Glasgow Mathematical Journal (2023), **65**, pp. 569–572
doi:10.1017/S0017089523000137

Glasgow Mathematical Journal

RESEARCH ARTICLE

Continuously many quasi-isometry classes of residually finite groups

Hip Kuen Chong and Daniel T. Wise

Department of Mathematics and Statistics, McGill University, Montreal, QC H3A 0B9, Canada
Corresponding author: Hip Kuen Chong; Email: chonghk1997@gmail.com

Received: 26 April 2022; **Revised:** 29 March 2023; **Accepted:** 4 May 2023; **First published online:** 19 June 2023

Keywords: Residually finite; small-cancellation groups; quasi-isometry

2020 Mathematics Subject Classification: *Primary* - 20E26; *Secondary* - 20F06

Abstract
We study a family of finitely generated residually finite small-cancellation groups. These groups are quotients of F_2 depending on a subset S of positive integers. Varying S yields continuously many groups up to quasi-isometry.

1. Introduction

Grigorchuk exhibited continuously many quasi-isometry classes of residually finite three-generator groups by producing continuously many growth types [4, Thm 7.2]. *Continuously many* means having the cardinality of $\mathbb{R}$. Here, we describe another family of such groups by building upon Bowditch's method for distinguishing quasi-isometry classes [2] and use consequences of the theory of special cube complexes to obtain residual finiteness [1].

Consider the rank-2 free group $F_2 = \langle a, b\rangle$. Let $w_n = [a, b^{2^{2^n}}][a^2, b^{2^{2^n}}] \cdots [a^{100}, b^{2^{2^n}}]$ for $n \in \mathbb{N}$.

Each subset $S \subseteq \mathbb{N}$ is associated to the following group:

$$G(S) \;=\; \langle a, b \mid w_n : n \in S\rangle$$

In Section 3, we show that $G(S)$ is residually finite when $S \subseteq \mathbb{N}_{>100}$. We also observe that $G(S)$ and $G(S')$ are not quasi-isometric when $S \Delta S'$ is infinite.

In fact, our proof of residual finiteness for $G(S)$ works in precisely the same way to prove the residual finiteness for the original examples of Bowditch having torsion. But it appears to fail for Bowditch's torsion-free examples. We refer to Remark 3.3.

We also produced an uncountable family of pairwise non-isomorphic residually finite groups in [3], and perhaps an appropriate subfamily also yields continuously many quasi-isometry classes.

Our simple approach arranges for certain infinitely presented small-cancellation groups to be residually finitely presented small-cancellation groups. This approach is likely to permit the construction of other interesting families of finitely generated groups.

2. Review of Bowditch's result

We first recall some small-cancellation background. See [5, Ch.V].

Definition 2.1. *For a presentation, a piece p is a word appearing in more than one way among the relators. Note that for a relator $r = q^n$, subwords that differ by a $\mathbb{Z}_n$-action are regarded as appearing in the same way. A presentation is $C'\left(\frac{1}{6}\right)$ if $|p| < \frac{1}{6}|r|$ for any piece p in a relator r.*

A *major subword* v of a relator r is a subword of a cyclic permutation of $r^{\pm}$ with $|v| > \frac{|r|}{2}$. A word u is *majority-reduced* if u does not contain a major subword of a relator. We will use the following well-known property for $C'\left(\frac{1}{6}\right)$ groups [5, Ch.V Thm 4.5].

Proposition 2.2. *Let $\langle x_1, x_2, \cdots \mid r_1, r_2, \ldots \rangle$ be a $C'\left(\frac{1}{6}\right)$ presentation. A non-empty cyclically reduced majority-reduced word in the generators must represent a nontrivial element in the group.*

We now recall definitions leading to the main theorem of [2]. Let $\mathbb{N}^+ = \{n \in \mathbb{Z} : n \geq 1\}$. Let $\mathbb{N}_{>k} = \{n \in \mathbb{N} : n > k\}$ for some $k \in \mathbb{N}^+$.

Definition 2.3. *Two subsets $L, L' \subseteq \mathbb{N}^+$ are* related *if for some $k \geq 1$:*

1. *for any $m \in L$ with $m > k$, there is $m' \in L'$ with $m' \in \left[\frac{m}{k}, km\right]$; and*
2. *for any $m' \in L'$ with $m' > k$, there is $m \in L$ with $m \in \left[\frac{m'}{k}, km'\right]$.*

We write $L \sim L'$ if L and L' are related, and write $L \not\sim L'$ otherwise.

Remark 2.4. *This is a simplified but equivalent form of Bowditch's definition* [2, *Def. before Lem* 3] *who used $m > (k+1)^2$ and $m' > (k+1)^2$. The equivalence is easy by proving relatedness via $(k+1)^2$ on one direction, and the other direction is clear since $(k+1)^2 > k$.*

Remark 2.5. *As pointed out by the referee, it is equivalent to say $L, L' \subseteq \mathbb{N}^+$ are related if there is $k \geq 1$ such that the sets $M = L \cap \mathbb{N}_{>k}$ and $M' = L' \cap \mathbb{N}_{>k}$ satisfy that $|\log M, \log M'| \leq k$. Here, $|Z, Z'| = \inf\{|z - z'| : z \in Z, z' \in Z'\}$ denotes the Hausdorff distance between sets Z and Z'. This observation could clarify the proofs below, especially Lemma 3.4, for some readers.*

Lemma 2.6. *The relation $\sim$ in Definition 2.3 is an equivalence relation on subsets of $\mathbb{N}^+$.*

Proof. The relation $\sim$ is reflexive via $k = 1$. The relation $\sim$ is symmetric by definition. Hence, it suffices to show $\sim$ is transitive.

Let $S, S', S'' \subseteq \mathbb{N}^+$. Suppose $S \sim S'$ via k and $S' \sim S''$ via k'. We claim that $S \sim S''$ via kk'. Let $m \in S$ with $m > kk'$. There is $m' \in S'$ with $m' \in \left[\frac{m}{k}, km\right]$ by $S \sim S'$ via k, hence $m' > k'$. Then there is $m'' \in S''$ with $m'' \in \left[\frac{m'}{k'}, k'm'\right]$ by $S' \sim S''$ via k'. Thus, $m'' \in \left[\frac{m}{kk'}, kk'm\right]$. Similarly, there is $m \in \left[\frac{m''}{kk'}, kk'm''\right]$ for any $m'' \in S''$ with $m'' > kk'$. $\square$

Example 2.7. *All finite sets are related. All uniform nets are related.* $\{2^n\}_{n\in\mathbb{N}} \sim \{3^n\}_{n\in\mathbb{N}}$.

For sets S, S', their *symmetric difference* is $S\Delta S' = (S - S') \cup (S' - S)$.

Example 2.8. *If $S, S' \subseteq \mathbb{N}^+$ with infinite $S\Delta S'$, then $\{2^{2^n}\}_{n\in S} \not\sim \{2^{2^m}\}_{m\in S'}$* [2, *Lem* 4].

With the notion of $\sim$, the following is a simplified version of the main theorem in [2].

Theorem 2.9. *Let G and G' be the finitely generated $C'\left(\frac{1}{6}\right)$ groups presented below. If G is quasi-isometric to G', then $\{|w_i|\}_{i\in I} \sim \{|w'_j|\}_{j\in J}$:*

$$G = \langle A \mid w_i : i \in I \rangle, \qquad G' = \langle A \mid w'_j : j \in J \rangle.$$

https://doi.org/10.1017/S0017089523000137 Published online by Cambridge University Press

3. Proving the family of groups have desired properties

3.1. Small cancellation

Proposition 3.1. *For any infinite subset* $S \subseteq \mathbb{N}_{>100}$*, the associated group* $G(S)$ *is* $C'\left(\frac{1}{6}\right)$*. Furthermore,* $G_k(S) = \left\langle a, b \,\middle|\, b^{2^{2^k}}, w_n : n \in S, n < k \right\rangle$ *is* $C'\left(\frac{1}{6}\right)$ *for each* $k \in \mathbb{N}$*.*

Proof. For the first statement, it suffices to show that w_n and w_m have small overlap for $n > m > 100$. The longest piece between w_n and w_m is $b^{-2^{2^m}} a^{100} b^{2^{2^m}}$. Thus, $C'\left(\frac{1}{6}\right)$ holds since:

$$\left| b^{-2^{2^m}} a^{100} b^{2^{2^m}} \right| \; = \; 100 + 2 \cdot 2^{2^m} \; < \; \frac{1}{6}\left(10100 + 200 \cdot 2^{2^m}\right) \; = \; \frac{1}{6}|w_m| \; < \; \frac{1}{6}|w_n|$$

For the second statement, we additionally show that w_n and $b^{2^{2^k}}$ satisfy the $C'\left(\frac{1}{6}\right)$ condition for $100 < n < k$. Their longest piece is $b^{2^{2^n}}$, which is shorter than $\frac{1}{6}$ of the lengths of w_n and $b^{2^{2^k}}$. □

3.2. Residual finiteness

Observe that $G_k(S) = G(S)/\langle\langle b^{2^{2^k}} \rangle\rangle$ since $w_m \in \langle\langle b^{2^{2^k}} \rangle\rangle$ for $m \geq k$. Indeed, $w_n = \left[a, b^{2^{2^n}}\right]\left[a^2, b^{2^{2^n}}\right] \cdots \left[a^{100}, b^{2^{2^n}}\right]$ is trivialised when $b^{2^{2^n}}$ becomes trivial.

Proposition 3.2. *For any infinite subset* $S \subseteq \mathbb{N}_{>100}$*, the associated group* $G(S)$ *is residually finite.*

Proof. Since $G_k(S)$ is a finitely presented $C'\left(\frac{1}{6}\right)$ group, the hyperbolic group $G_k(S)$ is cocompactly cubulated by [6]. Thus, $G_k(S)$ is residually finite by [1].

Each $g \in G(S) - \{1\}$ is represented by a cyclically reduced word v with minimal length. Then v is majority-reduced since otherwise, v contains a major subword of a relator, which can reduce the length of v. Moreover, v does not contain a majority subword of $b^{2^{2^{|v|}}}$ since $|v| < \frac{1}{2} \cdot 2^{2^{|v|}} = \frac{1}{2}\left|b^{2^{2^{|v|}}}\right|$. Hence, $v \neq 1_{G_{|v|}}$ by Proposition 2.2 since v is majority-reduced in $G_{|v|}$. Thus, $G(S)$ is residually residually finite and hence residually finite. □

Remark 3.3. *Bowditch's original examples were* $B(S) = \langle a, b \mid \left(a^{2^{2^n}} b^{2^{2^n}}\right)^7 : n \in S \subseteq \mathbb{N}\rangle$*. As in Proposition 3.2,* $B(S)$ *is residually finite since it is residually finitely presented* $C'\left(\frac{1}{6}\right)$ *using the quotients to* $B/\langle\langle a^{2^{2^n}}, b^{2^{2^n}} \rangle\rangle$ *for* $n \geq 3$*. However, the analogous argument fails for Bowditch's torsion-free examples* $B'(S) = \langle a, b \mid a\left(a^{2^{2^n}} b^{2^{2^n}}\right)^{12} : n \in S \subseteq \mathbb{N}\rangle$*.*

3.3. Pairwise non-quasi-isometric

We first prove a lemma about the relation $\sim$.

Lemma 3.4. $S \sim nS \sim (S + n)$ *for* $n \in \mathbb{N}^+$ *and* $S \subseteq \mathbb{N}^+$*.*

https://doi.org/10.1017/S0017089523000137 Published online by Cambridge University Press

Proof. First, $S \sim nS$ via n. Indeed, for any $s \in S$, $ns \in \left[\frac{s}{n}, ns\right]$; for any $ns \in nS$, $s \in \left[\frac{ns}{n}, n \cdot ns\right]$.

Moreover, $S \sim (S+n)$ via $n+1$. For any $s \in S$, $s+n \leq (n+1)s$, so $s+n \in \left[\frac{s}{n+1}, (n+1)s\right]$. On the other hand, for $s+n \in S+n$, $(n+1)s \geq s+n$ implies $s \geq \frac{s+n}{n+1}$. Hence, $s \in \left[\frac{s+n}{n+1}, (n+1)(s+n)\right]$. □

Proposition 3.5. *Let $S, S' \subseteq \mathbb{N}^+$ have infinite $S \Delta S'$, then $\{|w_n|\}_{n \in S} \not\sim \{|w_m|\}_{m \in S'}$.*

Proof. $\{|w_n| : n \in S\} = \{10100 + 200 \cdot 2^{2^n} : n \in S\} = 10100 + 200 \cdot \{2^{2^n} : n \in S\}$. By Lemma 3.4, $\{|w_n| : n \in S\} \sim \{2^{2^n} : n \in S\}$. Similarly, $\{|w_m| : m \in S'\} \sim \{2^{2^m} : m \in S'\}$. By Example 2.8, $\{2^{2^n} : n \in S\} \not\sim \{2^{2^m} : m \in S'\}$, so $\{|w_n|\}_{n \in S} \not\sim \{|w_m|\}_{m \in S'}$ by Lemma 2.6. □

Corollary 3.6. *If $S, S' \subseteq \mathbb{N}_{>100}$ have infinite $S \Delta S'$, then $G(S)$ and $G(S')$ are not quasi-isometric.*

Proof. $\{|w_n|\}_{n \in S} \not\sim \{|w_m|\}_{m \in S'}$, hence $G(S)$ and $G(S')$ are not quasi-isometric by Theorem 2.9. □

For $A, B \subseteq N$, declare $A \sim_\Delta B$ if $|A \Delta B| < \infty$. As noted by Bowditch, each $\sim_\Delta$ equivalence class is countable. Hence, there are continuously many $\sim_\Delta$ equivalence classes. Our construction thus produces continuously many pairwise non-quasi-isometric groups $G(S)$, which are $C'\left(\frac{1}{6}\right)$ and residually finite.

Acknowledgements. We are grateful to the referee for helpful comments. Research is supported by NSERC.

References

[1] I. Agol, The virtual Haken conjecture, *Doc. Math.* **18** (2013), 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning.

[2] B. H. Bowditch, Continuously many quasi-isometry classes of 2-generator groups, *Comment. Math. Helv.* **73**(2) (1998), 232–236.

[3] H. K. Chong and D. T. Wise, An uncountable family of finitely generated residually finite groups, *J. Group Theory*. **25**(2) (2022), 207–216.

[4] R. I. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, *Izv. Akad. Nauk SSSR Ser. Mat.* **48**(5) (1984), 939–985.

[5] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition,

[6] D. T. Wise, Cubulating small cancellation groups, *Geom. Funct. Anal.* **14**(1) (2004), 150–214.

https://doi.org/10.1017/S0017089523000137 Published online by Cambridge University Press