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Generalized Kähler–Einstein Metrics and Energy Functionals

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Abstract. In this paper, we consider a generalized Kähler–Einstein equation on a Kähler manifold M. Using the twisted $\mathcal{K}$-energy introduced by Song and Tian, we show that the existence of generalized Kähler–Einstein metrics with semi-positive twisting $(1,1)$-form θ is also closely related to the properness of the twisted $\mathcal{K}$-energy functional. Under the condition that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, we prove that the existence of generalized Kähler–Einstein metric implies a Moser–Trudinger type inequality.

1 Introduction

An important problem in Kähler geometry is that of finding a canonical Kähler metric in a given Kähler class. By Aubin and Yau's work [1, 23], we know that $[\omega]$ admits a Kähler–Einstein metric when $c_1(M) = 0$ and also when $c_1(M) < 0$ and $[\omega_0] = -kc_1(M)$. For the remaining case, *i.e.,* $c_1(M) > 0$, the existence question is still open. Important progress was made by Tian [18–20], Tian and Yau [21], Siu [14], Ding [8], and others. In [20], Tian introduced $\mathcal{K}$ stability and showed that the existence of Kähler–Einstein metrics is equivalent to the properness of the corresponding energy functionals. For the case where the given Kähler class is not proportional to the first Chern class, we can consider the constant scalar curvature Kähler metrics or, more generally, the extremal Kähler metrics, which were first considered by Calabi [5]. It is well known that the existence of the canonical Kähler metrics is related to the stability in the sense of Hilbert schemes and geometric invariant theory by a conjecture of Yau [24], Tian [20], and Donaldson [10].

Let (M, J) be a $2m$-dimensional complex manifold, let $[\omega_0] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$ be a Kähler class on (M, J), and let $\alpha := 2\pi c_1(M) - k[\omega_0]$, where k is a constant. Fixing a closed $(1,1)$-form $\theta \in \alpha$, we consider the following generalized Kähler–Einstein equation

$$\rho(\omega) - \theta = k\omega, \tag{1.1}$$

where $\rho(\omega)$ is the Ricci form of the Kähler metric $\omega \in [\omega_0]$. If $\theta \equiv 0$, equation (1.1) is just the Kähler–Einstein equation. A Kähler metric ω satisfying (1.1) will be called a generalized Kähler–Einstein metric. Let us denote by $\mathcal{H}_{\omega_0}$ the set of all smooth

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strictly ω_0-plurisubharmonic functions, *i.e.*,

$$\mathcal{H}_{\omega_0} = \{\varphi \in C^\infty(M) : \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi > 0\},$$

and by $\mathcal{K}_{\omega_0}$ the set of all Kähler forms on M cohomologous to ω_0. It is easy to see that solving the generalized Kähler–Einstein equation (1.1) is equivalent to solving the following complex Monge–Ampère equation,

$$\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^m}{\omega_0^m} = \exp(h_{\omega_0} - k\varphi), \tag{1.2}$$

where $\varphi \in \mathcal{H}_{\omega_0}$ and h_{ω_0} is a smooth function that satisfies

$$\rho(\omega_0) - \theta = k\omega_0 + \sqrt{-1}\partial\overline{\partial}h_{\omega_0} \quad \text{and} \quad \int_M \exp(h_{\omega_0})\omega_0^m = \int_M \omega_0^m = V.$$

If $k \le 0$, by Aubin and Yau's work [1, 23], the complex Monge–Ampère equation (1.2) can be solved. In this paper, we consider the remaining case $k > 0$ and there should be obstructions to admit generalized Kähler–Einstein metrics. Through the work of Bando and Mabuchi[4], Ding and Tian[9], Tian[20], Donaldson [11], and others, it is well known that the Mabuchi $\mathcal{K}$-energy is very useful in Kähler geometry. Let us recall the following twisted $\mathcal{K}$-energy, which was first introduced by Song and Tian in [15].

Definition 1.1 For every $(\varphi_0, \varphi_1) \in \mathcal{H}_{\omega_0} \times \mathcal{H}_{\omega_0}$, we define

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) = -\frac{1}{V}\int_0^1 \int_M \dot{\varphi}_t(S(\omega_t) - \Lambda_{\omega_{\varphi_t}}\theta - \overline{S}_\theta)\omega_{\varphi_t}^m\, dt, \tag{1.3}$$

where $\{\varphi_t | 0 \le t \le 1\}$ is an arbitrary piecewise smooth path in $\mathcal{H}_{\omega_0}$ such that $\varphi_t|_{t=0} = \varphi_0$ and $\varphi_t|_{t=1} = \varphi_1$, $S(\omega_{\varphi_t})$ is the scalar curvature of ω_{φ_t}, $\Lambda_{\omega_{\varphi_t}}$ is the contraction with ω_{φ_t}, and $\overline{S}_\theta = \frac{1}{V}\int_M m(2\pi c_1(M) - [\theta]) \cup [\omega_0]^{m-1}$. For every $\varphi \in \mathcal{H}_{\omega_0}$, we define

$$\mathcal{V}_{\theta,\omega_0}(\varphi) = \mathcal{M}_\theta(0, \varphi_1).$$

Song and Tian [15, Proposition 6.1] have shown that the integral in (1.3) is independent of the choice of the path φ_t. Thus, $\mathcal{M}_\theta$ is well defined in $\mathcal{H}_{\omega_0} \times \mathcal{H}_{\omega_0}$. By the definition, it is easy to check that $\mathcal{M}_\theta$ satisfies the 1-cocycle condition, *i.e.*,

$$\begin{gathered}
\mathcal{M}_\theta(\varphi_0, \varphi_1) + \mathcal{M}_\theta(\varphi_1, \varphi_0) = 0,\\
\mathcal{M}_\theta(\varphi_0, \varphi_1) + \mathcal{M}_\theta(\varphi_1, \varphi_2) + \mathcal{M}_\theta(\varphi_2, \varphi_0) = 0,\\
\mathcal{M}_\theta(\varphi_0 + C_0, \varphi_1 + C_1) = \mathcal{M}_\theta(\varphi_1, \varphi_0),
\end{gathered} \tag{1.4}$$

for any $\varphi_0, \varphi_1, \varphi_2 \in \mathcal{H}_{\omega_0}$ and $C_0, C_1 \in \mathbb{R}$. By the above properties, we know that $\mathcal{M}_\theta$ (or $\mathcal{V}_{\theta,\omega_0}$) can also be defined on the space $\mathcal{K}_{\omega_0} \times \mathcal{K}_{\omega_0}$ ($\mathcal{K}_{\omega_0}$).

We say the $\mathcal{K}$-energy functional $\mathcal{V}_{\theta,\omega_0}$ is *proper* if $\limsup_{i\to+\infty}\mathcal{V}_{\theta,\omega_0}(\varphi_i) = +\infty$ whenever $\lim_{i\to+\infty} J_{\omega_0}(\varphi_i) = +\infty$, where $\varphi_i \in \mathcal{H}_{\omega_0}$ and J_{ω_0} is the Aubin functional. In this paper, we follow Tian's method in [20] to show that the existence of generalized Kähler–Einstein metric is closely related to the properness of the twisted $\mathcal{K}$-energy functional, and we also follow the discussion in Tian and Zhu [22] and Phong, Song, Strum, and Weinkove [13] to deduce a Moser–Trudinger type inequality. In fact, we obtain the following theorem.

Theorem 1.2 *Let (M,ω_0) be a Kähler manifold, and let $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. If $\mathcal{V}_{\theta,\omega_0}$ is proper, then there exists a generalized Kähler–Einstein metric $\omega_{GKE} \in \mathcal{K}_{\omega_0}$. On the other hand, assuming that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, if there exists a generalized Kähler–Einstein metric in $\omega_{GKE} \in \mathcal{K}_{\omega_0}$, then $\mathcal{V}_{\theta,\omega_0}$ must be proper. In fact, there exist uniform positive constants C_2, C_3 depending only on k, θ and the geometry of (M,ω_0) such that*

$$\mathcal{V}_{\theta,\omega_0}(\varphi) \geq C_2 J_{\omega_0}(\varphi) - C_3 \tag{1.5}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$.

In [16], Stoppa discussed the so-called twisted cscK equation, *i.e.*, finding a metric $\omega \in [\omega_0]$ such that

$$S(\omega) - \Lambda_\omega \theta = \overline{S}_\theta, \tag{1.6}$$

where θ is a real closed semipositive $(1,1)$-form and $\overline{S}_\theta$ is a constant. In particular, if $\theta \in 2\pi c_1(M) - k[\omega_0]$, then the above twisted cscK equation is equivalent to the generalized Kähler–Einstein equation (1.1). By the definition of the twisted $\mathcal{K}$-energy, it is easy to check that the second derivative along a path $\varphi_t \in \mathcal{H}_{\omega_0}$ is given by

$$\begin{aligned} V\frac{d^2}{dt^2}\mathcal{V}_{\theta,\omega_0}(\varphi_t) &= \|\overline{\partial}\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t\|^2_{\omega_{\varphi_t}} + (\partial\dot{\varphi}_t \wedge \overline{\partial}\dot{\varphi}_t, \theta)_{\omega_{\varphi_t}} \\ &\quad - \int_M \Big(\ddot{\varphi}_t - \frac{1}{2}|\nabla^{1,0}_{\omega_{\varphi_t}}\dot{\varphi}_t|^2_{\omega_{\varphi_t}}\Big)\big(S(\omega_t) - \Lambda_{\omega_{\varphi_t}}\theta - \overline{S}_\theta\big)\,\omega^m_{\varphi_t}. \end{aligned}$$

If either the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field, then $\mathcal{V}_\theta$ is strictly convex along geodesics in $\mathcal{H}_{\omega_0}$. Then the results of Chen and Tian [7] on the regularity of weak geodesics imply uniqueness of solution of the twisted cscK equation (1.6) and that the twisted $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega_0}$ has a lower bound. The above facts were pointed out by Stoppa in [16], where he used the lower bound of $\mathcal{V}_{\theta,\omega_0}$ to get a slope stability condition.

Let $D \subset M$ be an effective divisor. The Seshadri constant of D with respect to the Kähler class $[\omega_0]$ is given by

$$\epsilon\big(D,[\omega_0]\big) = \sup\big\{\, x \mid [\omega_0] - x2\pi c_1(D) \in \mathcal{K}\big\},$$

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where $\mathcal{K}$ is the Kähler cone. Stoppa defined the twisted Ross–Thomas polynomial of $(M, [\omega_0])$ with respect to D and θ by

$$\mathcal{F}_{\theta,D}(\lambda) = \int_0^\lambda (\lambda - x)\alpha_2(x)dx + \frac{\lambda}{2}\alpha_1(0) - \frac{\overline{S}_\theta}{2}\int_0^\lambda (\lambda - x)\alpha_1(x)dx,$$

where

$$\alpha_1(x) = \frac{1}{(m-1)!}\int_M 2\pi c_1(D) \cup ([\omega_0] - 2x\pi c_1(D))^{m-1},$$
$$\alpha_2(x) = \frac{\int_M 2\pi c_1(D) \cup (2\pi c_1(M) - [\theta] - 2\pi c_1(D)) \cup ([\omega_0] - x2\pi c_1(D))^{m-2}}{2(m-2)!}.$$

In [16], Stoppa proved that if (1.6) is solvable in $[\omega_0]$, then $\mathcal{F}_{\theta,D}(\lambda) \geq 0$ for all effective divisors $D \subset M$ and $0 \leq \lambda \leq \epsilon(D, [\omega_0])$. In fact, see [16, Theorem 3.1], we can find a family of Kähler metrics $\omega_\epsilon \in [\omega_0]$ with $\omega_\epsilon|_{\epsilon=1} = \omega_0$ such that as $\epsilon \to 0$

$$\mathcal{V}_{\theta,\omega_0}(\omega_\epsilon) = -\pi\mathcal{F}_{\theta,D}(\lambda)\log(\epsilon) + l.o.t.$$

By the calculation in [16, Lemmas 3.12 and 3.15], we also have the following asymptotic behavior of the Aubin functional:

$$J_{\omega_0}(\omega_\epsilon) = -\frac{\pi}{2}\int_0^\lambda (\lambda - x)\alpha_1(x)dx \log(\epsilon) + l.o.t.$$

By the above Moser–Trudinger inequality (1.5) in Theorem 1.2, we can obtain a strictly slope stability. In fact, we have the following corollary.

Corollary 1.3 *Let (M, ω_0) be a Kähler manifold, and let $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. Assume that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there is a generalized Kähler–Einstein metric $\omega \in \mathcal{K}_{\omega_0}$, then there exists a uniform positive constant C_4 such that*

$$\mathcal{F}_{\theta,D}(\lambda) \geq C_4 \int_0^\lambda (\lambda - x)\alpha_1(x)dx > 0$$

for all effective divisors $D \subset M$ and $0 < \lambda \leq \epsilon(D, [\omega_0])$.

In a special case, if $\alpha = (1-k)[\omega_0]$ with $0 < k < 1$, we let $\theta = (1-k)\omega_0$. Then the generalized Kähler–Einstein equation (1.1) is just the Aubin equation

$$\rho(\omega) = (1-k)\omega_0 + k\omega. \tag{1.7}$$

The twisted $\mathcal{K}$-energy $\mathcal{V}_{(1-k)\omega_0,\omega_0}$ can be expressed by

$$\mathcal{V}_{(1-k)\omega_0,\omega_0}(\varphi) = \mathcal{V}_{\omega_0}(\varphi) + (1-k)(I_{\omega_0} - J_{\omega_0})(\varphi),$$

for all $\varphi \in \mathcal{H}_{\omega_0}$, where $\mathcal{V}_{\omega_0}$ is the Mabuchi $\mathcal{K}$-energy, I_{ω_0} and J_{ω_0} are the Aubin energy functionals. If there exists a Kähler metric $\omega \in [\omega_0]$ such that

$$\rho(\omega) - k\omega > 0,$$

and let $\theta = (1-k)\omega = \rho(\omega) - k\omega > 0$, we know that the generalized Kähler–Einstein equation (1.1) can be solved in $[\omega_0]$. By Theorem 1.2, it follows that $\mathcal{V}_{(1-k)\omega,\omega_0}$ is proper. Moreover, it satisfies the Moser–Trudinger type inequality (1.5). On the other hand, by Lemma 2.1, the cocycle identity of $\mathcal{M}_\theta$ and properties of I_ω, J_ω (see (1.4), (2.6), and (2.5)), it is easy to see that the properness of the twisted $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega}$ is independent on the choice of the twisting form $\theta \in \alpha$ and Kähler metric $\omega \in [\omega_0]$. So we have the following corollary, which was also proved by G. Székelyhidi in [17].

Corollary 1.4 *Let (M, ω_0) be a Kähler manifold with $2\pi c_1(M) = [\omega_0]$, and $0 < k < 1$. The following are equivalent.*

(i) *We can uniquely solve the equation (1.7).*
(ii) *There exists a Kähler metric $\omega \in [\omega_0]$ such that $\rho(\omega) > k\omega$.*
(iii) *For any Kähler metric $\omega \in [\omega_0]$, $\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi)$ is proper.*
(iv) *For any Kähler metric $\omega \in [\omega_0]$, there exist uniform positive constants C_5 and C_6 such that*

$$\mathcal{V}_\omega(\varphi) + (1-k)(I_\omega - J_\omega)(\varphi) \geq C_5 J_\omega(\varphi) - C_6,$$

for all $\varphi \in \mathcal{H}_\omega$.

This paper is organized as follows. In Section 2, we give some preliminary results about energy functionals. In Section 3, we give a existence result for generalized Kähler–Einstein metric; *i.e.*, the properness of twisted $\mathcal{K}$ energy implies the existence of the generalized Kähler–Einstein metrics. In Section 4, we obtain the Moser–Trudinger type inequality (1.5) and finish the proof of Theorem 1.2.

2 Twisted $\mathcal{K}$-energy Functional

Let (M, ω_0) be a Kähler manifold, and let $\alpha \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$. Fix a real closed $(1,1)$ form $\theta \in \alpha$. The twisted $\mathcal{K}$-energy functional can be expressed by

$$\mathcal{M}_\theta(\varphi_0, \varphi_1) = -\frac{1}{V}\int_M \sum_{j=0}^{m-1} (\varphi_1 - \varphi_0)(\rho(\omega_{\varphi_0}) - \theta) \wedge \omega_{\varphi_0}^j \wedge \omega_{\varphi_1}^{m-j-1}$$
$$+ \frac{\overline{S}_\theta}{(m+1)V} \sum_{j=0}^{m} \int_M (\varphi_1 - \varphi_0)\omega_{\varphi_0}^j \wedge \omega_{\varphi_1}^{m-j} + \frac{1}{V}\int_M \log \frac{\omega_{\varphi_1}^m}{\omega_{\varphi_0}^m} \omega_{\varphi_1}^m$$

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and

$$\mathcal{V}_{\theta,\omega_0}(\varphi) = -\frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi(\rho(\omega_0) - \theta) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} + \frac{1}{V}\int_M \log \frac{\omega_\varphi^m}{\omega_0^m}\omega_\varphi^m + \frac{\overline{S}_\theta}{(m+1)V}\sum_{j=0}^{m}\int_M \varphi\omega_0^j \wedge \omega_\varphi^{m-j} \tag{2.1}$$

for all $\varphi, \varphi_0, \varphi_1 \in \mathcal{H}_{\omega_0}$. Let us recall the Aubin functionals

$$I_{\omega_0}(\varphi) = \frac{1}{V}\int_M \varphi(\omega_0^m - \omega_\varphi^m), \quad J_{\omega_0}(\varphi) = \int_0^1 \frac{1}{t} I_{\omega_0}(t\varphi)dt. \tag{2.2}$$

Let φ_s be a smooth curve in $\mathcal{H}_{\omega_0}$, by direct calculation, we have

$$\begin{aligned} \frac{d}{ds}I_{\omega_0}(\varphi_s) &= \frac{1}{V}\int_M \dot{\varphi}_s(\omega_0^m - \omega_{\varphi_s}^m) - \frac{1}{2V}\int_M \varphi_s \triangle_{\varphi_s}\dot{\varphi}_s\omega_{\varphi_s}^m, \\ \frac{d}{ds}J_{\omega_0}(\varphi_s) &= \frac{1}{V}\int_M \dot{\varphi}_s(\omega_0^m - \omega_{\varphi_s}^m) \end{aligned} \tag{2.3}$$

and then

$$\frac{d}{ds}\big(I_{\omega_0}(\varphi_s) - J_{\omega_0}(\varphi_s)\big) = -\frac{1}{2V}\int_M \varphi_s(\triangle_s\dot{\varphi}_s)\omega_{\varphi_s}^m. \tag{2.4}$$

We also have the following properties for I and J (the proof can be found in [4]). For a constant C,

$$I_{\omega_0}(\varphi + C) = I_{\omega_0}(\varphi), \quad J_{\omega_0}(\varphi + C) = J_{\omega_0}(\varphi);$$

and

$$0 \leq I_{\omega_0}(\varphi) \leq (m+1)\{I_{\omega_0}(\varphi) - J_{\omega_0}(\varphi)\} \leq mI_{\omega_0}(\varphi) \tag{2.5}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$. Let ω' be an another Kähler form in $[\omega_0]$, and assume that $\omega' = \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi$ for some function ϕ. It is easy to check that

$$|I_{\omega'}(\varphi - \phi) - I_{\omega_0}(\varphi)| \leq (m+1)\operatorname{Osc}(\phi) \tag{2.6}$$

for all $\varphi \in \mathcal{H}_{\omega_0}$.

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If $\theta_1 - \theta_2 = \sqrt{-1}\partial\overline{\partial} f$, then we have

$$\begin{aligned}\mathcal{V}_{\theta_1,\omega_0}(\varphi) - \mathcal{V}_{\theta_2,\omega_0}(\varphi) &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi(\theta_1 - \theta_2) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} \varphi\sqrt{-1}\partial\overline{\partial} f \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M \sum_{j=0}^{m-1} f(\omega_\varphi - \omega_0) \wedge \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= \frac{1}{V}\int_M f(\omega_\varphi^m - \omega_0^m).\end{aligned}$$

This gives us the following lemma.

Lemma 2.1 *Let $\theta_1 - \theta_2 = \sqrt{-1}\partial\overline{\partial} f$ Then*

$$|\mathcal{V}_{\theta_1,\omega_0}(\varphi) - \mathcal{V}_{\theta_2,\omega_0}(\varphi)| \leq \operatorname{Osc}(f)$$

for all $\varphi \in \mathcal{H}_{\omega_0}$.

Now, we suppose that $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$. Let h_{ω_0} be the smooth function that satisfies

$$\rho(\omega_0) - \theta = k\omega_0 + \sqrt{-1}\partial\overline{\partial} h_{\omega_0} \quad \text{and} \quad \int_M \exp(h_{\omega_0})\omega_0^m = \int_M \omega_0^m = V.$$

Let us recall the Ding–Tian functional

$$F^0_{\omega_0}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi\omega_0^m, \tag{2.7}$$

$$F_{\omega_0}(\varphi) = F^0_{\omega_0}(\varphi) - k^{-1}\log\Big(\frac{1}{V}\int_M e^{h_{\omega_0} - k\varphi}\omega_0^m\Big).$$

Let φ_s be a smooth path in $\mathcal{H}_{\omega_0}$, then

$$\frac{d}{ds}F^0_{\omega_0}(\varphi_s) = -\frac{1}{V}\int_M \dot{\varphi}_s\omega_{\varphi_s}^m,$$

and

$$\frac{d}{ds}F_{\omega_0}(\varphi_s) = -\frac{1}{V}\int_M \dot{\varphi}_s\omega_{\varphi_s}^m + \Big(\int_M e^{h_{\omega_0} - k\varphi}\omega_0^m\Big)^{-1}\int_M \dot{\varphi}_s e^{h_{\omega_0} - k\varphi}\omega_0^m. \tag{2.8}$$

From (2.8), it is easy to check that the critical points of F_{ω_0} are generalized Kähler–Einstein metrics. As that in [18], one can also check that F_{ω_0} satisfies the following cocycle property:

$$\begin{aligned} F_{\omega_0}(\psi) + F_{\omega'}(\phi - \psi) &= F_{\omega_0}(\phi), \\ F_{\omega_0}(\psi) &= -F_{\omega'}(-\psi), \end{aligned} \tag{2.9}$$

for all $\phi, \psi \in \mathcal{H}_{\omega_0}$ and $\omega' = \omega_0 + \sqrt{-1}\partial\overline{\partial}\psi$. Moreover, $F^0_{\omega_0}$ also has the same cocycle condition.

By the definitions and direct calculation, we have

$$V(I_{\omega_0} - J_{\omega_0})(\varphi) = -\frac{m}{m+1}\int_M \varphi\omega_\varphi^m + \frac{1}{m+1}\int_M \sum_{j=1}^{m} \varphi\omega_0^j \wedge \omega_\varphi^{m-j}$$

and

$$\begin{aligned}\int_M h_{\omega_0}(\omega_0^m - \omega_\varphi^m) &= -\int_M h_{\omega_0}(\sqrt{-1}\partial\overline{\partial}\varphi) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= -\int_M \varphi(\sqrt{-1}\partial\overline{\partial}h_{\omega_0}) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1} \\ &= -\int_M \varphi(\rho(\omega_0) - \theta + k\omega_0) \wedge \sum_{j=0}^{m-1} \omega_0^j \wedge \omega_\varphi^{m-j-1}.\end{aligned}$$

Noting that $\overline{S}_\theta = km$, by (2.1),

$$\begin{aligned}\mathcal{V}_{\theta,\omega_0}(\varphi) &= -k(I_{\omega_0} - J_{\omega_0})(\varphi) \\ &\quad + \frac{1}{V}\int_M h_{\omega_0}(\omega_0^m - \omega_\varphi^m) + \frac{1}{V}\int_M \log(\frac{\omega_\varphi^m}{\omega_0})\omega_\varphi^m.\end{aligned} \tag{2.10}$$

We also have the following relation between the Ding–Tian functional and the twisted Mabuchi $\mathcal{K}$-energy functional.

Lemma 2.2 *Let (M, ω_0) be a Kähler manifold, and $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$. Then*

$$\mathcal{V}_{\theta,\omega_0}(\varphi) - kF_{\omega_0}(\varphi) = \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{1}{V}\int_M h_{\omega_\phi}\omega_\phi^m \tag{2.11}$$

for any $\varphi \in \mathcal{H}_{\omega_0}$, where h_ω is the smooth function that satisfies

$$\rho(\omega) - \theta = k\omega + \sqrt{-1}\partial\overline{\partial}h_\omega$$

and the normalized condition $\int_M \exp(h_\omega)\omega^m = V$. Furthermore, we have

$$\mathcal{V}_{\theta,\omega_0}(\varphi) \geq kF_{\omega_0}(\varphi) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m. \tag{2.12}$$

Proof By the definition of h_ω, it is easy to check that

$$-\log\frac{\omega_\varphi^m}{\omega_0^m} - k\phi + c_\varphi = h_{\omega_\varphi} - h_{\omega_0} \tag{2.13}$$

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for all $\varphi \in \mathcal{H}_{\omega_0}$ with $c_\varphi = -\log(\frac{1}{V}\int_M e^{h_{\omega_0}-k\phi}\omega_0)$. Then by (2.10) and (2.13) we have

$$\begin{aligned}
&\mathcal{V}_{\theta,\omega_0}(\varphi)\\
&\quad = kJ_{\omega_0}(\varphi) - kI_{\omega_0}(\varphi) - \frac{k}{V}\int_M \varphi\omega_\varphi^m + c_\varphi + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\phi^m\\
&\quad = k\Big(J_{\omega_0}(\varphi) - \frac{1}{V}\int_M \varphi\omega_0^m + k^{-1}c_\varphi\Big) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{1}{V}\int_M h_{\omega_\varphi}\omega_\phi^m,
\end{aligned}$$

which implies (2.11). By the normalized condition of h_{ω_φ}, we have $\int_M h_{\omega_\varphi}\omega_\varphi^m \leq 0$, and (2.12) follows. ■

3 Existence Result for the Generalized Kähler–Einstein Metrics

As in Kähler–Einstein case, finding generalized Kähler–Einstein metric can be reduced to solving the complex Monge–Ampère equation (1.2). We consider a family of complex Monge–Ampère equations

$$\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^m}{\omega_0^m} = \exp(h_{\omega_0} - tk\varphi) \tag{3.1}$$

and set

$$S = \{t \in [0,1] \mid (3.1) \text{ is solvable for } t\}.$$

By [23], we know that (3.1) is solvable for $t = 0$, thus S is not empty. If we can prove that S is open and closed, then we must have $S = [0,1]$ and the complex Monge–Ampère equation (1.2) can be solved. In the proof that S is open and closed, we need the assumption that θ is semipositive. The key point is that the semipositivity of θ will lead a lower bound of the Ricci curvature by a positive constant. Then we can use the Implicit Function theorem to prove the openness and obtain a lower bound of the Green's function that is crucial to getting the C^0 estimate. We follow Aubin's discussion [1] in the proof of openness and adopt Tian's method [20] to prove closedness. First, we have the following proposition for further discussion; the proof is similar to that in [4].

Proposition 3.1 *Let (M,ω_0) be a Kähler manifold, and $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ is a real closed semipositive $(1,1)$-form with $k > 0$. Let $0 < \tau \leq 1$ and suppose that (3.1) has a solution φ_τ at $t = \tau$.*

- *If $0 < \tau < 1$, then there exists some $\epsilon > 0$ such that φ_τ uniquely extends to a smooth family of solution $\{\varphi_t\}$ of (3.1) for $t \in (0,1) \cap (\tau-\epsilon, \tau+\epsilon)$.*
- *S is also open near $t = 0$,* i.e., *there exists a small positive number ϵ such that there is a smooth family of solutions of (3.1) for $t \in (0,\epsilon)$.*
- *If M admits no nontrivial Hamiltonian holomorphic vector field or the twisting form θ is strictly positive at a point, then φ_1 can also be extended uniquely to a smooth family of solutions $\{\varphi_t\}$ of (3.1) for $t \in (1-\epsilon, 1]$.*

Proof For $2 \leq \gamma \in \mathbb{Z}$ and $0 < \alpha < 1$, we define

$$H^{\gamma,\alpha} = \{\phi \in C^{\gamma,\alpha}(M) \mid \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi > 0\}.$$

Consider the operator $\Xi : H^{\gamma,\alpha} \times R \to C^{\gamma-2,\alpha}(M)$ defined by

$$\Xi(\varphi, t) := \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^m}{(\omega_0)^m} + tk\varphi - h_{\omega_0}.$$

The linearized operator is

$$D_\varphi \Xi(\psi) = \frac{1}{2}\triangle_\varphi \psi + tk\psi$$

for $\psi \in C^{\gamma,\alpha}(M)$. By the implicit function theorem, it is sufficient to prove that $D_\varphi\Xi$ is invertible. For further consideration, let us recall the Bochner–Kodaira formula

$$2\int_M |\nabla^{1,0}(\nabla^{1,0}_\omega u)|^2_\omega \omega^m = \int_M (\triangle_\omega u)^2 - 2\rho(\omega)(\nabla_\omega u, J(\nabla_\omega u))\omega^m \tag{3.2}$$

for any $u \in C^2(M)$ and $\omega \in [\omega_0]$.

In the case of $\tau \in (0, 1)$, we have

$$\rho_{\varphi_\tau} = \theta + k\omega_0 + \tau k\sqrt{-1}\partial\overline{\partial}\varphi_\tau > \tau k\omega_{\varphi_\tau},$$

since φ_τ is a solution of (3.1). Let $\psi \in \ker D_{\varphi_\tau}\Xi$, the Bochner–Kodaira formula (3.2) implies $\nabla_{\omega_{\varphi_\tau}}\psi \equiv 0$ and thus $\psi \equiv 0$. This shows that $D_{\varphi_\tau}\Xi$ is invertible. When $\tau = 0$, we consider the operator

$$\widetilde{\Xi}(\varphi, t) := \log \frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^m}{(\omega_0)^m} + tk\varphi - h_{\omega_0} + \beta\int_M \varphi\omega_0^m,$$

where the constant $\beta > 0$. Its linearized operator is given by

$$D_\varphi\widetilde{\Xi}(\psi) = \frac{1}{2}\triangle_\varphi\psi + tk\psi + \beta\int_M \psi\omega_0^m.$$

It is easy to check that $D_\varphi\widetilde{\Xi}$ is invertible at $t = 0$. By the implicit function theorem, there is a smooth one parameter family $\{\widetilde{\varphi}_t \mid t \in [0, \epsilon)\}$ such that $\widetilde{\Xi}(\widetilde{\varphi}_t, t) = 0$ and

$$\varphi_t = \widetilde{\varphi}_t + \frac{\beta}{tk}\int_M \widetilde{\varphi}_t\omega_0^m$$

is a family of solutions of (3.1) for $t \in (0, \epsilon)$. Thus, S is open near $t = 0$.

When $\tau = 1$, let φ_1 be a solution of (3.1) for $t = 1$, and let $\psi \in \ker D_{\varphi_1}\Xi$, *i.e.*, $\triangle_{\omega_{\varphi_1}}\psi = -2k\psi$. Replacing ω and u in (3.2) by ω_{φ_1} and ψ, we have

$$\int_M |\nabla^{1,0}(\nabla^{1,0}_{\omega_{\varphi_1}}\psi)|^2_{\omega_{\varphi_1}}\omega^m_{\varphi_1} = -\int_M \theta\big(\nabla_{\omega_{\varphi_1}}\psi, J(\nabla_{\omega_{\varphi_1}}\psi)\big)\,\omega^m_{\varphi_1}. \tag{3.3}$$

If θ is strictly positive at some point, then $\nabla_{\omega_{\varphi_1}}\psi = 0$ on some open domains. Because the Laplace–Beltrami operator $\triangle_{\omega_{\varphi_1}}$ is real, Aronszajin's unique continuation theorem implies $\nabla_{\omega_{\varphi_1}}\psi \equiv 0$. If M admits no nontrivial Hamiltonian holomorphic vector field, since θ is semi positive, then (3.3) implies that $\nabla^{1,0}_{\omega_{\varphi_1}}\psi \equiv 0$. So, $D_{\varphi_1}\Xi$ is invertible. ■

Using the generalized Aubin equations and proceeding as in Bando and Mabuchi's paper [4, Section 4], we can obtain the uniqueness of the solution of equation (1.2) (*i.e.,* the uniqueness of the generalized Kähler–Einstein metric). As we mentioned in the introduction, the uniqueness can also be implied by a result of Chen and Tian [7] on the regularity of weak geodesics. So we omit the proof of the following lemma.

Lemma 3.2 *Let (M, ω_0) be a Kähler manifold and let $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. If M admits no nontrivial Hamiltonian holomorphic vector field or the twisting form θ is strictly positive at a point, then there exists at most one solution of (1.2).*

Let $\{\varphi_t\}$ be a smooth family of solutions of (3.1) for $t \in (0,1]$. Differentiating (3.1) with respect to t, one can get

$$\frac{1}{2}\triangle_t\dot{\varphi}_t = -t(m+1)\dot{\varphi}_s - (m+1)\varphi_t. \tag{3.4}$$

Using (3.2) and (3.4), we have the following lemma. Since the proof is the same as that in [4], we also omit the proof.

Lemma 3.3 *If $\{\varphi_t\}$ is a smooth family of solutions of (3.1) for $t \in (0,1]$, then*

$$\frac{d}{dt}(I_{d\eta} - J_{d\eta})(\varphi_t) \geq 0. \tag{3.5}$$

Next, we consider the existence of the generalized Kähler–Einstein metrics, which is given by the following theorem.

Theorem 3.4 *Let (M, ω_0) be a Kähler manifold and let $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. If $\mathcal{V}_{\theta,\omega_0}$ (or F_{ω_0}) is proper then there exists a generalized Kähler–Einstein metric $\omega \in \mathcal{K}_{\omega_0}$.*

Proof From inequality (2.12) in Lemma 2.2, we only need to prove the theorem for the case where modified $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega_0}$ is proper. By Proposition 3.1, we suppose that there exists a smooth family of solution $\{\varphi_t\}$ of (3.1) for $t \in (0,\tau)$ with $\tau \in (0,1)$. From equation (3.1), we know that $\triangle_t\varphi_t \leq 2m$ and $\rho(\omega_{\varphi_t}) \geq tk\omega_{\varphi_t}$. Using Green's formula and the lower bound of the Green's function given by Bando and Mabuchi [4], we have

$$\frac{1}{V}\int_M \varphi_t(\omega_{\varphi_t})^m \leq \inf_M \varphi_t + \frac{\epsilon_1(m)}{tk}$$

for some positive constant $\epsilon_1(m)$ depending only on m. On the other hand, by the fact that $\triangle_{\omega_0}\varphi_t \geq -2m$ and Green's formula, we have

$$\sup_M \varphi_t \leq \frac{1}{V}\int_M \varphi_t(\omega_0)^m + \epsilon_2,$$

where ϵ_2 is a positive constant depending only on the geometry of (M, ω_0). By the normalization, it is easy to see that $\sup_M \varphi_t \geq 0$ and $\inf_M \varphi_t \leq 0$. Then

$$\|\varphi_t\|_{C^0} \leq \sup_M \varphi_t - \inf_M \varphi_t \leq I_{\omega_0}(\varphi_t) + \frac{\epsilon_1(m)}{tk} + \epsilon_2. \tag{3.6}$$

By (2.5) and (3.5), we have

$$I_{\omega_0}(\varphi_{t_1}) \leq (m+1)(I_{\omega_0} - J_{\omega_0})(\varphi_{t_2}) \tag{3.7}$$

for any $0 < t_1 \leq t_2 < \tau$. Combining (3.6) and (3.7), it follows that

$$t\|\varphi_t\|_{C^0} \leq t_0(m+1)(I_{\omega_0} - J_{\omega_0})(\varphi_{t_0}) + \epsilon_3$$

for any $0 < t \leq t_0 < \tau$, where ϵ_3 is a positive constant depending only on k and the geometry of (M, ω_0). Thus, we obtain a uniform bound on

$$\Big|\frac{(\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_t)^m}{(\omega_0)^m}\Big|$$

for $0 < t \leq t_0 < \tau$. By Yau's C^0 estimate [23] for complex Monge–Ampère equations, there exists a uniform constant ϵ_4 such that

$$\|\varphi_t\|_{C^0} \leq \epsilon_4 \text{ for } 0 < t \leq t_0 < \tau. \tag{3.8}$$

It is easy to see that along the solutions φ_t of (3.1),

$$S(\omega_{\varphi_t}) = k\Big(m - \frac{(1-t)}{2}\triangle_{\omega_{\varphi_t}}\varphi_t\Big) + \Lambda_{\omega_{\varphi_t}}\theta \tag{3.9}$$

and

$$\mathcal{V}_{\theta,\omega_0}(\varphi_t) = -k(I_{\omega_0} - J_{\omega_0})(\varphi_t) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m - \frac{tk}{V}\int_M \varphi_t\omega_{\varphi_t}^m. \tag{3.10}$$

Then, by (2.4) and (3.9), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{V}_{\theta,\omega_0}(\varphi_t) &= -\frac{1}{V}\int_M \dot{\varphi}_t\big(S(\omega_{\varphi_t}) - \Lambda_{\omega_{\varphi_t}}\theta - km\big)\,\omega_{\varphi_t}^m \\ &= \frac{k}{V}\int_M \dot{\varphi}_t\frac{(1-t)}{2}\triangle_{\omega_{\varphi_t}}\varphi_t\omega_{\varphi_t}^m = k(t-1)\frac{d}{dt}\big((I_{\omega_0} - J_{\omega_0})(\varphi_t)\big).\end{aligned}$$

Together with (3.10), this gives

$$\frac{d}{dt}\Big(\frac{t}{V}\int_M \varphi_t\omega_{\varphi_t}^m + t(I_{\omega_0} - J_{\omega_0})(\varphi_t)\Big) = (I_{\omega_0} - J_{\omega_0})(\varphi_t). \tag{3.11}$$

By the uniform estimate (3.8) near $t = 0$, we know

$$\frac{t}{V}\int_M \varphi_t\omega_{\varphi_t}^m + t(I_{\omega_0} - J_{\omega_0})(\varphi_t) \to 0, \quad \text{as } t \to 0.$$

The identity (3.11) implies

$$\frac{1}{V}\int_M \varphi_t\omega_{\varphi_t}^m + (I_{\omega_0} - J_{\omega_0})(\varphi_t) \geq 0$$

and

$$\mathcal{V}_{\theta,\omega_0}(\varphi_t) \leq -k(1-t)(I_{\omega_0} - J_{\omega_0})(\varphi_t) + \frac{1}{V}\int_M h_{\omega_0}\omega_0^m \leq \frac{1}{V}\int_M h_{\omega_0}\omega_0^m.$$

Then the properness of $\mathcal{V}_{\theta,\omega_0}$ implies that $J_{\omega_0}(\varphi_t)$ and $I_{\omega_0}(\varphi_t)$ are uniformly bounded. By (3.6)), we obtain a uniform C^0 estimate on φ_t for $t \in [\epsilon, \tau)$. By Yau's estimate ([23]) for complex Monge–Ampère equations, the C^0-estimate implies the $C^{2,\alpha}$-estimate and the elliptic Schauder estimates give higher order estimates. Therefore, the equation (1.2) can be solved, *i.e.,* there exist generalized Kähler–Einstein metrics in $\mathcal{K}_{\omega_0}$. ∎

4 A Moser–Trudinger Type Inequality

First, we consider the generalized Kähler–Ricci flow

$$\frac{\partial\omega_s}{\partial s} = -(\rho(\omega_s) - \theta - k\omega_s)$$

with $\omega_s|_{s=0} = \widetilde{\omega}_0 \in [\omega_0]$. Solving these equations can be reduced to studying the following parabolic version of the complex Monge–Ampère equation:

$$\frac{\partial v}{\partial s} = \log\frac{(\widetilde{\omega}_0 + \sqrt{-1}\partial\overline{\partial}v)^m}{\widetilde{\omega}_0^m} + kv - h_{\widetilde{\omega}_0} \tag{4.1}$$

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with $v|_{s=0} \equiv 0$. By Cao's result [6], we have the long-time existence for (4.1). Let v_s be a smooth solution of (4.1) and $\widetilde{\omega}_s = \widetilde{\omega}_0 + \sqrt{-1}\partial\overline{\partial} v_s$. By direct calculation, we have

$$
\begin{aligned}
&\frac{\partial}{\partial s}\dot{v}_s = \frac{1}{2}\triangle_{\widetilde{\omega}_s}\dot{v}_s + k\dot{v}_s,\\
&\frac{\partial}{\partial s}|d\dot{v}_s|^2_{\widetilde{\omega}_s} = \frac{1}{2}\triangle_{\widetilde{\omega}_s}|d\dot{v}_s|^2_{\widetilde{\omega}_s} + k|d\dot{v}_s|^2_{\widetilde{\omega}_s} - |\nabla_{\widetilde{\omega}_s} d\dot{v}_s|^2_{\widetilde{\omega}_s} - \theta(\nabla_{\widetilde{\omega}_s}\dot{v}_s, J(\nabla_{\widetilde{\omega}_s}\dot{v}_s)),\\
&\Big(\frac{\partial}{\partial s} - \frac{1}{2}\triangle_{\widetilde{\omega}_s}\Big)(\dot{v}_s^2 + s|d\dot{v}_s|^2_{\widetilde{\omega}_s})\\
&\quad = 2k\dot{v}_s^2 + sk|d\dot{v}_s|^2_{\widetilde{\omega}_s} - s|\nabla_{\widetilde{\omega}_s} d\dot{v}_s|^2_{\widetilde{\omega}_s} - s\theta(\nabla_{\widetilde{\omega}_s}\dot{v}_s, J(\nabla_{\widetilde{\omega}_s}\dot{v}_s))\\
&\quad \le 2k(\dot{v}_s^2 + s|d\dot{v}_s|^2_{\widetilde{\omega}_s}),\\
&\Big(\frac{\partial}{\partial s} - \frac{1}{2}\triangle_{\widetilde{\omega}_s}\Big)(\triangle_{\widetilde{\omega}_s}\dot{v}_s) = k\triangle_{\widetilde{\omega}_s}\dot{v}_s - |\partial\overline{\partial}\dot{v}_s|^2_{\widetilde{\omega}_s},
\end{aligned} \tag{4.2}
$$

where $\dot{v}_s = \frac{\partial}{\partial s} v_s$ and we have used the semi-positivity of θ in (4.2). Using the maximum principle and the above equalities, proceeding as in [3] (or [13, Lemma 4]), we have the following lemma.

Lemma 4.1 *The following inequalities hold for all $s \ge 0$:*

$$\Big\| \frac{\partial v_s}{\partial s} \Big\|_{C^0} \le e^{ks} \|h_{\widetilde{\omega}_0}\|_{C^0}, \tag{4.3}$$

$$\sup_M \big(|h_{\widetilde{\omega}_s}|^2 + s|dh_{\widetilde{\omega}_s}|^2_{\widetilde{\omega}_s} \big) \le 4e^{2ks} \|h_{\widetilde{\omega}_0}\|^2_{C^0}, \tag{4.4}$$

$$e^{-ks}\triangle_{\widetilde{\omega}_s} h_{\widetilde{\omega}_s} \ge \triangle_{\widetilde{\omega}_0} h_{\widetilde{\omega}_0}. \tag{4.5}$$

Lemma 4.2 *Suppose that there exists a generalized Kähler–Einstein metric $\omega_{GKE} \in [\omega_0]$. Let $v_{t,s}$ be a solution of (4.1) with $\widetilde{\omega}_0 = \omega_{\varphi_t}$ and $\widetilde{h} = h_{\widetilde{\omega}_1} - \frac{1}{V}\int_M h_{\widetilde{\omega}_1}(\widetilde{\omega}_1)^m$. We assume that*

$$\frac{1}{2}\omega_{GKE} \le \widetilde{\omega}_1 \le \omega_{GKE}. \tag{4.6}$$

Then for any $p > 2m$ there exists a positive constant $\overline{C}_1$ depending only on p, k, and (M, ω_{GKE}) such that

$$\|\widetilde{h}\|_{C^0} \le \overline{C}_1 (1-t)^{\frac{1}{p-1}} \|h_{\omega_{\varphi_t}}\|_{C^0}^{\frac{p-2}{p-1}}. \tag{4.7}$$

Proof By the condition $\widetilde{\omega}_0 = \omega_{\varphi_t}$, we have

$$\rho(\widetilde{\omega}_0) = \theta + k\omega_0 + tk\sqrt{-1}\partial\overline{\partial}\varphi_t \ge \theta + tk\widetilde{\omega}_0$$

and $\triangle_{\widetilde{\omega}_0} h_{\widetilde{\omega}_0} \ge 2mk(t-1)$. By (4.5), we have

$$-\triangle_{\widetilde{\omega}_1} h_{\widetilde{\omega}_1} \le 2mke^k(1-t).$$

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Integrating by parts, we have

$$\int_M |d\widetilde{h}|^2_{\widetilde{\omega}_1} \widetilde{\omega}_1^m = -\int_M \widetilde{h} \triangle_{\widetilde{\omega}_1} \widetilde{h} \widetilde{\omega}_1^m \leq \int_M (\widetilde{h} - \inf \widetilde{h}) \sup_M (-\triangle_{\widetilde{\omega}_1} \widetilde{h}) \widetilde{\omega}_1^m \leq \overline{C}_2 (1-t) \|\widetilde{h}\|_{C^0}, \tag{4.8}$$

where $\overline{C}_2$ depends only on k, m and the volume V. On the other hand, (4.4) implies that

$$\|\widetilde{h}\|_{C^0} \leq 4e^k \|h_{\widetilde{\omega}_0}\|_{C^0}.$$

If $p \geq 2m+1$, then by the Sobolev imbedding theorem [2, Lemma 2.22], the Poincaré inequality, (4.4), and condition (4.6), we have

$$\begin{aligned} \|\widetilde{h}\|^p_{C^0} &\leq \overline{C}_3 \int_M \big(|\widetilde{h}|^p + |d\widetilde{h}|^p_{\omega_{GKE}} \big) \omega^m_{GKE} \\ &\leq \overline{C}_4 \|h_{\widetilde{\omega}_0}\|^{p-2}_{C^0} \int_M \big(|\widetilde{h}|^2 + |d\widetilde{h}|^2_{\omega_{GKE}} \big) \omega^m_{GKE} \\ &\leq \overline{C}_5 \|h_{\widetilde{\omega}_0}\|^{p-2}_{C^0} \int_M |d\widetilde{h}|^2_{\omega_{GKE}} \omega^m_{GKE} \\ &\leq \overline{C}_6 \|h_{\widetilde{\omega}_0}\|^{p-2}_{C^0} \int_M |d\widetilde{h}|^2_{\widetilde{\omega}_1} \widetilde{\omega}_1^m, \end{aligned} \tag{4.9}$$

where constants $\overline{C}_i$ depends only on p, m and the geometry of (M, ω_{GKE}). Then (4.8) and (4.9) imply (4.7), and we are done. ■

Lemma 4.3 *Let $v_{t,s}$ be a solution of (4.1) with initial data $\widetilde{\omega}_0 = \omega_{\varphi_t}$ and $u_t = v_{t,1}$. We have the following estimate*

$$\|u_t\|_{C^0} \leq \frac{1}{k} e^k \|h_{\omega_{\varphi_t}}\|_{C^0} \quad \textit{for } t \in [0,1]. \tag{4.10}$$

Moreover, if we assume that

$$\frac{1}{2}\omega_{GKE} \leq \omega_{\varphi_t + u_t} \leq \omega_{GKE}$$

for all $t \in [t_1, 1]$ with $t_1 \in [0,1)$, then for any $p > 2m$ and $0 \leq \delta < 1$ there exists a constant $\overline{C}_7$ depending only on p, k, and (M, ω_{GKE}) such that

$$\|h_{\omega_{\varphi_t + u_t}}\|_{C^{0,\delta}(\omega_{GKE})} \leq \overline{C}_7 (1-t)^{1-\beta} \big(1 + \|h_{\omega_{\varphi_t}}\|_{C^0} \big)^{\beta} \tag{4.11}$$

for all $t \in [t_1, 1]$, where $\beta = \frac{p+\delta-2}{p-1}$.

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Proof Estimate (4.10) can be easily deduced from (4.3).

By the condition $\frac{1}{2}\omega_{GKE} \le \omega_{\varphi_t+u_t} \le \omega_{GKE}$, we have

$$|dh_{\omega_{\varphi_t+u_t}}|_{\omega_{GKE}} \le \sqrt{2}|dh_{\omega_{\varphi_t+u_t}}|_{\omega_{\varphi_t+u_t}} \quad \text{for } t \in [t_1, 1].$$

In the following proof, let $d(x, y)$ be the distance between x and y with respect to the metric ω_{GKE}.

If $d(x, y) \le (1-t)^{\frac{1}{p-1}}\big(1 + \|h_{\omega_{\varphi_t}}\|_{C^0}\big)^{-\frac{1}{p-1}}$, by (4.4) in Lemma 4.1, we have

$$\begin{aligned}(4.12)\qquad |h_{\omega_{\varphi_t+u_t}}(x) - h_{\omega_{\varphi_t+u_t}}(y)| &\le d(x,y)\sup_M |dh_{\omega_{\varphi_t+u_t}}|_{\omega_{GKE}}\\ &\le \sqrt{2}d(x,y)\sup_M |dh_{\omega_{\varphi_t+u_t}}|_{\omega_{\varphi_t+u_t}} \le 4\sqrt{2}e^k d(x,y)\big(1+\|h_{\omega_{\varphi_t}}\|_{C^0}\big)\\ &\le 4\sqrt{2}e^k(1-t)^{\frac{1-\delta}{p-1}}\big(1+\|h_{\omega_{\varphi_t}}\|_{C^0}\big)^{\frac{p+\delta-2}{p-1}} d(x,y)^{\delta}.\end{aligned}$$

If $d(x, y) \ge (1-t)^{\frac{1}{p-1}}\big(1 + \|h_{\omega_{\varphi_t}}\|_{C^0}\big)^{-\frac{1}{p-1}}$, then the estimate (4.7) in Lemma 4.2 implies

$$\begin{aligned}(4.13)\qquad |h_{\omega_{\varphi_t+u_t}}(x) - h_{\omega_{\varphi_t+u_t}}(y)| &\le 2\|\widetilde{h}\|_{C^0} \le 2\overline{C}_1(1-t)^{\frac{1}{p-1}}\|h_{\omega_{\varphi_t}}\|_{C^0}^{\frac{p-2}{p-1}}\\ &\le 2\overline{C}_1(1-t)^{\frac{1-\delta}{p-1}}\big(1+\|h_{\omega_{\varphi_t}}\|_{C^0}\big)^{\frac{p+\delta-2}{p-1}} d(x,y)^{\delta}.\end{aligned}$$

On the other hand, the normalization condition $\int_M e^{h_{\omega_{\varphi_t+u_t}}}(\omega_{\varphi_t+u_t})^m = V$ implies that $h_{\omega_{\varphi_t+u_t}}$ change signs. So we have

$$\begin{aligned}(4.14)\qquad \|h_{\omega_{\varphi_t+u_t}}\|_{C^0} &\le \operatorname{Osc}(h_{\omega_{\varphi_t+u_t}}) = \operatorname{Osc}(\widetilde{h}) \le 2\|\widetilde{h}\|_{C^0}\\ &\le 2\overline{C}_1(1-t)^{\frac{1}{p-1}}\|h_{\omega_{\varphi_t}}\|_{C^0}^{\frac{p-2}{p-1}}.\end{aligned}$$

It is easy to see that (4.12), (4.13), and (4.14) imply the estimate (4.11). ■

Set $\zeta := 1 - \frac{1}{4m} > \frac{1}{2}$ and define the function f_{ω_0} by

$$f_{\omega_0}(t) := (1-t)^{1-\zeta}\big(k^{-1} + 2(1-t)\|\varphi_t\|_{C^0}\big)^{\zeta}.$$

Proceeding as in [18] (or [13, Lemma 1]), we have the following proposition. We give the proof for reader's convenience.

Proposition 4.4 *Let φ_t be a smooth family of solutions of equation (3.1) for $t \in (0, 1]$, and $\omega_{GKE} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_1$. There exists a constant $D > 0$ depending only on k and (M, ω_{GKE}) such that*

$$\|\varphi_1 - \varphi_t\|_{C^0} \le A(1-t)\|\varphi_t\|_{C^0} + 1$$

for all $t \in [t_0, 1]$, where $t_0 \in [0, 1)$ satisfies $f_{\omega_0}(t_0) = \max_{[t_0,1]} f_{\omega_0} = D$ and A depends only on m and k.

Proof Let us rewrite (3.1) as the following complex Monge–Ampère equation with ω_{GKE} as the reference metric

$$\frac{(\omega_{GKE}+\sqrt{-1}\partial\overline{\partial}(\varphi_t-\varphi_1))^m}{(\omega_{GKE})^m}=\exp(-k(\varphi_t-\varphi_1)+(1-t)k\varphi_t). \tag{4.15}$$

It is easy to see that $h_{\omega_{\varphi_t}}=(t-1)k\varphi_t+c_t$ for some constant c_t. The integral normalization of the potential function $h_{\omega_{\varphi_t}}$ implies

$$|c_t|\le k(1-t)\|\varphi_t\|_{C^0}$$

and

$$\|h_{\omega_{\varphi_t}}\|_{C^0}\le 2k(1-t)\|\varphi_t\|_{C^0}. \tag{4.16}$$

Then it follows from Lemma 4.3 that

$$\|u_t\|_{C^0}\le 2e^k(1-t)\|\varphi_t\|_{C^0}. \tag{4.17}$$

Let us recall $\omega_{\varphi_t+u_t}=\omega_0+\sqrt{-1}\partial\overline{\partial}(\varphi_t+u_t)=\omega_{GKE}+\sqrt{-1}\partial\overline{\partial}(\varphi_t+u_t-\varphi_1)$, and then

$$\frac{(\omega_{GKE}+\sqrt{-1}\partial\overline{\partial}(\varphi_t+u_t-\varphi_1))^m}{\omega_{GKE}^m}= \exp\bigl(-k(\varphi_t+u_t-\varphi_1)-h_{\omega_{\varphi_t+u_t}}-\widetilde{c}_t\bigr) \tag{4.18}$$

for some constant $\widetilde{c}_t$.

Let $\widetilde{\varphi}_t=\varphi_t+u_t-\varphi_1+\frac{\widetilde{c}_t}{k}$. By (4.18), (4.15) and (4.17), we have

$$\int_M e^{h_{\omega_{\varphi_t+u_t}}}\omega_{\varphi_t+u_t}^m=\int_M e^{-k\widetilde{\varphi}_t}\omega_{GKE}^m=\int_M e^{-k\widetilde{\varphi}_t+tk\varphi_t-k\varphi_1}\omega_{\varphi_t}^m$$
$$=\int_M e^{(t-1)k\varphi_t-ku_t-\widetilde{c}_t}\omega_{\varphi_t}^m.$$

It follows that

$$|\widetilde{c}_t|\le(1-t)k\|\varphi_t\|_{C^0}+k\|u_t\|_{C^0}\le(1-t)k(1+2e^k)\|\varphi_t\|_{C^0}. \tag{4.19}$$

Recall that $\varphi_t-\varphi_1=\widetilde{\varphi}_t-u_t-\frac{\widetilde{c}_t}{k}$, from (4.17) and (4.19), we have

$$\|\varphi_t-\varphi_1\|_{C^0}=\bigl\|\widetilde{\varphi}_t\bigr\|_{C^0}+(1-t)(4e^k+1)\|\varphi_t\|_{C^0}.$$

Thus, it is enough to get the estimate $\|\widetilde{\varphi}_t\|_{C^0}\le 1$.

Let us consider the following complex Monge–Ampère equation

$$\log\frac{(\omega_{GKE}+\sqrt{-1}\partial\overline{\partial}\psi)^m}{\omega_{GKE}^m}+k\psi=\widetilde{\psi}. \tag{4.20}$$

The linearized operator of the left side of (4.20) at $\psi = 0$ is

$$\delta\psi \mapsto \frac{1}{2}\triangle_{\omega_{GKE}}\delta\psi + k\delta\psi.$$

If M does not have non-trivial Hamiltonian holomorphic vector fields or θ is strictly positive at a point, then by (3.3), we have

$$\ker\Big(\frac{1}{2}\triangle_{\omega_{GKE}} + k\Big) = 0.$$

Then the operator $(\frac{1}{2}\triangle_{\omega_{GKE}} + k)\colon C^{i+2,\epsilon}(M) \to C^{i+2,\epsilon}(M)$ is invertible. Applying the implicit function theorem, there exist positive constants $\epsilon(\omega_{GKE})$ and $C^*(\omega_{GKE})$ that depend only on δ and the geometry of (M, ω_{GKE}) such that

$$\text{if } \|\widetilde{\psi}\|_{C^{0,\delta}} \leq \epsilon(\omega_{GKE}), \text{ then } \|\psi\|_{C^{2,\delta}} \leq C^*(\omega_{GKE})\|\widetilde{\psi}\|_{C^{0,k}}. \tag{4.21}$$

Let

$$D = \frac{\epsilon k^{-\zeta}}{2(\overline{C}_7 + 1)(C^* + 1)(\epsilon + 1)}$$

with $\epsilon = \epsilon(\omega_{GKE})$, $C^* = C^*(\omega_{GKE})$ chosen as in (4.21), $\zeta = 1 - \frac{1}{4m}$ and $\overline{C}_7$ is defined as in Lemma 4.3 (by choosing $\delta = \frac{1}{2}$ and $p = 2m + 1$). Let $t_0 \in [0, 1)$ satisfies $f_{\omega_0}(t_0) = \max_{[t_0,1]} f_{\omega_0} = D$. Now, we only need to prove the following claim.

Claim For all $t \in [t_0, 1]$, we have

$$\|\widetilde{\varphi}_t\|_{C^{2,\frac{1}{2}}} < \frac{1}{2}.$$

We argue by contradiction. Because $\widetilde{\varphi_1} = 0$, there exists $t_1 \in [t_0, 1)$ such that

$$\|\widetilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(\omega_{GKE})} = \frac{1}{2} \quad \text{and} \quad \|\widetilde{\varphi}_t\|_{C^{2,\frac{1}{2}}(\omega_{GKE})} < \frac{1}{2}, \qquad t_1 < t < 1.$$

In particular, one has $-\frac{1}{4}\omega_{GKE} \leq \sqrt{-1}\partial\overline{\partial}\widetilde{\varphi}_t \leq \frac{1}{4}\omega_{GKE}$, and then $\frac{3}{4}\omega_{GKE} \leq \omega_{\varphi_t + u_t} \leq \frac{5}{4}\omega_{GKE}$ for all $t \in [t_1, 1]$. By applying (4.11) in Lemma 4.3 (choosing $p = 2m + 1$) and (4.16), we have

$$\begin{aligned}
\|h_{\omega_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(\omega_{GKE})} &\leq \overline{C}_7(1-t)^{1-\zeta}\big(1 + \|h_{\omega_{\varphi_t}}\|_{C^0}\big)^{\zeta} \\
&\leq \overline{C}_7(1-t)^{1-\zeta}\big(1 + 2(1-t)k\|\varphi_t\|_{C^0}\big)^{\zeta} \\
&\leq \overline{C}_7 k^{\zeta}(1-t)^{1-\zeta}\big(k^{-1} + 2(1-t)\|\varphi_t\|_{C^0}\big)^{\zeta} \\
&\leq \overline{C}_7 k^{\zeta} D = \frac{\overline{C}_7\epsilon}{2(\overline{C}_7 + 1)(C^* + 1)(\epsilon + 1)} < \epsilon
\end{aligned}$$

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for all $t \in [t_1, 1]$. Using (4.21) again, we get

$$\begin{aligned}\|\widetilde{\varphi}_{t_1}\|_{C^{2,\frac{1}{2}}(d\eta_{SE})} &\leq C^* \|h_{d\eta_{\varphi_t+u_t}}\|_{C^{0,\frac{1}{2}}(\omega_{GKE})} \\ &\leq \frac{C^*\overline{C}_7\epsilon}{2(\overline{C}_7+1)(C^*+1)(\epsilon+1)} < \frac{1}{2}.\end{aligned}$$

This gives a contradiction and complete the proof of the claim. ∎

Using Proposition 4.4 and proceeding as in [13, Theorem 1], we can establish a Moser–Trudinger type inequality for functional $F_{\omega_{GKE}}$. In fact, we obtain the following theorem.

Theorem 4.5 *Let (M, ω_0) be a Kähler manifold and $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. Assume that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler–Einstein metric $\omega_{GKE} \in \mathcal{K}_{\omega_0}$, then there exist uniform positive constants $\widetilde{C}_1, \widetilde{C}_2$ depending only on θ, k, and the geometry of (M, ω_{GKE}) such that*

$$F_{\omega_{GKE}}(\varphi) \geq \widetilde{C}_1 J_{\omega_{GKE}}(\varphi) - \widetilde{C}_2 \tag{4.22}$$

for all $\varphi \in \mathcal{H}_{\omega_{GKE}}$.

Proof Fix a function $\phi \in \mathcal{H}_{\omega_{GKE}}$ and let $\omega_0 = \omega_{GKE} + \sqrt{-1}\partial\overline{\partial}\phi$. Now we consider the complex Monge–Ampère equation (3.1). Since M admits no nontrivial Hamiltonian holomorphic vector fields or the twisting form θ is strictly positive at a point, by the uniqueness of generalized Kähler–Einstein structure (Lemma 3.2) and Proposition 3.1, a unique solution φ_t exists for all $t \in (0,1]$ and $\omega_{\varphi_1} = \omega_{GKE}$. Moreover, φ_1 and $-\phi$ differ by a constant.

For further consideration, we give the following estimates for functionals F, I, and J. By (2.2), (2.3), and (3.4), we have

$$\frac{d}{ds}(I_{\omega_0} - J_{\omega_0})(\varphi_s) = -\frac{d}{ds}\Big(\frac{1}{V}\int_M \varphi_s \omega_{\varphi_s}^m\Big) - \frac{1}{sV}\int_M \varphi_s \omega_{\varphi_s}^m.$$

The uniform C^0 estimate of φ_t (3.8) implies that

$$s\frac{1}{V}\int_M \varphi_s(\omega_{\varphi_s})^m \to 0, \quad \text{as } s \to 0.$$

Integrating on $[0,t]$ for both sides of (4.40), we get

$$t(I_{\omega_0} - J_{\omega_0})(\varphi_t) - \int_0^t (I_{\omega_0} - J_{\omega_0})(\varphi_s)ds = -\frac{t}{V}\int_M \varphi_t \omega_{\varphi_t}^m, \tag{4.23}$$

and then

$$\begin{aligned}F^0_{\omega_0}(\varphi_t) &= -(I_{\omega_0} - J_{\omega_0})(\varphi_t) - \frac{1}{V}\int_M \varphi_t \omega_{\varphi_t}^m \\ &= -\frac{1}{t}\int_0^t (I_{\omega_0} - J_{\omega_0})(\varphi_s)ds.\end{aligned} \tag{4.24}$$

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Taking $t = 1$ and using the fact $F_{\omega_0}(\varphi_1) = -F_{\omega_{GKE}}(\phi)$, we obtain

$$F_{\omega_{GKE}}(\phi) = \int_0^1 (I_{\omega_0} - J_{\omega_0})(\varphi_s) ds. \tag{4.25}$$

By the definitions given in (2.7) and the cocycle property of $F^0_{\omega_0}$, it is easy to check that

$$|J_{\omega_0}(\varphi_1) - J_{\omega_0}(\varphi_t)| \leq \operatorname{Osc}(\varphi_1 - \varphi_t) \tag{4.26}$$

and

$$|(I_{\omega_0} - J_{\omega_0})(\varphi_t) - (I_{\omega_0} - J_{\omega_0})(\varphi_1)| \leq m \cdot \operatorname{Osc}(\varphi_1 - \varphi_t).$$

Again, by the fact that $F_{\omega_0}(\varphi_1) = -F_{\omega_{GKE}}(\phi)$, we have

$$\begin{aligned} J_{\omega_0}(\varphi_1) &= F_{\omega_0}(\varphi_1) + \frac{1}{V}\int_M \varphi_1 \omega_0^m = -F_{\omega_{GKE}}(\phi) + \frac{1}{V}\int_M \varphi_1 \omega_0^m \\ &= -J_{\omega_{GKE}}(\phi) + \frac{1}{V}\int_M \phi(\omega_{GKE}^m - \omega_0^m) \\ &= (I_{\omega_{GKE}} - J_{\omega_{GKE}})(\phi) \geq \frac{1}{m} J_{\omega_{GKE}}(\phi), \end{aligned} \tag{4.27}$$

where we have used the inequality (2.5). Notice that $(I_{\omega_0} - J_{\omega_0})(\varphi_t)$ is nondecreasing in t, so (4.25) implies that

$$F_{\omega_{GKE}}(\phi) \geq (1-t)(I_{\omega_0} - J_{\omega_0})(\varphi_t) \geq \frac{1-t}{m} J_{\omega_0}(\varphi_t).$$

Combining this with (4.27) and (4.26), we have

$$F_{\omega_{GKE}}(\phi) \geq \frac{1-t}{m^2} J_{\omega_{GKE}}(\phi) - \frac{1-t}{m} \operatorname{Osc}(\varphi_t - \varphi_1). \tag{4.28}$$

In the following, we choose t_0 as in Proposition 4.4.

If $2(1-t_0)\|\varphi_{t_0}\|_{C^0} \leq k^{-1}$, the definition of t_0 gives $D \leq (1-t_0)^{1-\zeta} 2^{\zeta} k^{-\zeta}$, *i.e.*,

$$(1-t_0) \geq 2^{-\frac{\zeta}{1-\zeta}} k^{\frac{\zeta}{1-\zeta}} D^{\frac{1}{1-\zeta}}.$$

If $2(1-t_0)\|\varphi_{t_0}\|_{C^0} \geq k^{-1}$, we have $D \leq 4^{\zeta}(1-t_0)\|\varphi_t\|^{\zeta}_{C^0}$. Then

$$(1-t_0) \geq \frac{D}{4^{\zeta}\|\varphi_{t_0}\|^{\zeta}_{C^0}}.$$

In the second case, we may assume that $1 - t_0 < \frac{A^{-1}}{2}$, which implies that

$$\|\varphi_{t_0}\|_{C^0} \leq 2\|\varphi_1\|_{C^0} + 2.$$

Then

$$(1-t_0) \geq \frac{D}{4^{\zeta}(2\|\varphi_1\|_{C^0}+2)^{\zeta}}.$$

Again, by the fact that $\sup \varphi_1 \cdot \inf \varphi_1 \leq 0$, we always have

$$(1-t_0) \geq \frac{C'}{(\|\varphi_1\|_{C^0}+1)^{\zeta}} \geq \frac{C'}{(\mathrm{Osc}(\phi)+1)^{\zeta}}, \tag{4.29}$$

where C' is a positive constant depending only on θ, k and (M, ω_{GKE}). On the other hand, by Proposition 4.4 again, we have

$$(1-t_0)\|\varphi_1-\varphi_{t_0}\|_{C^0} \leq (1-t_0)^2 A\|\varphi_{t_0}\|_{C^0}+1 \leq AD^{\frac{1}{\zeta}}+1.$$

Together with (4.28) and (4.29), this estimate gives

$$F_{\omega_{GKE}}(\phi) \geq \widetilde{C}_3 \frac{J_{\omega_{GKE}}(\phi)}{(\mathrm{Osc}(\phi)+1)^{\zeta}} - \widetilde{C}_4 \tag{4.30}$$

for all $\phi \in \mathcal{H}_{\omega_{GKE}}$, where $\widetilde{C}_3$ and $\widetilde{C}_4$ are positive constants depending only on θ, k, and the geometry of (M, ω_{GKE}).

Since $\varphi_t - \varphi_1 \in \mathcal{H}_{\omega_{GKE}}$ and $\rho(\omega_{\varphi_t}) \geq \theta + tk\omega_{\varphi_t}$, we can use (3.6) to obtain the estimate

$$\mathrm{Osc}(\varphi_t-\varphi_1) \leq I_{\omega_{GKE}}(\varphi_t-\varphi_1)+\widetilde{C}_5 \quad \text{for } t \in [\frac{1}{2}, 1], \tag{4.31}$$

where $\widetilde{C}_5$ is a constant depending only on on k and the geometry of (M, ω_{GKE}). By(2.5), (4.30), and (4.31), we have

$$F_{\omega_{GKE}}(\varphi_t-\varphi_1) \geq \widetilde{C}_6 \frac{J_{\omega_{GKE}}(\varphi_t-\varphi_1)}{(J_{\omega_{GKE}}(\varphi_t-\varphi_1)+1)^{\zeta}} - \widetilde{C}_4 \quad \text{for } t \in [\frac{1}{2}, 1], \tag{4.32}$$

where $\widetilde{C}_6$ is a positive constant depending only on θ, k and the geometry of (M, ω_{GKE}).

By the cocycle property of the functional F, (4.23), (4.24), (3.6), nondecreasingness of $(I_{\omega_0} - J_{\omega_0})(\varphi_t)$ and the concavity of the log function, we have

$$\begin{aligned} F_{\omega_{GKE}}(\varphi_t-\varphi_1) &= F_{\omega_0}(\varphi_t) - F_{\omega_0}(\varphi_1) \\ &\leq m(1-t)\Big((m+1)J_{\omega_{GKE}}(\varphi_t-\varphi_1) + \frac{\widetilde{C}_7}{tk} + \widetilde{C}_8\Big) \end{aligned} \tag{4.33}$$

By a similar discussion to that in [13, p. 1083], we know that (4.28), (4.31), (4.32), and (4.33) imply the Moser–Trudinger inequality (4.22). ∎

In view of the cocycle identity for F_ω and properties of I_ω, J_ω (see Section 2, (2.9), (2.6) and (2.5)), the inequality (1.5) holds for every Kähler metric ω that is cohomologous to ω_{GKE} . On the other hand, (2.12) implies that the Moser–Trudinger type inequality (4.22) is also valid for the $\mathcal{K}$-energy $\mathcal{V}_{\theta,\omega}$.

Corollary 4.6 *Let (M, ω_0) be a Kähler manifold and let $\theta \in \alpha = 2\pi c_1(M) - k[\omega_0]$ be a real closed semipositive $(1,1)$-form with $k > 0$. Assume that the twisting form θ is strictly positive at a point or M admits no nontrivial Hamiltonian holomorphic vector field. If there exists a generalized Kähler–Einstein metric in $\mathcal{K}_{\omega_0}$, then for any Kähler metric $\omega \in \mathcal{K}_{\omega_0}$ there exist uniform positive constants $\{\widetilde{D}_i\}_{i=1}^4$ depending only on k, θ and the geometry of (M, ω), such that*

$$F_\omega(\varphi) \geq \widetilde{D}_1 J_\omega(\varphi) - \widetilde{D}_2, \quad \textit{and} \quad \mathcal{V}_{\theta,\omega}(\varphi) \geq \widetilde{D}_3 J_\omega(\varphi) - \widetilde{D}_4,$$

for all $\varphi \in \mathcal{H}_\omega$.

Remark 4.7 Finally, Theorem 3.4 and Corollary 4.6 imply Theorem 1.2.

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