X^*. If $\mathfrak{F} = \dot{x}$, then there is $\mathfrak{G} \to x$ in X such that $\mathfrak{G}^* \subseteq \mathcal{A}$. By continuity of f, $f(\mathfrak{G}) \to f_*(\dot{x}) = f(x)$, and by Proposition 3.2, $cl_Y f(\mathfrak{G}) \subseteq f_*(\mathfrak{G}^*)$. Since Y is regular, $cl_Y f(\mathfrak{G}) \to f(x)$, and consequently $f_*(\mathcal{A}) \to f_*(\mathfrak{F}) = f(x)$. In case $\mathfrak{F} \in X'$, we have $\mathfrak{F}^* \subseteq \mathcal{A}$, $f(\mathfrak{F}) \to y_{\mathfrak{F}} = f_*(\mathfrak{F})$ in Y, and $cl_Y f(\mathfrak{F}) \to y_{\mathfrak{F}}$ as well. Again applying Proposition 3.2, $cl_Y f(\mathfrak{F}) \subseteq f_*(\mathfrak{F}^*) \subseteq f_*(\mathcal{A})$, and thus $f_*(\mathcal{A}) \to f_*(\mathfrak{G})$. □

The preceding theorem shows that, whenever X^* is regular and T_2-ordered, (X^*, ϕ) is the largest such compactification of X. Unfortunately, X^* is very rarely T_3. It is, however, "relatively-T_3," and we shall show that relative to this weaker property, (X^*, ϕ) is the largest convergence ordered compactification of X if X is also strongly T_2-ordered.

If (Y, f) is a T_2 compactification of a space X, and $A \subseteq Y$, let $p_Y(A) = A \cup \{(cl_Y A) - f(X)\}$. If $\mathcal{A}$ is a filter on Y, let $p_Y(\mathcal{A})$ be the filter generated by $\{p_Y(A) : A \in \mathcal{A}\}$. A T_2 compactification (Y, f) of X is *relatively-T_3* if $p(\mathcal{A}) \to y$ whenever $\mathcal{A} \to y$ in Y.

LEMMA 3.4. *If X is a c.o.s. and A a convex subset of X, then $cl_{X^*}(A^*) = cl_{X^*}\phi(A) = \phi(cl_X(A)) \cup A'$, where $A' = A^* \cap X'$.*

Proof. It is clear that $\phi(cl_X A) \cup A' \subseteq cl_{X^*}\phi(A) \subseteq cl_{X^*}A^*$. Let $\mathfrak{G} \in cl_{X^*}A^*$. Since $A^* = \phi(A) \cup A'$, $cl_{X^*}A^* = cl_{X^*}\phi(A) \cup cl_{X^*}A'$.

Case 1. $\mathfrak{G} \in cl_{X^*}\phi(A)$. There is $\mathcal{A} \xrightarrow{*} \mathfrak{G}$ such that $\phi(A) \in \mathcal{A}$, which implies $\mathcal{A} = \phi(\mathcal{H})$ for some $\mathcal{H}$ in $F(X)$, where $A \in \mathcal{H}$. If $\mathfrak{G} = \dot{x}$, then $\mathcal{H} \to x$, and therefore $\dot{x} \in \phi(cl_X(A))$. If $\mathfrak{G} \in X'$, then $\mathfrak{G}^* \subseteq \mathcal{A}$, implying $\mathfrak{G} \subseteq \mathcal{H}$ and $A \in \mathfrak{G}$ (since A is convex and $\mathfrak{G}$ is a maximal convex filter). Thus $\mathfrak{G} \in A'$.

Case 2. $\mathfrak{G} \in cl_{X^*}A'$. There is an ultrafilter $\mathcal{A} \xrightarrow{*} \mathfrak{G}$ such that $A' \in \mathcal{A}$. If $\mathfrak{G} = \dot{x}$ there is a maximal convex filter $\mathcal{H} \to x$ in X such that $\mathcal{H}^* \subseteq \mathcal{A}$. Thus $A' \cap H' \neq \varnothing$ for all $H \in \mathcal{H}$, and it follows that $A \in \mathcal{H}$. Hence $x \in cl_X A$, and $\dot{x} \in \phi(cl_X A)$. If $\mathfrak{G} \in X'$, then we can again conclude that $A \in \mathfrak{G}$, and thus that $\mathfrak{G} \in A'$. □

For the compactification (X^*, ϕ) of a T_2 c.o.s. X, we shall write p^* in place of p_{X^*} for the "partial closure operator" defined above. If $A \subseteq X$ is convex, then Lemma 3.4 implies that $p^*(A^*) = A^* \cup \{cl_{X^*}A^* - \phi(X)\} = A^* \cup A' = A^*$. Thus for every convex filter $\mathfrak{F}$ on X, $p^*(\mathfrak{F}^*) = \mathfrak{F}^*$; since filters of this type form a base for the convergence structure of X^*, we have established the following results.

COROLLARY 3.5. *If X is a T_2 c.o.s., then (X^*, ϕ) is a relatively-T_3 compactification of X.*

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If (Y, f) and (Z, g) are T_2 convergence ordered compactification of a c.o.s. X, we shall, following usual conventions, say that $(Y, f) \le (Z, g)$ if there is a continuous, increasing function $h: Z \to Y$ such that $h \circ g = f$.

THEOREM 3.6. *If X is a T_2 c.o.s. and (Y, f) is a T_2-ordered, relatively T_3 convergence ordered compactification of X, then $(Y, f) \le (X^*, \phi)$.*

Proof. Let $f_*: X^* \to Y$ be defined as in the paragraph preceding Proposition 3.2. By the latter proposition, f_* is increasing. We next assert that, for $A \subseteq X$, $f_*(A^*) \subseteq p_Y f(A)$. For if $y \in f_*(A^*)$ and $y = f_*(\dot{x})$, then $x \in A$ and $y \in f(A)$. If $y_{\mathfrak{G}} = f_*(\mathfrak{G})$, for $\mathfrak{G} \in A'$, then $y_{\mathfrak{G}} \in cl_Y(f(A)) - f(X) \subseteq p_Y f(A)$; for otherwise $\mathfrak{G}$ would converge to $f^{-1}(y_{\mathfrak{G}})$ in X, contradicting the assumption $\mathfrak{G} \in A'$.

We now know that $p_Y(f(\mathfrak{F})) \subseteq f_*(\mathfrak{F}^*)$ for each $\mathfrak{F} \in F(X)$. The proof that f_* is continuous can be completed by following the steps in the proof of Theorem 3.3, replacing "$cl_Y f(\mathfrak{F})$" by "$p_Y f(\mathfrak{F})$." □

COROLLARY 3.7. *If X is a strongly-T_2* c.o.s., then (X^*, ϕ) is the largest relatively-T_3, convergence ordered compactification of X.

4. **A topological ordered compactification.**

In [2], we showed that the convergence space compactification of [4] gives rise to a topological compactification with interesting properties. A similar procedure is used here to construct a topological ordered compactification of an arbitrary topological ordered space.

If $(X, \le, \tau)$ is a poset equipped with a topology τ with the property that open monotone members of τ form a subbase for τ, then the resulting space X will be called a *topological ordered space* (abbreviated t.o.s.). Since any topological ordered space has an open base of convex sets, such a space is a special case of a c.o.s. Furthermore, every c.o.s. gives rise to a t.o.s. in a natural way. If X is a c.o.s., a subset U is *open* in X if, whenever $\mathfrak{F} \to x$ in X and $x \in U$, it follows that $U \in \mathfrak{F}$. The set of all open sets forms a topology on X and the resulting topological space, often denoted by λX, is called the *topological modification* of X. For our purposes we consider not λX, but rather the topological space σX, equipped with the topology whose subbase consists of all open, monotone sets in X, and also equipped with the same partial order defined on X. In general, σX has a coarser topology than λX; σX will be called the *topological-ordered modification* of a convergence ordered space X. (In the case of the trivial order relation on X, note that σX and λX coincide.)

PROPOSITION 4.1. *If X is a T_1-ordered c.o.s., then σX is also T_1-ordered.*

Proof. Sets of the form $\imath(x)$ and $d(x)$, for $x \in X$, are closed relative to X. Their complements are subbasic open sets in σX, and consequently $\imath(x)$ and $d(x)$ are also closed in σX. □

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PROPOSITION 4.2. *If X and Y are convergence ordered spaces and $f: X \to Y$ is continuous and increasing, then $f: \sigma X \to \sigma Y$ is also continuous and increasing.*

Proof. The inverse image of an open, monotone set under a continuous, increasing function is again an open, monotone set. □

THEOREM 4.3. *If X is a c.o.s., then σX^* is a topological-ordered compactification of σX. If X is strongly T_1-ordered, then σX^* is a T_1-ordered compactification of σX.*

Proof. Let Y be the t.o.s. obtained by restricting σX^* to the set $\phi(X)$. Then $\phi: \sigma X \to Y$ is continuous by Proposition 4.2. If A is closed and monotone in X, then by Lemma 3.4, A^* is closed in X^*. It is straightforward to verify that A^* is monotone in X^* and thus $\phi(A)$ is closed relative to Y. It follows that $\phi: \sigma X \to Y$ is a homeomorphism, and thus σX^* is a compactification of σX. The second assertion follows by Propositions 2.7 and 4.1. □

The corollary is obtained by considering the case $X = \sigma X$.

COROLLARY 4.4. *Let X be a topological ordered space. Then $(\sigma X^*, \phi)$ is topological ordered compactification of X which is T_1-ordered if X is strongly T_1-ordered. If Y is a compact, T_2-ordered t.o.s., and $f: X \to Y$ is continuous and increasing, then there is a unique, continuous, increasing extension $f_*: \sigma X^* \to Y$.*

If X has the trivial order relation, then $(\sigma X^*, \phi)$ is the topological compactification described in [2]. Note that when σX^* is T_2, this compactification coincides with the topological Stone Čech compactification. However σX^* is rarely T_2; for circumstances under which this occurs, see Theorem 2.5, [2].

The T_1-order (T_2-order) property is referred to as semi-closed (closed) order by Choe and Park [1]. It is shown in [1] that a T_1-ordered t.o.s. X has an order compactification $(W_0(X), \imath)$, where $W_0(X)$ is a T_1 topological space, with the unique extension property of continuous increasing maps from X into T_2-ordered, compact topological spaces. Let X be a t.o.s. such that $(W_0(X), \imath)$ is a T_2-ordered compactification of X. It follows from Corollary 4.4 that there exists a unique, continuous, increasing extension of $\imath$ to $\imath_*: \sigma X^* \to W_0(X)$ such that $\imath_* \circ \phi = \imath$. In this sense, $(\sigma X^*, \phi)$ is a larger ordered compactification of X than $(W_0(X), \imath)$.

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https://doi.org/10.4153/CMB-1984-081-6 Published online by Cambridge University Press