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ON A RESULT OF A. M. MACBEATH ON NORMAL SUBGROUPS OF A FUCHSIAN GROUP

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A. M. Macbeath, in November 1965, communicated the following theorem to me which he proved with the aid of the Lefschetz fixed point formula.

THEOREM. *If Γ is a Fuchsian group and N a torsion free normal subgroup, then the rank of $N/[\Gamma, N]$ is twice the genus of the orbit space D/Γ where D denotes the hyperbolic plane which Γ acts.*

This theorem will follow from a consideration of the exact sequence

$$(*) \qquad H_2(\Gamma, Z) \to H_2(Q, Z) \to H_1(N, Z)_\Gamma \to H_1(\Gamma, Z) \to H_1(Q, Z) \to 0.$$

associated with edge homorphisms of a spectral sequence [1, 3]. Here Γ is a group, N is normal in Γ, and $Q = \Gamma/N$; moreover, $H_i(X, A)$ is the ith homology group of X with coefficients in A. Considered as a left X-module, M_X is the largest quotient module of the left X-module M acted upon trivially by X.

The following well-known results will be needed as well:

(1) $H_1(X, Z) \cong X/X'$; $\quad H_1(N, Z)_\Gamma = N/[N, \Gamma]$
where $[X, Y] = gp\langle x^{-1}y^{-1}xy \mid x \in X, y \in Y\rangle$ and $X' = [X, X]$.

(2) If $|X| < \infty$ then $|H_i(X, A)| < \infty$ for $i > 0$ and A finitely generated.

(3) If X and Y are finitely generated abelian groups and $Y \leq X$, then the rank of Y is no greater than the rank of X. Moreover, if $|X: Y| < \infty$ then rank $X =$ rank Y. Furthermore, if $Z \leq X$ and $|Z| < \infty$ then rank $X = \text{rank}\,(X/Z)$. The following theorem includes Macbeath's as a special case:

THEOREM. *Let Γ be a finitely generated group and N a normal subgroup of finite index. Then* rank $\Gamma/\Gamma' = \text{rank } N/[\Gamma, N]$.

Proof. In the exact sequence (*), $H_2(Q, Z)$ and $H_1(Q, Z)$ are finite in view of (2) since $|Q| = |\Gamma/N| < \infty$. Substituting from (1) gives:

$$H_2(Q, Z) \xrightarrow{\tau} N/[\Gamma, N] \xrightarrow{\sigma} \Gamma/\Gamma' \xrightarrow{\rho} Q/Q' \to 0$$

exact. Because $H_2(Q, Z)$ is finite, its image lies in the torsion subgroup T of $N/[\Gamma, N]$. Note that since the quotient of $N/[\Gamma, N]$ by the finite group $\tau H_2(Q)$ is a subgroup of the finitely generated abelian group Γ/Γ' it is immediate that $N/[\Gamma, N]$ is itself finitely generated.

Since Ker $\sigma = \text{Im } \tau$, rank $N/[\Gamma, N] = \text{rank Im } \sigma$.

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Since ρ maps Γ/Γ' onto a finite group, $|\Gamma/\Gamma' : \text{Ker } \rho| < \infty$ thus rank Ker $\rho =$ rank Γ/Γ'. Noticing that exactness gives Ker $\rho = \text{Im } \sigma$ finishes the proof.

Macbeath's theorem is a consequence of the fact that rank Γ/Γ' is twice the genus of the orbit space.

References

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