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# A NEW CONGRUENCE MODULO 25 FOR 1-SHELL TOTALLY SYMMETRIC PLANE PARTITIONS

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**Abstract**

For any positive integer $n$, let $f(n)$ denote the number of 1-shell totally symmetric plane partitions of $n$. Recently, Hirschhorn and Sellers ['Arithmetic properties of 1-shell totally symmetric plane partitions', *Bull. Aust. Math. Soc.* **89** (2014), 473–478] and Yao ['New infinite families of congruences modulo 4 and 8 for 1-shell totally symmetric plane partitions', *Bull. Aust. Math. Soc.* **90** (2014), 37–46] proved a number of congruences satisfied by $f(n)$. In particular, Hirschhorn and Sellers proved that $f(10n+5) \equiv 0 \pmod{5}$. In this paper, we establish the generating function of $f(30n+25)$ and prove that $f(250n+125) \equiv 0 \pmod{25}$.

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## 1. Introduction

A plane partition of $n$ is a two-dimensional array of integers $\pi_{i,j}$ (with positive integer indices $i$ and $j$) that are weakly decreasing in both indices and that add up to the given number $n$, that is, $\pi_{i,j} \geq \pi_{i+1,j}$, $\pi_{i,j} \geq \pi_{i,j+1}$ and $\sum \pi_{i,j} = n$. A plane partition is called a totally symmetric plane partition (TSPP) if it is invariant under any permutation of the three axes. (For more details about TSPPs, the reader may wish to see Andrews *et al.* [1] and Stembridge [7].) In 2012, Blecher [3] introduced a special class of totally symmetric plane partitions, called 1-shell totally symmetric plane partitions. A totally symmetric plane partition is a 1-shell totally symmetric plane partition if this partition has a self-conjugate first row and column (as an ordinary partition) and all other entries are 1. For example, the following totally symmetric plane partitions are 1-shell totally symmetric plane partitions:

$$\begin{matrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{matrix}\quad , \qquad \begin{matrix} 4 & 4 & 2 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 1 & & \\ 2 & 1 & & \end{matrix}\ .$$

For any positive integer $n$, let $f(n)$ denote the number of 1-shell totally symmetric plane partitions of $n$. As usual, set $f(0) = 1$. Blecher [3] proved that

$$\sum_{n=0}^{\infty} f(n)q^n = 1 + \sum_{n=1}^{\infty} q^{3n-2} \prod_{i=0}^{n-2} (1 + q^{6i+3}).$$

Recently, utilising elementary generating function manipulations and some well-known results due to Ramanujan and Watson, Hirschhorn and Sellers [6] proved a number of congruences satisfied by $f(n)$. More precisely, they proved that for $n \geq 1$,

$$f(3n) = f(3n-1) = 0, \tag{1.1}$$

$$f(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } 3 \nmid n \text{ and } n = k^2 \text{ for some integer } k, \\ 0 \pmod 2 & \text{otherwise} \end{cases}$$

and

$$f(10n-5) \equiv 0 \pmod 5. \tag{1.2}$$

Very recently, Yao [8] proved several infinite families of congruences modulo 4 and 8 satisfied by $f(n)$.

In this paper, we establish the generating function of $f(30 + 25)$ and a new congruence modulo 25 for $f(n)$ by employing some well-known results due to Hirschhorn [5], Hirschhorn and Sellers [6] and Ramanujan [2].

In order to state and prove the main results of this paper, we introduce some notation and terminology on $q$-series. In this paper, we adopt the common notation

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

The Ramanujan theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{1.3}$$

where $|ab| < 1$. The Jacobi triple product identity can be restated as

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \tag{1.4}$$

Three special cases of (1.3) are

$$\psi(q) : = f(q, q^3) = \sum_{n=0}^{\infty} q^{(n^2+n)/2}, \tag{1.5}$$

$$\varphi(q) : = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{1.6}$$

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and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

By (1.4),

$$f(-q) = (q; q)_\infty.$$

For any positive integer $k$, we use $f_k$ to denote $f(-q^k)$, that is,

$$f_k = (q^k; q^k)_\infty = \prod_{n=1}^{\infty} (1 - q^{nk}).$$

By (1.4)–(1.6) and the notation $f_k$,

$$\psi(q) = \frac{f_2^2}{f_1} \tag{1.7}$$

and

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}. \tag{1.8}$$

Replacing $q$ by $-q$ in (1.8),

$$\varphi(-q) = \frac{f_1^2}{f_2}.$$

The following two theorems are the main results of this paper.

THEOREM 1.1. *For all $n \geq 0$,*

$$\sum_{n=0}^{\infty} f(30n + 25) q^n = 5 \frac{f_2^2 f_5^2 f_{10}}{f_1^4}. \tag{1.9}$$

In view of (1.1), for $n \geq 0$,

$$f(30n + 5) = f(30n + 15) = 0. \tag{1.10}$$

Therefore, congruence (1.2) follows from (1.9) and (1.10).

THEOREM 1.2. *For $n \geq 0$,*

$$f(250n + 125) \equiv 0 \pmod{25}.$$

## 2. Proof of Theorem 1.1

Hirschhorn and Sellers [6] proved that for $n \geq 1$,

$$f(3n - 2) = h(n), \tag{2.1}$$

where $h(n)$ is defined by

$$\sum_{n=1}^{\infty} h(n) q^n = \sum_{n=1}^{\infty} q^n \prod_{i=0}^{n-2} (1 + q^{2i+1}).$$

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Employing some well-known results due to Ramanujan [1, Entry 9.5.2, page 238] and Watson [4, (26.82), page 61], Hirschhorn and Sellers [6] proved that

$$\sum_{n=0}^{\infty} h(2n+1)q^n = \prod_{n=1}^{\infty}(1+q^n)^3(1-q^n). \tag{2.2}$$

Employing the notation $f_k$, we can rewrite (2.2) as:

$$\sum_{n=0}^{\infty} h(2n+1)q^n = \frac{f_2^3}{f_1^2}. \tag{2.3}$$

From [2, Entry 10(v), page 262],

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{2.4}$$

By (1.4) and (1.7), we can rewrite (2.4) as

$$\frac{f_2^4}{f_1^2} = \frac{f_2 f_5^3}{f_1 f_{10}} + q\frac{f_{10}^4}{f_5^2}. \tag{2.5}$$

Substituting (2.5) into (2.3),

$$\sum_{n=0}^{\infty} h(2n+1)q^n = \frac{1}{f_2}\left(\frac{f_2 f_5^3}{f_1 f_{10}} + q\frac{f_{10}^4}{f_5^2}\right) = \frac{f_5^3}{f_1 f_{10}} + q\frac{f_{10}^4}{f_2 f_5^2}. \tag{2.6}$$

Hirschhorn [5] also established the following identity:

$$\begin{aligned}\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\bigg(&\frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} \\ &+ 5q^4 - 3q^5R(q^5) + 2q^6R^2(q^5) - q^7R^3(q^5) + q^8R^4(q^5)\bigg),\end{aligned} \tag{2.7}$$

where $R(q)$ is defined by

$$R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2.8}$$

If we substitute (2.7) into (2.6) and then extract those terms in which the power of $q$ is congruent to 4 modulo 5, and use (2.5), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h(10n+9)q^n &= 5\frac{f_5^5}{f_1^3 f_2} + 5q\frac{f_{10}^5}{f_1^2 f_2^2} = \frac{5f_5^2 f_{10}}{f_1^2 f_2^2}\left(\frac{f_2 f_5^3}{f_1 f_{10}} + q\frac{f_{10}^4}{f_5^2}\right) \\ &= \frac{5f_5^2 f_{10}}{f_1^2 f_2^2}\cdot\frac{f_2^4}{f_1^2} = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}.\end{aligned} \tag{2.9}$$

Theorem 1.1 follows from (2.1) and (2.9). This completes the proof. □

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## 3. Proof of Theorem 1.2

By the binomial theorem, it is easy to see that, for any positive integer $k$,

$$f_k^5 \equiv f_{5k} \pmod 5. \tag{3.1}$$

It follows from (2.9) and (3.1) that

$$\sum_{n=0}^{\infty} h(10n+9)q^n \equiv 5\frac{f_1^2 f_5^4}{f_2} + 5q\frac{f_1^3 f_2^3 f_{10}^4}{f_5} \pmod{25}. \tag{3.2}$$

It is trivial to show that

$$f_1^3 \equiv f(-q^{10}, -q^{15}) - 3qf(-q^5, -q^{20}) \pmod 5 \tag{3.3}$$

and

$$\frac{f_1^2}{f_2} = \frac{f_{25}^2}{f_{50}} - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45}). \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2),

$$\begin{aligned}\sum_{n=0}^{\infty} h(10n+9)q^n \equiv 5f_5^4\bigg(\frac{f_{25}^2}{f_{50}} - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45})\bigg) \\ + 5q\frac{f_{10}^4}{f_5}\big(f(-q^{10}, -q^{15}) - 3qf(-q^5, -q^{20})\big) \\ \times \big(f(-q^{20}, -q^{30}) - 3q^2 f(-q^{10}, -q^{40})\big) \pmod{25}.\end{aligned}$$

If we extract those terms in which the power of $q$ is congruent to 0 modulo 5, and replace $q^5$ by $q$, and then employ (3.1), we deduce that

$$\sum_{n=0}^{\infty} h(50n+9)q^n \equiv 5\frac{f_1^5 f_5^2}{f_1 f_{10}} \equiv 5\frac{f_5^3}{f_1 f_{10}} \pmod{25}. \tag{3.5}$$

Substituting (2.7) into (3.5), for $n \geq 0$,

$$h(250n+209) \equiv 0 \pmod{25}.$$

Replacing $n$ by $250n + 209$ in (2.1), for $n \geq 0$,

$$f(750n+625) \equiv 0 \pmod{25}. \tag{3.6}$$

By (1.1), for $n \geq 0$,

$$f(750n+125) = f(750n+375) = 0. \tag{3.7}$$

Theorem 1.2 follows from (3.6) and (3.7). The proof is complete. □

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