

RESEARCH ARTICLE

Families of similar simplices inscribed in most smoothly embedded spheres

Jason Cantarella ¹, Elizabeth Denne ² and John McCleary ³

¹Department of Mathematics, University of Georgia, Athens, GA 30602, USA; E-mail: jason.cantarella@uga.edu.

²Department of Mathematics, Washington & Lee University, Lexington, VA 24450, USA; E-mail: dennee@wlu.edu.

³Department of Mathematics & Statistics, Vassar College, Poughkeepsie, NY 12604, USA; E-mail: mccleary@vassar.edu.

Received: 5 July 2021; **Revised:** 7 June 2022; **Accepted:** 24 September 2022

2020 Mathematics Subject Classification: *Primary* – 55R80; *Secondary* – 51M04, 51K99, 57Q65, 58A20

Abstract

Let Δ denote a nondegenerate k -simplex in \mathbb{R}^k . The set $\text{Sim}(\Delta)$ of simplices in \mathbb{R}^k similar to Δ is diffeomorphic to $O(k) \times [0, \infty) \times \mathbb{R}^k$, where the factor in $O(k)$ is a matrix called the *pose*. Among $(k - 1)$ -spheres smoothly embedded in \mathbb{R}^k and isotopic to the identity, there is a dense family of spheres, for which the subset of $\text{Sim}(\Delta)$ of simplices inscribed in each embedded sphere contains a similar simplex of every pose $U \in O(k)$. Further, the intersection of $\text{Sim}(\Delta)$ with the configuration space of $k + 1$ distinct points on an embedded sphere is a manifold whose top homology class maps to the top class in $O(k)$ via the pose map. This gives a high-dimensional generalisation of classical results on inscribing families of triangles in plane curves. We use techniques established in our previous paper on the square-peg problem where we viewed inscribed simplices in spheres as transverse intersections of submanifolds of compactified configuration spaces.

Contents

1	Introduction	1
2	Configuration spaces	4
3	Finding special configurations with multijet transversality	6
4	Simplices	8
4.1	Nondegenerate simplices	8
4.2	Simplices and configuration spaces	9
5	Simplices inscribed in spheres	12
Appendix A	Results from linear algebra	15

1. Introduction

There is a general type of problem of finding special geometric configurations on families of manifolds. Quite often such problems seek to find some kind of polyhedron or polytope inscribed in a circle or sphere embedded in space. Our paper seeks to find constructible nondegenerate simplices inscribed in spheres smoothly embedded in \mathbb{R}^k . Specifically, for a given nondegenerate k -simplex Δ , we show that among all smoothly embedded $(k - 1)$ spheres in \mathbb{R}^k isotopic to the identity through a differentiable

© The Author(s), 2022. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (<https://creativecommons.org/licenses/by/4.0/>), which permits unrestricted re-use, distribution, and reproduction in any medium, provided the original work is properly cited.

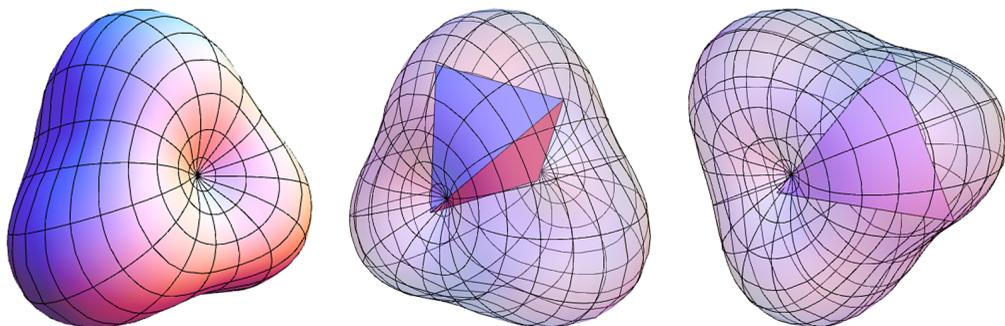


Figure 1. On the left, we see an irregular embedding of S^2 in \mathbb{R}^3 described in spherical coordinates as a graph over the unit sphere by the function $r(\phi, \theta) = 1 + \sin^3 \phi \sin 3\theta/5 - |\cos^7 \phi|$. The centre and right images show different views of a single regular tetrahedron inscribed in this surface with edge lengths close to 1.15. If this embedding of S^2 is transverse to the submanifold of regular tetrahedra, this tetrahedron is a member of the family of inscribed regular tetrahedra predicted by Theorem 5.4. This tetrahedron was found by computer search. Its vertices have spherical (ϕ, θ) coordinates $(0.224399, 0.224399)$, $(1.5708, 3.36599)$, $(1.5708, 2.0196)$, $(2.91719, 0.224399)$

isotopy in \mathbb{R}^k , there is a dense family of spheres, such that each sphere has an inscribed simplex similar to Δ corresponding to each $U \in O(k)$. An example of an embedded 2-sphere in \mathbb{R}^3 with an inscribed equilateral tetrahedra is found in Figure 1. We, in fact, prove more: If we let $\text{Sim}(\Delta)$ denote the configuration space of simplices in \mathbb{R}^k similar to a nondegenerate k -simplex Δ , then in $\text{Sim}(\Delta)$, the top homology class of the inscribed simplices similar to Δ on an embedded sphere maps to the top class in $O(k)$.

In 1969, Gromov [16] showed that every C^1 -smooth embedding of a $(k - 1)$ -sphere in \mathbb{R}^k contains an inscribed simplex similar to Δ for each pose $U \in O(k)$. Our results go further than Gromov. We show that the inscribed simplices in embedded spheres form a manifold cobordant to $O(k)$ by using configuration spaces and transversality as in [10].

Another closely related result is the 2009 work of Blagojević and Ziegler [6]. They prove that for every injective continuous map of $f: S^2 \rightarrow \mathbb{R}^3$, there are four distinct points in the image of f with the property that two opposite edges have the same length and the other four edges are also of equal length. While this result holds for injective maps, our theorem recovers an even more general result for generic smooth embeddings. In addition, Matschke [31] proved that any smoothly embedded compact surface S inscribes a particular type of tetrahedron.

All of these theorems generalise the classical results on inscribing families of triangles in planar and spatial curves. In 1978, Meyerson [34] proved that for a fixed arbitrary triangle, every simple closed curve in the plane contains the vertices of a triangle similar to the given one. He also proved that for every simple closed curve γ in the plane, then for all, except perhaps two, points x on γ , we can find points y and z in γ , such that xyz is an equilateral triangle. In 1992, Nielsen [35] proved that for any triangle T and any simple closed curve γ in the plane, there are infinitely many triangles similar to T inscribed in γ . In fact, he proves that the set of vertices corresponding to the vertex of the smallest angle in T is dense in γ . In 2011, Matschke [31] proved a more general result. He looked at n -gons whose edge lengths are in a prescribed ratio. He showed such polygons are inscribed in generic C^∞ -smooth embeddings of S^1 into a (complete) Riemannian manifold, and, moreover, there is a 1-parameter family of such polygons. He proves that there are an odd number of loops of such polygons that wind an odd number of times around the embedded S^1 . Our main theorem and Matschke's results are very similar for triangles inscribed in embedded curves in the plane, except that we prove (Corollary 5.6) that the degree of the map is 1. In 2021, Gupta and Rubinstein-Salzedo [18] generalised Nielsen's result to any Jordan curve embedded in \mathbb{R}^n for a restricted set of triangles dependent on certain geometric conditions. They

then get a wider class of inscribed triangles by adding in regularity conditions. Namely, they show that if the curve is differentiable at a point, then any triangle can be inscribed at that point.

We pause to note that many of the inscribed triangle results were inspired by the square-peg problem: finding four points on any Jordan curve (a simple closed curve in the plane) which are the vertices of a square. This question was posed by Toeplitz in 1911 [44], and progress on this problem has chiefly been an extension of the regularity class of simple closed curves for which the square can be found. The interested reader can find numerous articles [9, 26, 32, 36, 42] summarising the problem and describing the classes of curves for which the square-peg problem has been proved. There have also been many papers [1, 3, 15, 22, 23, 28, 33, 39, 45] examining quadrilaterals and polygons inscribed in curves and, more recently, making progress towards solving the rectangular-peg problem (finding rectangles of any aspect ratio inscribed in Jordan curves).

There are yet more results about inscribed and circumscribed polyhedra in spheres. For example, Kakutani's theorem [24] that a compact convex body in \mathbb{R}^3 has a circumscribed cube, that is, a cube each of whose faces touch the convex body. As Matschke [32] notes, most smooth embeddings $S^{k-1} \hookrightarrow \mathbb{R}^k$ do not inscribe a k -cube for $k \geq 3$ (intuitively, the number of equations to fulfil is larger than the degrees of freedom). Instead, asking whether crosspolytopes¹ are inscribed in such an embedding might be a better generalisation of inscribing a square. Makeev [30] proved the $k = 3$ case, and Karasev [25] generalised the proof to arbitrary odd prime powers. Later, Akoyan and Karasev [2] used a limit argument to show that a simple convex polytope admits an inscribed regular octahedron. In addition, there is the work of Kuperberg [27] and Makeev [29] on inscribed and circumscribed polyhedra in convex bodies and spheres.

The inscribed simplex problem can be framed in terms of compactified configuration spaces. We give a short overview of the basics about these spaces in Section 2. We consider the compactified configuration space $C_{k+1}[\mathbb{R}^k]$ of ordered $(k + 1)$ -tuples of points in \mathbb{R}^k as a manifold-with-boundary (and corners). The existence of inscribed simplices in embedded spheres can be viewed as finding the intersection of two submanifolds of $C_{k+1}[\mathbb{R}^k]$. The first is the submanifold of $(k + 1)$ -tuples of points on a C^∞ -smooth embedding $\gamma: S^{k-1} \hookrightarrow \mathbb{R}^k$ of a $(k - 1)$ -sphere in \mathbb{R}^k ; and the second is the submanifold of simplices in \mathbb{R}^k which are similar to a given simplex Δ , denoted by $\text{Sim}(\Delta)$.

Theorems about intersections of manifolds are often proved by transversality arguments. There are many examples in the literature [17, 37, 43]. In [10], we provide a framework for this kind of argument adapted to configuration spaces, which we will use again in this paper. Section 3 gives a description of this method, and we give a very quick overview here. We consider a smooth embedding $\gamma: S^l \hookrightarrow \mathbb{R}^k$ of S^l in \mathbb{R}^k . Then, in the compactified configuration space $C_n[\mathbb{R}^k]$ of n points in \mathbb{R}^k , we consider $C_n[\gamma(S^l)]$, the compactified configuration space of n points on $\gamma(S^l)$; and Z , the subspace of tuples of n points in \mathbb{R}^k that satisfy the conditions of a special configuration. Now, suppose there is a different, well-known, smooth embedding $i: S^l \hookrightarrow \mathbb{R}^k$ of S^l in \mathbb{R}^k , and assume that the configuration space $C_n[i(S^l)]$ is transverse to Z in $C_n[\mathbb{R}^k]$. We can use Haefliger's theorem [19] to find a differentiable isotopy between $i(S^l)$ and $\gamma(S^l)$ (the differentiable isotopy may need to go through \mathbb{R}^K for $K \geq k$). The key idea is that we ought to be able to vary $C_n[i(S^l)]$ to $C_n[\gamma(S^l)]$ while maintaining the transversality of the intersection with Z . To do so, we need to make several assumptions about Z and $C_n[\gamma(S^l)]$ (see Section 3) and to use technical tools like multijet transversality. In the end, we are able to deduce that there is, for all m , a C^m -dense² set of smooth embeddings $\gamma': S^l \hookrightarrow \mathbb{R}^k$, such that the corresponding embeddings $C_n[\gamma']$ on configuration spaces are C^0 -close to $C_n[\gamma]$, and that $C_n[\gamma'(S^l)]$ is transverse to Z . Moreover, $C_n[i(S^l)] \cap Z$ and $C_n[\gamma'(S^l)] \cap Z$ represent the same homology class in Z . We apply this method to the inscribed simplices problem in the final two sections of the paper.

In Section 4, we first describe a nondegenerate simplex in terms of the distances between distinct vertices. We then use the Cayley-Menger determinant (see Theorem 4.2) to give a description of when it

¹The regular k -dimensional polytope is the convex hull of $\{\pm e_i\}$, where e_i are the standard basis vectors in \mathbb{R}^k .

²The density is with respect to the Whitney C^∞ -topology, described in Section 3.

is possible to construct a simplex from a set of distances. Secondly, we change our perspective and view a simplex Δ in \mathbb{R}^k as an ordered $(k + 1)$ -tuple of distinct points in the configuration space $C_{k+1}(\mathbb{R}^k)$. We then define $\text{Sim}(\Delta)$ to be the space of simplices similar to Δ (through a translation, rotation or nonzero scaling of \mathbb{R}^k). In Theorems 4.8 and 4.9, we prove that $\text{Sim}(\Delta)$ is a submanifold of $C_{k+1}[\mathbb{R}^k]$ diffeomorphic to $O(k) \times [0, \infty) \times \mathbb{R}^k$.

In Section 5, we apply our method from Section 3. Given a C^∞ -smooth embedding $\gamma: S^{k-1} \hookrightarrow \mathbb{R}^k$ of S^{k-1} in \mathbb{R}^k , we let $C_{k+1}[\gamma(S^{k-1})]$ be the submanifold of $C_{k+1}[\mathbb{R}^k]$ corresponding to $(k + 1)$ -tuples of points on the embedded sphere $\gamma(S^{k-1})$. In Proposition 5.1, we show that $\text{Sim}(\Delta)$ and $C_{k+1}[\gamma(S^{k-1})]$ are boundary disjoint in $C_{k+1}[\mathbb{R}^k]$. We then restrict our attention to the standard embedding of S^{k-1} in \mathbb{R}^k , with corresponding configuration space $C_{k+1}[S^{k-1}]$. In Proposition 5.2, we show that $\text{Sim}(\Delta)$ intersects $C_{k+1}[S^{k-1}]$ transversally and the intersection $\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]$ is diffeomorphic to $O(k)$. Since $O(k)$ is disconnected, we restrict our attention to $\text{Sim}^+(\Delta)$, which is the submanifold diffeomorphic to $SO(k) \times [0, \infty) \times \mathbb{R}^k$. We then show in Proposition 5.3 that the top class in $\text{Sim}^+(\Delta)$ corresponds to the top homology class of $SO(k)$, which also corresponds to the top class of the intersection $\text{Sim}^+(\Delta) \cap C_{k+1}[S^{k-1}]$. Finally in Theorem 5.4, we prove there is a dense set of smooth embeddings of S^{k-1} in \mathbb{R}^k which are isotopic to the identity through a differentiable isotopy in \mathbb{R}^k , such that the subset of $\text{Sim}^+(\Delta)$ of simplices inscribed in each sphere contains a similar simplex corresponding to each $U \in SO(k)$. While we chose to restrict our attention to $\text{Sim}^+(\Delta)$, the same results hold for $\text{Sim}^-(\Delta) := \text{Sim}(\Delta) \setminus \text{Sim}^+(\Delta)$. In the special case of a smooth embedding of a circle in the plane, our results show that there are loops of triangles inscribed on an embedded circle C^0 -close to the given embedding (see Corollary 5.6 for a precise statement).

2. Configuration spaces

The compactified configuration space of $k + 1$ points in \mathbb{R}^k is the natural setting for finding inscribed simplices in embedded spheres. In this section, we give a very brief overview of compactified configuration spaces. There are many versions of this classical material (see, e.g., [4, 13]), but we follow Sinha [40], as this approach is more appropriate to our work. A discussion similar to the one found below is found in [10].

A reader familiar with configuration spaces may skip much of this section. However, we recommend paying attention to the notation we have used for the spaces. Definitions 2.2, 2.3, 2.9, Remark 2.5 and Theorem 2.7 are particularly useful.

Definition 2.1 ([40]). Given an m -dimensional smooth manifold M , let $M^{\times n}$ denote the n -fold product M with itself, and define $C_n(M)$ to be the subspace of points $\mathbf{p} = (p_1, \dots, p_n) \in M^{\times n}$, such that $p_j \neq p_k$ if $j \neq k$. Let ι denote the inclusion map of $C_n(M)$ in $M^{\times n}$.

The space $C_n(M)$ is an open submanifold of $M^{\times n}$. We next compactify $C_n(M)$ to a closed manifold-with-boundary and corners, which we will denote $C_n[M]$, without changing its homotopy type. The resulting manifold will be homeomorphic to $M^{\times n}$ with an open neighborhood of the fat diagonal removed. Recall that the fat diagonal is the subset of $M^{\times n}$ of n -tuples for which (at least) two entries are equal, that is, where some collection of points comes together at a single point. The construction of $C_n[M]$ preserves information about the directions and relative rates of approach of each group of collapsing points.

Definition 2.2 ([8, 40]). Given an ordered pair (i, j) of distinct elements from $\{1, \dots, n\}$, let the map $\pi_{ij}: C_n(\mathbb{R}^k) \rightarrow S^{k-1}$ send $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ to $\frac{\mathbf{p}_i - \mathbf{p}_j}{|\mathbf{p}_i - \mathbf{p}_j|}$, the unit vector in the direction of $\mathbf{p}_i - \mathbf{p}_j$. Let $[0, \infty]$ be the one-point compactification of $[0, \infty)$. Given an ordered triple (i, j, l) of distinct elements in $\{1, \dots, n\}$, let $r_{ijl}: C_n(\mathbb{R}^k) \rightarrow [0, \infty]$ be the map which sends \mathbf{p} to $\frac{|\mathbf{p}_i - \mathbf{p}_j|}{|\mathbf{p}_i - \mathbf{p}_l|}$, the ratio of distances between \mathbf{p}_i and \mathbf{p}_j , and \mathbf{p}_i and \mathbf{p}_l .

We then compactify $C_n(\mathbb{R}^k)$ as follows:

Definition 2.3 ([40]).

1. Let $A_n[\mathbb{R}^k]$ be the product $(\mathbb{R}^k)^n \times (S^{k-1})^{n(n-1)} \times [0, \infty]^{n(n-1)(n-2)}$. Define $C_n[\mathbb{R}^k]$ to be the closure of the image of $C_n(\mathbb{R}^k)$ under the map

$$\alpha_n = \iota \times (\pi_{ij}) \times (r_{ijl}) : C_n(\mathbb{R}^k) \rightarrow A_n[\mathbb{R}^k].$$

2. We assume that all manifolds M are smoothly embedded in \mathbb{R}^k , which allows us to define the restrictions of the maps π_{ij} and r_{ijl} . Then $C_n(M)$ is smoothly embedded in $C_n(\mathbb{R}^k)$, and we define $C_n[M]$ to be the closure of $C_n(M)$ in $M^n \times (S^{k-1})^{n(n-1)} \times [0, \infty]^{n(n-1)(n-2)}$. We denote the boundary of $C_n[M]$ by $\partial C_n[M] = C_n[M] \setminus C_n(M)$.

We now summarise some of the important features of this construction, including the fact that $C_n[M]$ does not depend on the choice of embedding of M in \mathbb{R}^k .

Theorem 2.4 ([8, 40]).

1. $C_n[M]$ is a manifold-with-boundary and corners with interior $C_n(M)$ having the same homotopy type as $C_n[M]$. The topological type of $C_n[M]$ is independent of the embedding of M in \mathbb{R}^k , and $C_n[M]$ is compact when M is.
2. The inclusion of $C_n(M)$ in $M^{\times n}$ extends to a surjective map from $C_n[M]$ to $M^{\times n}$, which is a homeomorphism over points in $C_n(M)$.

Remark 2.5. When discussing points in $C_n[\mathbb{R}^k]$ or $C_n[M]$, it is easy to become confused. We pause to clarify notation.

- o A point in \mathbb{R}^k is denoted by $\mathbf{x} = (x_1, \dots, x_k)$, where each $x_i \in \mathbb{R}$.
- o Points in $(\mathbb{R}^k)^{\times n}$ are also denoted by \mathbf{x} , where $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ and each $\mathbf{x}_i \in \mathbb{R}^k$ (it will be clear from context which is meant).
- o A point in $C_n[\mathbb{R}^k]$ or $C_n[M]$, is denoted $\vec{\mathbf{x}}$.

The space $C_n[M]$ may be viewed as a polytope with a combinatorial structure based on the different ways groups of points in M can come together. This structure defines a stratification of $C_n[M]$ into a collection of closed faces of various dimensions whose intersections are members of the collection. Full details can be found in [8, 40]. The main structure we will consider is the $(0, 1, \dots, k)$ -face of $\partial C_n[M]$. This is the boundary component where all the points come together at the same time.

Any pair \mathbf{p}, \mathbf{q} of disjoint points in \mathbb{R}^k has a direction $(\mathbf{p} - \mathbf{q})/|\mathbf{p} - \mathbf{q}|$ associated to it, while every triple of disjoint points $\mathbf{p}, \mathbf{q}, \mathbf{r}$ has a corresponding distance ratio $|\mathbf{p} - \mathbf{q}|/|\mathbf{p} - \mathbf{r}|$. One way to think of the coordinates of $C_n[M]$ is that they extend the definition of these directions and ratios to the boundary.

Theorem 2.6 ([8, 40]). *Given a manifold $M \subset \mathbb{R}^k$, then in any configuration of points $\vec{\mathbf{p}} \in C_n[M]$, the following holds.*

1. Each pair of points $\mathbf{p}_i, \mathbf{p}_j$ has associated to it a well-defined unit vector in \mathbb{R}^k giving the direction from \mathbf{p}_i to \mathbf{p}_j . If the pair of points project to the same point \mathbf{p} of M , this vector lies in $T_{\mathbf{p}}M$.
2. Each triple of points $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ has associated to it a well-defined scalar in $[0, \infty]$ corresponding to the ratio of the distances $|\mathbf{p}_i - \mathbf{p}_j|$ and $|\mathbf{p}_i - \mathbf{p}_k|$. If any pair of $\{\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k\}$ projects to the same point in M (or all three do), this ratio is a limiting ratio of distances.
3. The functions π_{ij} and r_{ijl} are continuous on all of $C_n[M]$ and smooth on each face of $\partial C_n[M]$.

It turns out that for connected manifolds of dimension at least 2, the combinatorial structure of the strata of $C_n[M]$ depends only on the number of points. Regardless of dimension, this construction and division of $\partial C_n[M]$ into strata is functorial in the following sense.

Theorem 2.7 ([40]). *Suppose M and N are embedded submanifolds of \mathbb{R}^k and $f : M \hookrightarrow N$ is an embedding. This induces an embedding of manifolds-with-corners called the evaluation map*

$C_n[f]: C_n[M] \hookrightarrow C_n[N]$ that respects the stratifications. This map is defined by choosing the ambient embedding of M in \mathbb{R}^k to be the composition of f with the ambient embedding of N .

For an embedding $f: M \hookrightarrow N$, the image of the induced embedding $C_n[f]: C_n[M] \hookrightarrow C_n[N]$ will be denoted by $C_n[f(M)]$.

Corollary 2.8. *Let $f: \mathbb{R}^k \rightarrow \mathbb{R}^k$ be a smooth diffeomorphism. Then the induced map of configuration spaces $C_n[f]: C_n[\mathbb{R}^k] \rightarrow C_n[\mathbb{R}^k]$ is also a smooth diffeomorphism (on each face of $C_n[\mathbb{R}^k]$).*

Proof. This is an immediate corollary of the previous theorem. □

Finally, we will need a metric on the set of evaluation maps $C_n[f]: C_n[M] \hookrightarrow C_n[N]$. The definition of compactified configuration spaces allows us to view $C_n[N] \subset (\mathbb{R}^k)^n \times (S^{k-1})^{n(n-1)} \times [0, \infty]^{n(n-1)(n-2)}$ as a metric space with the sup norm. If we define the mapping pr_i to be the projection onto the i th space of the product, then this naturally leads to a metric on the set of continuous functions $C^0(C_n[M], C_n[N])$.

Definition 2.9. With the above assumptions, the metric on the set $C^0(C_n[M], C_n[N])$ is given by

$$\|F - G\|_0 = \sup_{\vec{p} \in C_n[M]} \{ \|\text{pr}_i(F(\vec{p})) - \text{pr}_i(G(\vec{p}))\| \mid \text{for all } i \}.$$

Thus, given the embeddings $f, g: M \hookrightarrow N$, we say that the corresponding maps on configuration spaces $C_n[f], C_n[g]: C_n[M] \hookrightarrow C_n[N]$ are C^0 -close if for all $\epsilon > 0$, we have $\|C_n[f] - C_n[g]\|_0 < \epsilon$.

3. Finding special configurations with multijet transversality

In this section, we set up a general method for tackling problems where we seek a special configuration of n points on a compact manifold M that is smoothly embedded in \mathbb{R}^k (cf. [10]). In both the square-peg problem and the inscribed simplex problem, M is a sphere (either S^1 or S^{k-1}). We thus denote the C^∞ smooth embedding by $\gamma: S^l \hookrightarrow \mathbb{R}^k$, and the corresponding compactified configuration space of n points on $\gamma(S^l)$ is $C_n[\gamma(S^l)]$. We let Z denote the subspace of tuples of n points in \mathbb{R}^k that satisfy the conditions of a special configuration. For example, in [10], Z is the set of all square-like quadrilaterals in \mathbb{R}^k . In this paper, we are interested in $Z = \text{Sim}(\Delta)$ (see Section 4), which is the set of all $(k + 1)$ -simplices in \mathbb{R}^k which are similar to a given nondegenerate simplex Δ .

The central idea is as follows: suppose there is a different, well-understood, smooth embedding of S^l in \mathbb{R}^k (via $i: S^l \hookrightarrow \mathbb{R}^k$), and assume that the corresponding configuration space $C_n[i(S^l)]$ is transverse to Z in $C_n[\mathbb{R}^k]$. Also assume that $i(S^l)$ is smoothly homotopy equivalent to $\gamma(S^l)$ in \mathbb{R}^k . Standard transversality arguments should allow us to vary $C_n[i(S^l)]$ to $C_n[\gamma(S^l)]$ while maintaining the transversality of the intersection with Z . There are various technical obstacles to overcome, all of which are handled in detail in [10]. Here are the key steps.

Step 1: It is possible that special configurations on $\gamma(S^l)$ shrink away to the boundary of $C_n[\mathbb{R}^k]$ during the isotopy. To prevent this, we first prove

- (a) that n -tuples of points in \mathbb{R}^k satisfying the geometric condition, Z , are a submanifold in $C_n(\mathbb{R}^k)$, and that $\partial Z \subset \partial C_n[\mathbb{R}^k]$;
- (b) that $C_n[\gamma(S^l)]$ and Z are boundary-disjoint.

For $Z = \text{Sim}(\Delta)$, we prove 1(a) in Theorems 4.8 and 4.9. We prove 1(b) in Proposition 5.1.

Step 2: For a standard embedding $i: S^l \hookrightarrow \mathbb{R}^k$, we need to do two things

- (a) prove that the intersection between $C_n[i(S^l)]$ and Z is nonempty and transverse (in other words, $C_n[i] \pitchfork Z$).
- (b) compute the homology class of the intersection $C_n[i(S^l)] \cap Z$ in Z .

For our inscribed simplex problem, we use the standard embedding $\text{id}: S^{k-1} \hookrightarrow \mathbb{R}^k$ of S^{k-1} in \mathbb{R}^k . We prove 2(a) in Proposition 5.2 and 2(b) in Proposition 5.3.

Step 3: Note that in order to apply our transversality arguments, we need to be able to perturb $C_n[\gamma(S^l)]$ so the intersection of Z remains transverse. However, there is no guarantee that the perturbed submanifold consists of configurations on a perturbed smooth embedding of S^l in \mathbb{R}^k . We deal with this issue by applying the multijet transversality theorem [14, Theorem II.4.13]. This allows us to conclude (see [10] Theorem 17) that for any $\epsilon > 0$, there is a C^∞ -open neighborhood of γ in which there is, for all m , a C^m -dense set of smooth embeddings $\gamma': S^l \hookrightarrow \mathbb{R}^k$, such that $\|C_n[\gamma'] - C_n[\gamma]\|_0 < \epsilon$, and $C_n[\gamma'] \pitchfork Z$, and for which ∂Z and $\partial C_n[\gamma'(S^l)]$ are disjoint in $\partial C_n[\mathbb{R}^k]$.

In order to fully appreciate Step 3, first recall that the metric on the set $C^0(C_n[S^l], C_n[\mathbb{R}^k])$ was given in Definition 2.9. Second, to understand the statement about density, we need to give the topology of the spaces we are working in. In general, for manifolds $M(= S^l)$ and $N(= \mathbb{R}^k)$, the space $C^\infty(M, N)$ has the Whitney C^∞ -topology (see, e.g., [20]). The sets of the form

$$\mathcal{N}^t(f; (U, \phi), (V, \psi), \delta)$$

give a subbasis for the Whitney C^t -topology on $C^t(M, N)$ (where t is finite). This subbasis consists of subsets of functions $g: M \rightarrow N$ that are smooth, and for coordinate charts $\phi: (U' \subset M) \rightarrow (U \subset \mathbb{R}^m)$ and $\psi: (V' \subset N) \rightarrow (V \subset \mathbb{R}^k)$ and $K \subset U$ compact with $g(\phi(K)) \subset V'$, then we have, for all $s \leq r$, and all $x \in \phi(K)$,

$$\|D^s(\psi g \phi^{-1})(x) - D^s(\psi f \phi^{-1})(x)\| < \delta.$$

Here, $D^s F$ for a function $F: (U \subset \mathbb{R}^m) \rightarrow (V \subset \mathbb{R}^k)$ is the k -tuple of the s th homogeneous parts of the Taylor series representations of the projections of F . Finally, the subspace $C^\infty(M, N)$ has the Whitney C^∞ -topology generated by taking the union of all subbases for all $t \geq 0$.

Note that Step 3 can be applied immediately to the inscribed simplex problem.

Step 4: We need to deform standard spheres into spheres of interest and then consider what happens on the level of configuration spaces. We know precisely when such a deformation of spheres exists due to the following result of Haefliger (which we have stated in a form useful to us).

Theorem 3.1 ([19]). *Any two differentiable embeddings of S^l in \mathbb{R}^k are homotopic through a differentiable isotopy in $\mathbb{R}^K \supset \mathbb{R}^k$ when $K > 3(l + 1)/2$.*

We use Theorem 3.1 to find a smooth map $E: S^l \times I \rightarrow \mathbb{R}^K$ with $E(-, 0) = i$ our standard embedding and $E(-, 1) = \gamma'$ (where K may be greater than our original k). Recalling that both $C_n[i]$ and $C_n[\gamma']$ are transverse to Z allows us to conclude that using functoriality, we get a homotopy $H: C_n(S^l) \times I \rightarrow C_n(\mathbb{R}^K)$ with $H(-, 0) = C_n[i]$ and $H(-, 1) = C_n[\gamma']$. In [10], we then use the Transversality Homotopy Extension Theorem (see [17]), to find a map H' homotopic to H and with $H'(-, 0) = H(-, 0)$, $H'(-, 1) = H(-, 1)$ and H' is transverse to Z . This then implies that in Z , the intersections $C_n[i(S^l)] \cap Z$ and $C_n[\gamma'(S^l)] \cap Z$ represent the same homology class. In other words, we have sketched a proof of the following:

Theorem 3.2 ([10] Theorem 20). *Suppose there are two embeddings $\eta, i: S^l \hookrightarrow \mathbb{R}^k$ of an l -sphere in \mathbb{R}^k . Assume that Z is a closed topological space contained in $C_n[\mathbb{R}^k]$, such that $Z \cap C_n(\mathbb{R}^k)$ is a submanifold of $C_n(\mathbb{R}^k)$, $\partial Z \subset \partial C_n[\mathbb{R}^k]$, and ∂Z is disjoint from $\partial C_n[i(S^l)]$. Also assume that both $C_n[i]$ and $C_n[\eta]$ are transverse to Z . Then in Z , the homology class of $C_n[i(S^l)] \cap Z$ and $C_n[\eta(S^l)] \cap Z$ are equal.*

In this paper, we will not be making full use of Haefliger’s Theorem in Step 4. Instead, we will restrict our attention to smooth embeddings of S^{k-1} in \mathbb{R}^k which are differentiably isotopic to the identity through a differentiable isotopy in \mathbb{R}^k . The conclusion of Theorem 3.2 still holds for such embeddings. Putting all of our steps together allows us to conclude the following theorem.

Theorem 3.3 ([10] Theorem 21). *Suppose $\gamma: S^l \hookrightarrow \mathbb{R}^k$ is a smooth embedding of S^l in \mathbb{R}^k , with a corresponding embedding of compactified configuration spaces $C_n[\gamma]: C_n[S^l] \hookrightarrow C_n[\mathbb{R}^k]$. Assume*

that Z is a closed topological space contained in $C_n[\mathbb{R}^k]$, such that $Z \cap C_n(\mathbb{R}^k)$ is a submanifold of $C_n(\mathbb{R}^k)$, and $\partial Z \subset \partial C_n[\mathbb{R}^k]$. Also assume that $C_n[\gamma(S^l)]$ and Z are boundary-disjoint. Suppose there is a standard embedding $i: S^l \hookrightarrow \mathbb{R}^k$, such that $C_n[i] \pitchfork Z$ in $C_n[\mathbb{R}^k]$.

Then for all $\epsilon > 0$, there is a C^∞ -open neighborhood of γ , in which there is, for all m , a C^m -dense set of smooth embeddings $\gamma': S^l \hookrightarrow \mathbb{R}^k$, such that $\|C_n[\gamma'] - C_n[\gamma]\|_0 < \epsilon$, and $C_n[\gamma'] \pitchfork Z$. Moreover, $C_n[i(S^l)] \cap Z$ and $C_n[\gamma'(S^l)] \cap Z$ represent the same homology class in Z .

Intuitively, this theorem shows that any smooth embedding γ of S^l in \mathbb{R}^k has a neighborhood in which there is a dense set of smooth embeddings γ' for which $C_n[\gamma'(S^l)]$ is guaranteed to have certain intersections with various submanifolds of $C_n[\mathbb{R}^k]$ defined by geometric conditions. Stated in a different way, we have the idea that a dense set of embeddings of S^l always contain certain inscribed configurations of points.

4. Simplices

4.1. Nondegenerate simplices

Definition 4.1. By a *simplex* Δ in \mathbb{R}^k , we mean a set of $k + 1$ distinct points $\{\mathbf{p}_0, \dots, \mathbf{p}_k\}$ in \mathbb{R}^k in general position.

By general position, we mean that no hyperplane in \mathbb{R}^k contains more than k points of Δ . As a first consequence of this definition, we note that the volume of the simplex Δ is nonzero. This means that for us, a **simplex is nondegenerate**. In addition, to each simplex Δ , we can associate several sets:

- the nonzero distances $\{d_{ij}(\Delta)\} = \{\|\mathbf{p}_i - \mathbf{p}_j\|\}$, and observe that $d_{ij} = d_{ji}$;
- the unit vectors $\{\pi_{ij}(\Delta)\} = \{\frac{\mathbf{p}_i - \mathbf{p}_j}{\|\mathbf{p}_i - \mathbf{p}_j\|}\}$;
- the ratios $\{r_{ijl}(\Delta)\} = \{\frac{\|\mathbf{p}_i - \mathbf{p}_j\|}{\|\mathbf{p}_i - \mathbf{p}_l\|}\} = \{\frac{d_{ij}}{d_{il}}\}$.

Here, we assume that $i \neq j$ (and $j \neq l, i \neq l$), so the set of $k(k + 1)/2$ distances consists of nonzero values, and the set of ratios has values in $(0, \infty)$.

A natural question to ask is, given a set of nonzero distances $\mathcal{D} = \{d_{ij}\}$, is there a simplex that can be constructed with those distances? The theory of distance geometry allows us to decide which sets of distances \mathcal{D} are constructible. We start by defining the Cayley-Menger determinant for $\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k\}$, or equivalently for $\mathcal{D} = \{d_{ij}\} = \{\|\mathbf{p}_i - \mathbf{p}_j\|\}$ (see, e.g., [7, 41]):

$$\text{CM}(\{\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k\}) = \text{CM}(\mathcal{D}) = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{01}^2 & \dots & d_{0k}^2 \\ 1 & d_{10}^2 & 0 & \dots & d_{1k}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{k0}^2 & d_{k1}^2 & \dots & 0 \end{vmatrix}.$$

Theorem 4.2 ([5] Theorems 9.7.3.4 and 9.14.23). *Given a set of nonzero distances $\mathcal{D} = \{d_{ij}\}$ for $i, j = 0, 1, \dots, k$, a necessary and sufficient condition for the existence of a simplex $\{\mathbf{p}_0, \dots, \mathbf{p}_k\}$ with $d_{ij} = \|\mathbf{p}_i - \mathbf{p}_j\|$ is that for every $h = 2, \dots, k$, and every h -element subset of $\{0, 1, \dots, k\}$, the corresponding Cayley-Menger determinant be nonzero and its sign be $(-1)^h$.*

Moreover, when $\mathcal{D} = \{d_{ij}\} = \{\|\mathbf{p}_i - \mathbf{p}_j\|\}$ for $\mathbf{p}_0, \dots, \mathbf{p}_k \in \mathbb{R}^k$, the volume V of the simplex with vertices $\mathbf{p}_0, \dots, \mathbf{p}_k$ obeys

$$V^2 = \frac{(-1)^{k+1}}{2^k (k!)^2} \text{CM}(\mathcal{D}).$$

Recall that by definition, a simplex is nondegenerate. This means that both the volume of the simplex and Cayley-Menger determinant are nonzero.

The Cayley-Menger determinant generalises standard facts in triangle geometry: for instance, for a triangle with side lengths a, b and c , we can write this determinant explicitly as

$$CM(\{a, b, c\}) = a^4 - 2a^2b^2 - 2a^2c^2 + b^4 - 2b^2c^2 + c^4 = -(a + b + c)(a + b - c)(a - b + c)(-a + b + c),$$

and conclude that

$$\text{Area}(\{a, b, c\})^2 = \frac{1}{16}(a + b + c)(a + b - c)(a - b + c)(-a + b + c).$$

This is Heron’s formula for the area of the triangle. We can see the triangle inequality (a criteria for constructability of a triangle), in these formulas: Theorem 4.2 holds if and only if one of the side lengths is greater than the sum of the other two.

4.2. Simplices and configuration spaces

We will now shift our viewpoint and, with an abuse of notation, view the simplex Δ in \mathbb{R}^k as an ordered $(k + 1)$ -tuple of points $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k)$. Thus, the simplex is a point in the open configuration space $C_{k+1}(\mathbb{R}^k)$. We now wish to understand the set of points $\{(\mathbf{q}_0, \dots, \mathbf{q}_k)\}$ in $C_{k+1}(\mathbb{R}^k)$ that correspond to simplices similar to our given simplex Δ . By similar, we mean there is a translation, rotation and nonzero scaling of \mathbb{R}^k which maps $(\mathbf{q}_0, \dots, \mathbf{q}_k)$ to Δ .

Definition 4.3. For a given nondegenerate simplex $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k)$, we let $\text{Sim}(\Delta) \subset C_{k+1}(\mathbb{R}^k)$ denote the space of simplices similar to Δ . Then $\text{Sim}(\Delta)$ is the set of all $\vec{\mathbf{q}} \in C_{k+1}(\mathbb{R}^k)$, such that

$$r_{ijl}(\vec{\mathbf{q}}) = r_{ijl}(\Delta) \quad \text{for all } i \neq j \neq l \neq i.$$

That is, when $\vec{\mathbf{q}} = (\mathbf{q}_0, \dots, \mathbf{q}_k)$ is extrinsically similar to Δ , then we have

$$r_{ijl}(\vec{\mathbf{q}}) = \frac{\|\mathbf{q}_i - \mathbf{q}_j\|}{\|\mathbf{q}_i - \mathbf{q}_l\|} = \frac{\|\mathbf{p}_i - \mathbf{p}_j\|}{\|\mathbf{p}_i - \mathbf{p}_l\|} = r_{ijl}(\Delta).$$

Our next aim is to show that $\text{Sim}(\Delta)$ is a submanifold of $C_{k+1}(\mathbb{R}^k)$, by proving that $\text{Sim}(\Delta)$ is diffeomorphic to $O(k) \times (0, \infty) \times \mathbb{R}^k$. We also aim to understand how the boundary of $\text{Sim}(\Delta)$ sits inside of $C_{k+1}[\mathbb{R}^k]$. In order to do this, we will associate a matrix to $\text{Sim}(\Delta)$ and look at the polar decomposition of that matrix. So we will need to use some results from linear algebra along the way. These are all found in Appendix A.

Definition 4.4. Given a configuration $\vec{\mathbf{q}} = (\mathbf{q}_0, \dots, \mathbf{q}_k)$ in $\text{Sim}(\Delta)$, we define the $k \times k$ matrix $\Pi(\vec{\mathbf{q}})$ by

$$\Pi(\vec{\mathbf{q}}) = [\pi_{10}(\vec{\mathbf{q}}) \dots \pi_{k0}(\vec{\mathbf{q}})].$$

Proposition 4.5. Given a configuration $\vec{\mathbf{q}} = \{\mathbf{q}_0, \dots, \mathbf{q}_k\}$ in $\text{Sim}(\Delta)$, the matrix $\Pi(\vec{\mathbf{q}})$ has rank k , and the $k \times k$ matrix $\Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}})$ is given by

$$\Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}}) = [\cos(\theta_{ij})] = \left[\frac{1}{2}(r_{0ij}(\vec{\mathbf{q}}) + r_{0ji}(\vec{\mathbf{q}}) - r_{ij0}(\vec{\mathbf{q}})r_{ji0}(\vec{\mathbf{q}})) \right] = P^2(\Delta),$$

where $P(\Delta)$ is a uniquely determined symmetric positive-definite matrix.

The matrix $\Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}})$ is called the Gram matrix, and it consists of the dot products of the $\pi_{ij}(\vec{\mathbf{q}})$, which are the cosines of the angles between the unit vectors $\pi_{ij}(\vec{\mathbf{q}})$.

Proof. Recall the law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C$, where c is the length of the side of a triangle $\triangle ABC$ opposite angle C and a and b are the lengths of the sides subtending angle C . In the case

$a = \|\mathbf{q}_i - \mathbf{q}_0\|$, $b = \|\mathbf{q}_j - \mathbf{q}_0\|$ and $c = \|\mathbf{q}_i - \mathbf{q}_j\|$, angle C is θ_{ij} , and we have

$$\begin{aligned} \cos \theta_{ij} &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2} \left(\frac{a}{b} + \frac{b}{a} - \frac{c^2}{ab} \right) \\ &= \frac{1}{2} (r_{0ij}(\vec{\mathbf{q}}) + r_{0ji}(\vec{\mathbf{q}}) - r_{ij0}(\vec{\mathbf{q}})r_{ji0}(\vec{\mathbf{q}})). \end{aligned}$$

Because the r_{ijl} are the same for all configurations in $\text{Sim}(\Delta)$, this matrix is the same as for $\Pi(\Delta)$ associated with $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k) \in C_{k+1}(\mathbb{R}^k)$. The following shows $\Pi(\Delta)$ has rank k :

$$\begin{aligned} 0 < \text{Vol}_k(\Delta) &= \frac{1}{k!} \det[\mathbf{p}_1 - \mathbf{p}_0 \ \dots \ \mathbf{p}_k - \mathbf{p}_0] \\ &= \frac{\|\mathbf{p}_1 - \mathbf{p}_0\| \times \dots \times \|\mathbf{p}_k - \mathbf{p}_0\|}{k!} \det[\pi_{10}(\vec{\mathbf{p}}) \ \dots \ \pi_{k0}(\vec{\mathbf{p}})] \\ &= \frac{\|\mathbf{p}_1 - \mathbf{p}_0\| \times \dots \times \|\mathbf{p}_k - \mathbf{p}_0\|}{k!} \det \Pi(\Delta). \end{aligned}$$

By construction, $\Pi(\Delta)^T \Pi(\Delta)$ is a symmetric matrix. We have just shown that $\Pi(\Delta)$ has full column rank. Thus, by Theorem A.3, we know $\Pi(\Delta)^T \Pi(\Delta)$ is positive-definite. Since $\Pi(\Delta)^T \Pi(\Delta) = \Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}})$, we again use Theorem A.3 to deduce $\Pi(\vec{\mathbf{q}})$ also has rank k . Now, using other standard results from linear algebra (Theorem A.1 and Remark A.2), we can deduce there is a unique symmetric positive-definite matrix $P(\Delta)$, such that $P(\Delta) = (\Pi(\Delta)^T \Pi(\Delta))^{1/2}$. \square

We now use the polar decomposition theorem for a matrix to get a better understanding of $\text{Sim}(\Delta)$. Recall, from Theorem A.4, that the polar decomposition of a $k \times k$ real matrix A is a factorisation of the form $A = UP$, where U is orthogonal and P is a positive semidefinite symmetric matrix. This decomposition is unique when A is nonsingular (intuitively, if A is interpreted as a linear transformation of \mathbb{R}^k , then the polar decomposition separates it into a rotation or reflection U of \mathbb{R}^k , and a scaling of the space along a set of k orthogonal axes).

Proposition 4.6. *If $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, then $\Pi(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})P(\Delta)$, where $U(\vec{\mathbf{q}})$ is a uniquely determined $k \times k$ orthogonal matrix which is a smooth function of $\vec{\mathbf{q}}$. The $k \times k$ matrix $P(\Delta)$ depends only on Δ , and is a symmetric positive-definite matrix.*

Proof. In our case, since $\text{rank } \Pi(\vec{\mathbf{q}}) = k$, the polar decomposition theorem (Theorem A.4) tells us that for the $k \times k$ matrix $\Pi(\vec{\mathbf{q}})$, there is a unique decomposition into $k \times k$ matrices: $\Pi(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})P$. Here, matrix $U(\vec{\mathbf{q}})$ is orthogonal, and $P = (\Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}}))^{1/2}$. From Proposition 4.5, we know

$$P = (\Pi(\vec{\mathbf{q}})^T \Pi(\vec{\mathbf{q}}))^{1/2} = (\Pi(\Delta)^T \Pi(\Delta))^{1/2} = P(\Delta).$$

Thus, for each $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, we have the same symmetric positive-definite $P(\Delta)$ matrix. The dependence of $\Pi(\vec{\mathbf{q}})$ on $\vec{\mathbf{q}}$ is clearly smooth; that $U(\vec{\mathbf{q}})$ depends smoothly on $\Pi(\vec{\mathbf{q}})$, and, hence, on $\vec{\mathbf{q}}$, and $U(\vec{\mathbf{q}})$ is smooth, were shown in [12] (see Remark A.5). \square

Let $O(k)$ be the set of all orthogonal $k \times k$ matrices. Roughly speaking, Proposition 4.6 says that for a (nondegenerate) simplex Δ , we can obtain a different configuration in $\text{Sim}(\Delta)$ by multiplying $P(\Delta)$ on the left by a matrix in $O(k)$. This leads us to define the following map.

Definition 4.7. The *pose* map $ps : \text{Sim}(\Delta) \rightarrow O(k)$ is defined by $ps(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})$, where $U(\vec{\mathbf{q}})$ is the unique orthogonal matrix $U(\vec{\mathbf{q}})$, such that $\Pi(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})P(\Delta)$ (as found in Proposition 4.6). For a simplex $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, we call the corresponding element $U(\vec{\mathbf{q}}) \in O(k)$ the *pose* of $\vec{\mathbf{q}}$.

We can now deduce the structure of $\text{Sim}(\Delta)$ as a submanifold of $C_{k+1}[\mathbb{R}^k]$.

Theorem 4.8. *If Δ is a nondegenerate k -simplex in \mathbb{R}^k , then $\text{Sim}(\Delta) \subset C_{k+1}(\mathbb{R}^k)$ is a submanifold diffeomorphic to $O(k) \times (0, \infty) \times \mathbb{R}^k$.*

Proof. We will define the following pair of maps:

$$O(k) \times (0, \infty) \times \mathbb{R}^k \xrightarrow{i_\Delta} \text{Sim}(\Delta) \subset C_{k+1}(\mathbb{R}^k) \xrightarrow{p_{S\Delta}} O(k) \times (0, \infty) \times \mathbb{R}^k.$$

We will then prove that both maps are smooth, that the first map i_Δ is onto $\text{Sim}(\Delta)$ and the composition is the identity. This will prove that i_Δ is a diffeomorphism onto $\text{Sim}(\Delta)$.

We have been given $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k) \in C_{k+1}(\mathbb{R}^k)$. We can then define the set of ratios $\{r_{ijl}(\Delta)\}$, the set of unit vectors $\{\pi_{ij}(\Delta)\}$ and the matrix $\Pi(\Delta)$. We know that $\Pi(\Delta) = U(\Delta)P(\Delta)$, from Proposition 4.6. Moreover, without loss of generality, we can assume that Δ is chosen such that $U(\Delta) = I_k$ (the identity matrix), and so $\Pi(\Delta) = P(\Delta)$.

Let $A = [\mathbf{v}_1 \dots \mathbf{v}_k] \in O(k)$, and define a new set of unit vectors by $\pi_{ij} = [\mathbf{v}_1 \dots \mathbf{v}_k]\pi_{ij}(\Delta)$ (intuitively, the new π_{ij} are $\pi_{ij}(\Delta)$ rotated/reflected by A). If we also let $r_{ijl} = r_{ijl}(\Delta)$, then we define the map i_Δ by

$$i_\Delta(A, \lambda, \mathbf{q}_0) = (\mathbf{q}_0, \mathbf{q}_0 + \lambda\pi_{10}, \mathbf{q}_0 + \lambda r_{021}\pi_{20}, \dots, \mathbf{q}_0 + \lambda r_{0k1}\pi_{k0}) = \vec{\mathbf{q}}.$$

This map i_Δ is clearly smooth in A , λ and \mathbf{q}_0 . Furthermore, $i_\Delta(A, \lambda, \mathbf{q}_0)$ has $r_{ijl}(\vec{\mathbf{q}}) = r_{ijl}(\Delta)$ by construction, and so lies in $\text{Sim}(\Delta)$.

We prove that i_Δ is onto $\text{Sim}(\Delta)$. Given $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, we use Proposition 4.6, and our assumption that $U(\Delta) = I_k$, to write

$$\Pi(\vec{\mathbf{q}}) = [\pi_{10}(\vec{\mathbf{q}}) \dots \pi_{k0}(\vec{\mathbf{q}})] = U(\vec{\mathbf{q}})P(\Delta) = U(\vec{\mathbf{q}})\Pi(\Delta).$$

This means that $\pi_{j0}(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})\pi_{j0}(\Delta)$ for $j = 1, \dots, k$. Remembering that $\pi_{j0} = -\pi_{0j}$, we then deduce the other values of $\pi_j(\vec{\mathbf{q}})$ satisfy the same relationship:

$$\begin{aligned} \pi_{ij}(\vec{\mathbf{q}}) &= \frac{\mathbf{q}_j - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|} = \frac{\mathbf{q}_j - \mathbf{q}_0}{\|\mathbf{q}_j - \mathbf{q}_i\|} + \frac{\mathbf{q}_0 - \mathbf{q}_i}{\|\mathbf{q}_j - \mathbf{q}_i\|} \\ &= \frac{\|\mathbf{q}_j - \mathbf{q}_0\|}{\|\mathbf{q}_j - \mathbf{q}_i\|} \frac{\mathbf{q}_j - \mathbf{q}_0}{\|\mathbf{q}_j - \mathbf{q}_0\|} + \frac{\|\mathbf{q}_0 - \mathbf{q}_i\|}{\|\mathbf{q}_j - \mathbf{q}_i\|} \frac{\mathbf{q}_0 - \mathbf{q}_i}{\|\mathbf{q}_0 - \mathbf{q}_i\|} \\ &= r_{j0i}(\vec{\mathbf{q}})\pi_{j0}(\vec{\mathbf{q}}) + r_{i0j}(\vec{\mathbf{q}})\pi_{0i}(\vec{\mathbf{q}}) \\ &= r_{j0i}(\vec{\mathbf{q}})U(\vec{\mathbf{q}})\pi_{j0}(\Delta) + r_{i0j}(\vec{\mathbf{q}})U(\vec{\mathbf{q}})\pi_{0i}(\Delta) = U(\vec{\mathbf{q}})\pi_{ij}(\Delta). \end{aligned}$$

Thus, for $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, we have $\pi_{ij}(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})\pi_{ij}(\Delta)$. For the map i_Δ , we let $\lambda = \|\mathbf{q}_1 - \mathbf{q}_0\|$, then $i_\Delta(U(\vec{\mathbf{q}}), \|\mathbf{q}_1 - \mathbf{q}_0\|, \mathbf{q}_0) = (\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_k) = \vec{\mathbf{q}}$. To see this last equation, note that

$$\mathbf{q}_1 = \mathbf{q}_0 + \lambda \frac{\mathbf{q}_1 - \mathbf{q}_0}{\|\mathbf{q}_1 - \mathbf{q}_0\|} \quad \text{and} \quad \mathbf{q}_2 = \mathbf{q}_0 + \lambda r_{021}(\vec{\mathbf{q}})\pi_{20}(\vec{\mathbf{q}}) = \mathbf{q}_0 + \lambda \frac{\|\mathbf{q}_2 - \mathbf{q}_0\|}{\|\mathbf{q}_1 - \mathbf{q}_0\|} \frac{\mathbf{q}_2 - \mathbf{q}_0}{\|\mathbf{q}_2 - \mathbf{q}_0\|}.$$

We can now define the map $p_{S\Delta}: \text{Sim}(\Delta) \rightarrow O(k) \times (0, \infty) \times \mathbb{R}^k$ using the pose map from Definition 4.7. We define $p_{S\Delta}(\vec{\mathbf{q}}) := (ps(\vec{\mathbf{q}}), \|\mathbf{q}_1 - \mathbf{q}_0\|, \mathbf{q}_0) = (U(\vec{\mathbf{q}}), \|\mathbf{q}_1 - \mathbf{q}_0\|, \mathbf{q}_0)$. From Proposition 4.6, we know $U(\vec{\mathbf{q}})$ depends smoothly on $\vec{\mathbf{q}}$, and, hence, $p_{S\Delta}$ depends smoothly on $\vec{\mathbf{q}}$. A moment's thought shows that, by construction, we have $p_{S\Delta} \circ i_\Delta = \text{id}$.

This means that i_Δ (and, hence, $p_{S\Delta}$) is a diffeomorphism, and $\text{Sim}(\Delta)$ is a submanifold of $C_{k+1}(\mathbb{R}^k)$. □

We note that this theorem shows that the polar decomposition is a diffeomorphism for matrices of full rank. This is analogous to the smoothness result for the polar decomposition of Dieci and Eirola [12]. This theorem is not true for rank-deficient matrices, which explains why we have restricted our attention to spheres S^{k-1} isotopic to each other in \mathbb{R}^k ; Haefliger's theorem might guarantee the

existence of an isotopy of the spheres in a higher-dimensional \mathbb{R}^K , but $\text{Sim}(\Delta)$ would still consist of rank $k < K$ matrices and, hence, be harder to control.

The next theorem shows $\text{Sim}(\Delta)$ has a well-understood structure in the boundary $\partial C_{k+1}[\mathbb{R}^k]$.

Theorem 4.9. *If Δ is a nondegenerate k -simplex in \mathbb{R}^k , then the boundary of $\text{Sim}(\Delta)$ corresponds to configurations in the interior of the $(0, \dots, k)$ face of $\partial C_{k+1}[\mathbb{R}^k]$, and is diffeomorphic to $O(k) \times \{0\} \times \mathbb{R}^k$.*

Proof. By assumption, the given simplex $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k)$ is nondegenerate, so none of the vertices of Δ coincide, and the ratios $r_{ijl}(\Delta)$ are never 0 nor ∞ . If we take $(A, \lambda, \mathbf{p}) \in O(k) \times [0, \infty) \times \mathbb{R}^k$ and consider the points in $i_\Delta(A, \lambda, \mathbf{p}_0)$, we see these points coincide if and only if $\lambda = 0$, in which case, they all coincide. This means $i_\Delta(A, 0, \mathbf{p}_0) \subset \partial C_{k+1}[\mathbb{R}^k]$ and is in the $(0, 1, \dots, k)$ face. Since the boundary of the $(0, 1, \dots, k)$ face consists of configurations with $r_{ijl}(\Delta) = 0$, or $r_{ijl}(\Delta) = \infty$, it follows immediately that $i_\Delta(A, 0, \mathbf{p}_0)$ is in the interior of this face. \square

Corollary 4.10. *If Δ is a nondegenerate k -simplex in \mathbb{R}^k , then $\text{Sim}(\Delta)$ is a submanifold of $C_{k+1}[\mathbb{R}^k]$ that is diffeomorphic to $O(k) \times [0, \infty) \times \mathbb{R}^k$.*

5. Simplices inscribed in spheres

We next move to Step 1 of the method described in Section 3. We will be looking at C^∞ -smooth embeddings of S^{k-1} in \mathbb{R}^k . We will always assume that these embeddings are *regular*, that is, the embedding induces an injection of tangent spaces everywhere. This is particularly relevant for embeddings of S^1 in \mathbb{R}^2 , where we assume the tangent vector is nowhere zero (otherwise, it is possible to smoothly describe an embedded curve with corners, by allowing the tangent vector to smoothly change to zero at each corner). Now, given any such regular C^∞ -smooth embedding $\gamma: S^{k-1} \hookrightarrow \mathbb{R}^k$, we can view the corresponding configuration space $C_{k+1}[\gamma(S^{k-1})]$ as a submanifold of $C_{k+1}[\mathbb{R}^k]$.

Proposition 5.1. *Given a nondegenerate simplex $\Delta \in C_{k+1}(\mathbb{R}^k)$, and a regular C^∞ -smooth embedding $\gamma: S^{k-1} \hookrightarrow \mathbb{R}^k$ with corresponding configuration space $C_{k+1}[\gamma(S^{k-1})]$, then $\partial \text{Sim}(\Delta)$ and $\partial C_{k+1}[\gamma(S^{k-1})]$ are disjoint in $\partial C_{k+1}[\mathbb{R}^k]$.*

Proof. From Theorem 4.9, we know that $\partial \text{Sim}(\Delta)$ is in the $(0, 1, \dots, k)$ face of $\partial C_{k+1}[\mathbb{R}^k]$, so we restrict our attention to that boundary face. Since Δ is nondegenerate, no hyperplane in \mathbb{R}^k contains more than k points of Δ , and so not all of the $\pi_{ij}(\Delta)$ vectors are coplanar. However, for a point in $\partial C_{k+1}[\gamma(S^{k-1})]$, all of the π_{ij} vectors must be coplanar. Hence, $\text{Sim}(\Delta)$ and $C_{k+1}[\gamma(S^{k-1})]$ are boundary disjoint. \square

We next move to Step 2. We take the standard embedding of the $(k - 1)$ sphere in \mathbb{R}^k , that is, $\text{id}: S^{k-1} \hookrightarrow \mathbb{R}^k$, and consider the corresponding configuration spaces. In this special case, we use the notation $C_{k+1}[S^{k-1}] = C_{k+1}[\text{id}(S^{k-1})]$.

We need to show that there is a transverse intersection between $C_{k+1}[S^{k-1}]$ and $\text{Sim}(\Delta)$ in $C_{k+1}[\mathbb{R}^k]$, in other words, $C_{k+1}(\text{id}) \pitchfork \text{Sim}(\Delta)$. We also need to compute the homology class of the intersection of $C_{k+1}[S^{k-1}] \cap \text{Sim}(\Delta)$ in $\text{Sim}(\Delta)$.

Proposition 5.2. *Given the configuration space $C_{k+1}[S^{k-1}]$ corresponding to the standard embedding of S^{k-1} in \mathbb{R}^k , and given a nondegenerate simplex $\Delta \in C_{k+1}(\mathbb{R}^k)$, then $\text{Sim}(\Delta)$ intersects $C_{k+1}[S^{k-1}]$ transversally, and the intersection $\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]$ is diffeomorphic to $O(k)$.*

Proof. Since $\text{Sim}(\Delta)$ and $C_{k+1}[S^{k-1}]$ are boundary disjoint (Proposition 5.1), we just need to consider $\text{Sim}(\Delta) \cap C_{k+1}(S^{k-1})$. Now, every simplex Δ has a unique circumsphere; a $(k - 1)$ -sphere passing through all of the $k + 1$ vertices. The radius of the circumsphere and coordinates of the circumcentre are well known (see, for instance, Proposition 9.7.3.7 [5] or [11, 38]). Indeed, the circumradius R of the

simplex Δ is given by

$$R^2 = -\frac{\begin{vmatrix} 0 & d_{01}^2 & \cdots & d_{0k}^2 \\ d_{10}^2 & 0 & \cdots & d_{1k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ d_{k0}^2 & d_{k2}^2 & \cdots & 0 \end{vmatrix}}{2 \text{CM}(\{\mathbf{p}_0, \dots, \mathbf{p}_k\})}.$$

Given Δ , we can scale and translate it to give a new $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$ with circumradius $R = 1$, and circumcentre $\mathbf{0}$. Thus, $\vec{\mathbf{q}} \in \text{Sim}(\Delta) \cap C_{k+1}(S^{k-1})$ and the intersection is nonempty. Intuitively, we obtain all other points of $\text{Sim}(\Delta) \cap C_{k+1}(S^{k-1})$ by rotating/reflecting $\vec{\mathbf{q}}$ via $A\vec{\mathbf{q}}$ for $A \in O(k)$. More formally, we define the diffeomorphism $r: O(k) \rightarrow \text{Sim}(\Delta) \cap C_{k+1}(S^{k-1})$ by $r(A) = A\vec{\mathbf{q}} = (A\mathbf{q}_0, \dots, A\mathbf{q}_k)$.

We now want to show that $\text{Sim}(\Delta)$ intersects $C_{k+1}[S^{k-1}]$ transversally. Specifically, we take any inscribed simplex $\vec{\mathbf{q}} \in \text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]$, and we want to show that

$$T_{\vec{\mathbf{q}}}(\text{Sim}(\Delta)) \oplus T_{\vec{\mathbf{q}}}(C_{k+1}[S^{k-1}]) = T_{\vec{\mathbf{q}}}(C_{k+1}[\mathbb{R}^k]).$$

We first note that the orthogonal complement of $T_{\vec{\mathbf{q}}}(C_{k+1}[S^{k-1}])$ in $T_{\vec{\mathbf{q}}}(C_{k+1}[\mathbb{R}^k])$ is the $(k + 1)$ -dimensional space with orthonormal basis $\mathcal{B} = \{(\mathbf{q}_0, 0, \dots, 0), \dots, (0, \dots, \mathbf{q}_k)\}$. Next, observe that $T_{\vec{\mathbf{q}}}(\text{Sim}(\Delta)) \cong T_{\vec{\mathbf{q}}}(O(k)) \oplus T_{\vec{\mathbf{q}}}[0, \infty) \oplus T_{\vec{\mathbf{q}}}(\mathbb{R}^k)$. The tangent space $T_{\vec{\mathbf{q}}}(\text{Sim}(\Delta))$ thus contains the vectors $(\mathbf{e}_1, \dots, \mathbf{e}_1), \dots, (\mathbf{e}_k, \dots, \mathbf{e}_k)$ from the translational component of $\text{Sim}(\Delta)$, as well as the vector $(\mathbf{q}_0, \dots, \mathbf{q}_k)$ from scaling the configuration $\vec{\mathbf{q}}$. Writing these vectors in the basis \mathcal{B} , we get the matrix:

$$M = \begin{pmatrix} q_{0,0} & q_{1,0} & \cdots & q_{k,0} \\ q_{0,1} & q_{1,1} & \cdots & q_{k,1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{0,k} & q_{1,k} & \cdots & q_{k,k} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Subtracting the last column from the rest, we get

$$M' = \begin{pmatrix} \mathbf{q}_0 - \mathbf{q}_k & \mathbf{q}_1 - \mathbf{q}_k & \cdots & \mathbf{q}_{k-1} - \mathbf{q}_k & \mathbf{q}_k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The determinant of this matrix is ± 1 multiplied by the determinant of the upper-left $k \times k$ principal minor. But that determinant is positive because $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$ and is nondegenerate. Thus, the $k + 1$ tangent vectors are linearly independent, and we have proven transversality. \square

Before we move on to determine the homology class corresponding to inscribed simplices, we pause to remember that $O(k)$ has two connected components. One component, $SO(k)$, is a subgroup of $O(k)$, and consists of all orthogonal matrices with determinant $+1$. The other component contains all orthogonal matrices with determinant -1 .

In order to sensibly discuss homology classes, we restrict our attention to the submanifold of simplices diffeomorphic to $SO(k) \times [0, \infty) \times \mathbb{R}^k$, which we denote $\text{Sim}^+(\Delta)$. We could equally restrict our attention to $\text{Sim}^-(\Delta) := \text{Sim}(\Delta) \setminus \text{Sim}^+(\Delta)$.

Proposition 5.3. *Given a nondegenerate simplex $\Delta = (\mathbf{p}_0, \dots, \mathbf{p}_k) \in C_{k+1}(\mathbb{R}^k)$, and given the configuration space $C_{k+1}[S^{k-1}]$ corresponding to the standard embedding of S^{k-1} in \mathbb{R}^k , then in $\text{Sim}(\Delta)$*

$$H_*(O(k); \mathbb{Z}) \cong H_*(\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]; \mathbb{Z}).$$

Moreover, the pose map is a diffeomorphism $ps: \text{Sim}^+(\Delta) \cap C_{k+1}[S^{k-1}] \rightarrow \text{SO}(k)$ which induces an isomorphism on integral homology taking the top class of $\text{Sim}^+(\Delta) \cap C_{k+1}[S^{k-1}]$ in $\text{Sim}(\Delta^+)$ to the top class of $\text{SO}(k)$. A similar result holds for $\text{Sim}^-(\Delta) \cap C_{k+1}[S^{k-1}]$.

Proof. Without loss of generality, we can normalise $\Delta = (\mathbf{p}_1, \dots, \mathbf{p}_k)$ so that it lies on S^{k-1} ; that is Δ has circumcentre $\mathbf{0}$ and circumradius $R = 1$. We then have the following maps

$$\text{O}(k) \xrightarrow{r} \text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}] \xrightarrow{ps} \text{O}(k).$$

Here, the map r is the rotation/reflections diffeomorphism previously defined in Proposition 5.2. That is, for $A \in \text{O}(k)$, we define $r(A) = A\Delta = (A\mathbf{p}_0, \dots, A\mathbf{p}_k)$. The map ps is the pose map from Definition 4.7: for $\vec{\mathbf{q}} \in \text{Sim}(\Delta)$, we define $ps(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})$ (where $U(\vec{\mathbf{q}})$ is the unique orthogonal matrix, such that $\Pi(\vec{\mathbf{q}}) = U(\vec{\mathbf{q}})\Pi(\Delta)$). From Proposition 5.2, we know that r is a diffeomorphism, and by construction, $ps \circ r = \text{id}$. Hence, the pose map is also a diffeomorphism.

From Proposition 5.2, we recall that $\text{Sim}(\Delta)$ intersects $C_{k+1}[S^{k-1}]$ transversally, and, moreover, that the intersection is diffeomorphic to $\text{O}(k)$. Now, since $\text{Sim}(\Delta) \cong \text{O}(k) \times [0, \infty) \times \mathbb{R}^k$, we have $\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]$ is a deformation retract of $\text{Sim}(\Delta)$.

When we put this together and take the homology of the spaces, we see

$$H_*(\text{O}(k)) \xrightarrow{r_*} H_*(\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]) \cong H_*(\text{Sim}(\Delta)) \xrightarrow{ps_*} H_*(\text{O}(k)).$$

Since $ps_* \circ r_* = \text{id}$, we know ps_* is an isomorphism. Hence,

$$H_*(\text{O}(k)) \cong H_*(\text{Sim}(\Delta) \cap C_{k+1}[S^{k-1}]).$$

We could also choose to restrict r and ps as follows:

$$\text{SO}(k) \xrightarrow{r} \text{Sim}^+(\Delta) \cap C_{k+1}(S^{k-1}) \xrightarrow{ps} \text{SO}(k).$$

Then, repeating the previous argument gives

$$ps_*([\text{Sim}^+(\Delta) \cap C_{k+1}(S^{k-1})]) = [r(\text{SO}(k))] = [\text{Sim}^+(\Delta)] \in H_{\frac{k(k-1)}{2}}(\text{Sim}^+(\Delta)). \quad \square$$

We now have all the pieces needed for our main theorem.

Theorem 5.4. *Suppose $\gamma: S^{k-1} \hookrightarrow \mathbb{R}^k$ is a C^∞ -smooth embedding of S^{k-1} in \mathbb{R}^k isotopic to the identity through a differentiable isotopy in \mathbb{R}^k , with a corresponding embedding of compactified configuration spaces $C_{k+1}[\gamma]: C_{k+1}[S^{k-1}] \hookrightarrow C_{k+1}[\mathbb{R}^k]$. Assume that $\Delta \in C_{k+1}(\mathbb{R}^k)$ is a nondegenerate simplex.*

Then for all $\epsilon > 0$, there is a C^∞ -open neighborhood of γ , in which there is, for all m , a C^m -dense set of smooth embeddings $\gamma': S^{k-1} \hookrightarrow \mathbb{R}^k$ isotopic to the identity through a differentiable isotopy in \mathbb{R}^k , such that $\|C_{k+1}[\gamma'] - C_{k+1}[\gamma]\|_0 < \epsilon$, and $C_{k+1}[\gamma'] \pitchfork \text{Sim}(\Delta)$. Moreover, in $\text{Sim}(\Delta^+)$, both $C_{k+1}[S^{k-1}] \cap \text{Sim}^+(\Delta)$ and $C_{k+1}[\gamma'(S^{k-1})] \cap \text{Sim}^+(\Delta)$ represent the top homology class of $\text{SO}(k)$. A similar result holds for $\text{Sim}^-(\Delta)$.

Proof. This is Corollary 4.10 and Propositions 5.2 and 5.3 applied to Theorem 3.3. □

We immediately recover Gromov’s result [16] (but for C^∞ -smooth embeddings) as a corollary.

Corollary 5.5. *For any nondegenerate k -simplex Δ in \mathbb{R}^k , there is a dense family of smoothly embedded $(k - 1)$ -spheres in \mathbb{R}^k isotopic to the identity through a differentiable isotopy in \mathbb{R}^k , such that the subset of $\text{Sim}(\Delta)$ of simplices inscribed in each embedded sphere contains a similar simplex corresponding to each $U \in \text{O}(k)$.*

In fact, we have proved more than Gromov.

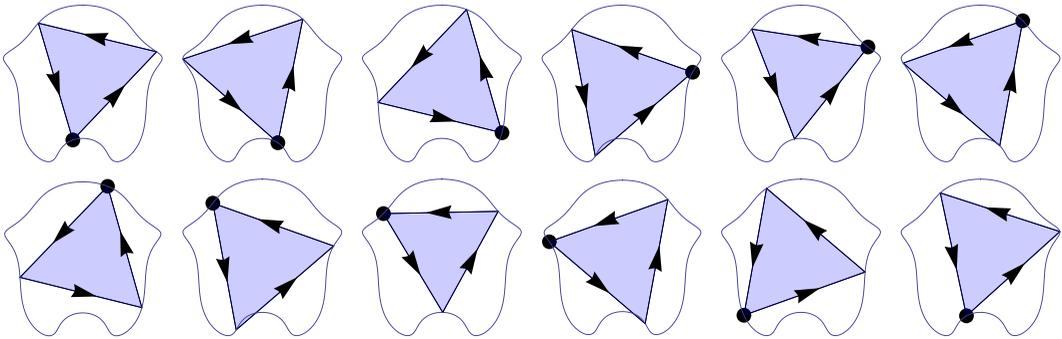


Figure 2. In this irregular embedding of a circle in the plane, we see a series of inscribed equilateral triangles. By following the highlighted vertex around the curve, we see there is a loop of such inscribed triangles.

Corollary 5.6. For any nondegenerate k -simplex Δ in \mathbb{R}^k , there is a dense family of smoothly embedded $(k - 1)$ -spheres in \mathbb{R}^k isotopic to the identity through a differentiable isotopy in \mathbb{R}^k , such that the subset $\text{Sim}^+(\Delta) \cap C_{k+1}[\gamma(S^{k-1})]$ of simplices inscribed in each embedded sphere is a smooth orientable submanifold of $\text{Sim}^+(\Delta)$. Furthermore, the pose map $ps: \text{Sim}^+(\Delta) \cap C_{k+1}[\gamma(S^{k-1})] \rightarrow \text{SO}(k)$ is onto and is a degree 1 map.

In particular, the set of triangles similar to a given triangle ΔABC inscribed in a generic smooth plane curve is a collection of loops L_1, \dots, L_n . Furthermore, the sum of the degrees of the pose maps ps_1, \dots, ps_n is one.

Proof. The first statement follows from Theorem 3.2. That the pose map is onto follows from Corollary 5.5. For the standard embedding of S^{k-1} in \mathbb{R}^k , the pose map is a diffeomorphism, and, hence, of degree 1. Proposition 5.3 and homotopy invariance show the pose map is always of degree 1 for our embeddings.

When $k = 2$, we know $\text{SO}(2) \cong S^1$. Hence, the set of inscribed triangles similar to Δ is a collection of loops. □

In [31], Matschke proves that for generic C^∞ -smooth embeddings of the circle in the plane, there are an odd number of loops of inscribed triangles similar to Δ that wind an odd number of times around the embedded circle. Figure 2 shows such a loop of inscribed equilateral triangles in an irregular embedding of a circle in the plane. We recover Matschke’s result in Corollary 5.6, and strengthen it with the fact that the degree of the map is 1.

We end by, again, noting that Meyerson [34] and Nielsen [35] have results about inscribed triangles that are for Jordan curves (just assuming continuity of the embedding). By adding both a generic and smoothness assumption on our embeddings, we are able to provide more information about the structure of the set of inscribed triangles.

Appendix A. Results from linear algebra

We will let M_n denote the set of all $n \times n$ matrices, and let $M_{n,m}$ denote the set of all $n \times m$ matrices. Recall that a symmetric matrix is a square matrix that is equal to its transpose: $A = A^T$. That is, $A = [a_{ij}]$ is symmetric if and only if $a_{ij} = a_{ji}$. Also recall that

- an $n \times n$ symmetric real matrix M is *positive-definite* if $\mathbf{x}^* M \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$.
- an $n \times n$ symmetric real matrix M is *positive semidefinite* or *nonnegative-definite* if $\mathbf{x}^* M \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
- similar definitions hold for negative-definite and negative-semidefinite.

We now review some theorems about symmetric matrices.

Theorem A.1 (see [21] Theorem 7.2.6). *Let A be an $n \times n$ symmetric and positive semidefinite matrix, let $r = \text{rank}(A)$ and let $k = \{2, 3, \dots\}$. Then there is a unique symmetric positive semidefinite matrix B , such that $B^k = A$.*

Remark A.2. While this theorem only guarantees a unique positive *semidefinite* square root for a *semidefinite* matrix A , an examination of the proof in [21] shows that more has been proven. The proof shows there is a unique positive *definite* square root for a positive *definite* matrix A .

Theorem A.3 (see [21] Theorem 7.2.7). *Let A be an $n \times n$ symmetric matrix. If $A = B^T B$, with B an $m \times n$ matrix, then A is positive definite if and only if B has full column rank.*

The polar decomposition of a matrix is incredibly useful. Below, we give the version that best applies to our work.

Theorem A.4 (see [21] Theorem 7.3.1 Polar decomposition). *Let A be an $n \times n$ real matrix. Then $A = PU = UQ$, in which $P, Q \in M_n$ are positive semidefinite and $U \in M_n$ is orthogonal. The factors $P = (AA^T)^{1/2}$ and $Q = (A^T A)^{1/2}$ are uniquely determined; P is a polynomial in AA^T , and Q is a polynomial in $A^T A$. The factor U is uniquely determined if A is nonsingular.*

Remark A.5 (see [12] Section 2.3 (c)). Suppose A is an $m \times n$ matrix with full rank and $A(t)$ depends C^k -smoothly on t . Then we can write $A(t) = O(t)P(t)$, where O is orthonormal, P is symmetric positive definite and O and P are as smooth as A .

Acknowledgments. We would like to thank all the people who have discussed these problems with us over the years, especially Jordan Ellenberg, Benjamin Matschke and Gunter Ziegler. This research was supported by a grant from the Simons Collaboration Grant #524120 (Cantarella). All authors contributed equally to the research and writing.

Conflict of Interest. The authors have no conflict of interest to declare.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

Supplementary material. None.

References

- [1] A. Akopyan and S. Avvakumov, ‘Any cyclic quadrilateral can be inscribed in any closed convex smooth curve’, *Forum Math. Sigma*. **6**(9) (2018), doi:[10.1017/fms.2018.7](https://doi.org/10.1017/fms.2018.7).
- [2] A. Akopyan and R. Karasev, ‘Inscribing a regular octahedron into polytopes’, *Discrete Math.* **313**(1) (2013), 122–128.
- [3] J. Aslam, S. Chen, F. Frick, S. Saloff-Coste, L. Setiabrata and H. Thomas, ‘Splitting loops and necklaces: variants of the square peg problem’, *Forum Math. Sigma* **8** (2020), e5.
- [4] S. Axelrod and I. M. Singer, ‘Chern-Simons perturbation theory. II’, *J. Differential Geom.* **39**(1) (1994), 173–213.
- [5] M. Berger, *Geometry. I*, Universitext (Springer-Verlag, Berlin, 1987). Translated from the French by M. Cole and S. Levy.
- [6] P. V. M. Blagojević and G. M. Ziegler, ‘Tetrahedra on deformed spheres and integral group cohomology’, *Electron. J. Combin.* **16**(2) (2009). Special volume in honor of Anders Björner. [arXiv:0808.3841](https://arxiv.org/abs/0808.3841).
- [7] L. M. Blumenthal and B. E. Gilliam, ‘Distribution of points in n -space’, *Amer. Math. Monthly*. **50**(3) (1943), 181–185.
- [8] R. Budney, J. Conant, K. P. Scannell and D.P. Sinha, ‘New perspectives on self-linking’, *Adv. Math.* **191**(1) (2005), 78–113.
- [9] J. Cantarella, E. Denne and J. McCleary, ‘Square-like quadrilaterals inscribed in embedded space curves’, Preprint, 2021, [arXiv:2103.13848](https://arxiv.org/abs/2103.13848).
- [10] J. Cantarella, E. Denne and J. McCleary, ‘Configuration spaces, multijet transversality, and the square-peg problem’, *Illinois. J. Math.* **66**(3) (2022), 385–420. doi:[10.1215/00192082-10120454](https://doi.org/10.1215/00192082-10120454).
- [11] H. S. M. Coxeter, ‘The circumradius of the general simplex’, *Math. Gaz.* **15**(210) (1930), 229–231.
- [12] L. Dieci and T. Eirola, ‘On smooth decompositions of matrices’, *SIAM J. Matrix Anal. Appl.* **20**(3) (1999), 800–819.
- [13] W. Fulton and R. MacPherson, ‘A compactification of configuration spaces’, *Ann. of Math. (2)*, **139**(1) (1994), 183–225.
- [14] M. Golubitsky and V. Guillemin, ‘Stable mappings and their singularities’. In *Graduate Texts in Mathematics*, vol. 14 (Springer-Verlag, New York, 1973), 72–90.
- [15] J. E. Greene and A. Lobb, ‘The rectangular peg problem’, *Ann. of Math. (2)* **194**(2) (2021), 509–517.
- [16] M. L. Gromov, ‘Simplexes inscribed on a hypersurface’, *Mat. Zametki* **5** (1969), 81–89.

- [17] V. Guillemin and A. Pollack, *Differential Topology*. (AMS Chelsea Publishing, Providence, RI, 2010). Reprint of the 1974 original.
- [18] A. Gupta and S. Rubinstein-Salzedo, 'Inscribed triangles of Jordan curves in \mathbb{R}^n ', Preprint, 2021, [arXiv:2102.03953](https://arxiv.org/abs/2102.03953).
- [19] A. Haeffliger, 'Differentiable imbeddings', *Bull. Amer. Math. Soc.* **67**(1) (1961), 109–112.
- [20] M. W. Hirsch, 'Differential topology'. In *Graduate Texts in Mathematics*, vol. 33 (Springer-Verlag, New York, 1994). Corrected reprint of the 1976 original.
- [21] R. A. Horn and C. R. Johnson, *Matrix Analysis*, second edn (Cambridge University Press, Cambridge, 2013).
- [22] C. Hugelmeyer, 'Inscribed rectangles in a smooth Jordan curve attain at least one third of all aspect ratios', *Ann. of Math. (2)* **194**(2) (2021), 497–508.
- [23] C. Hugelmeyer, 'Every smooth Jordan curve has an inscribed rectangle with aspect ratio equal to $\sqrt{3}$ ', Preprint, 2018, [arXiv:1803.07417](https://arxiv.org/abs/1803.07417).
- [24] S. Kakutani, 'A proof that there exists a circumscribing cube around any bounded closed convex set in R^3 ', *Ann. of Math. (2)* **43** (1942), 739–741.
- [25] R. N. Karasev, 'Inscribing a regular crosspolytope', Preprint, 2009, [arXiv:0905.2671](https://arxiv.org/abs/0905.2671).
- [26] V. Klee and S. Wagon, 'Old and new unsolved problems in plane geometry and number theory', in *The Dolciani Mathematical Expositions*, vol. 11 (Mathematical Association of America, Washington, D.C., 1991).
- [27] G. Kuperberg, 'Circumscribing constant-width bodies with polytopes', *New York J. Math.* **5** (1999), 91–100.
- [28] V. V. Makeev, 'On quadrangles inscribed in a closed curve and the vertices of the curve', *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, **299**(8) (2003), 241–251, 331. MR2038265 (2005a:51021).
- [29] V. V. Makeev, 'Inscribed and circumscribed polyhedra for a convex body and a problem on continuous functions on a sphere in Euclidean space', *Algebra i Analiz*, **18**(6) (2006), 187–204.
- [30] V.V. Makeev, 'Universally inscribed and outscribed polytopes', PhD thesis, St. Petersburg State University, 2003.
- [31] B. Matschke, 'Equivariant topology methods in discrete geometry', PhD thesis, Freie Universität, 2011.
- [32] B. Matschke, 'A survey on the square peg problem', *Notices Amer. Math. Soc.* **61**(4) (2014), 346–352.
- [33] B. Matschke, 'Quadrilaterals inscribed in convex curves', *Trans. Amer. Math. Soc.* **374**(8) (2021), 5719–5738.
- [34] M. D. Meyerson, 'Equilateral triangles and continuous curves' *Fund. Math.* **110** (1980), 1–9.
- [35] M. J. Nielsen, 'Triangles inscribed in simple closed curves', *Geom. Dedicata* **43**(3) (1992), 291–297.
- [36] I. Pak, 'Lectures on discrete and polyhedral geometry', (2010), Free online text.
- [37] H. Samelson, 'Note on vector fields in manifolds', *Proc. Amer. Math. Soc.* **36** (1972), 272–274.
- [38] I. J. Schoenberg, 'Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe d'espace distanciés vectoriellement applicable sur l'espace de Hilbert"', *Ann. of Math. (2)* **36**(3) (1935), 724–732.
- [39] R. E. Schwartz, 'A trichotomy for rectangles inscribed in Jordan loops', *Geom. Dedicata* **208** (2020), 177–196.
- [40] D. P. Sinha, 'Manifold-theoretic compactifications of configuration spaces', *Selecta Math. (N.S.)*, **10**(3) (2004), 391–428.
- [41] M. J. Sippl and H. A. Scheraga, 'Cayley-Menger coordinates', *Proc. Nat. Acad. Sci. U.S.A.* **83**(8) (1986), 2283–2287.
- [42] T. Tao, 'An integration approach to the Toeplitz square peg problem', *Forum Math. Sigma*, **5**(e30) (2017), 63.
- [43] R. Thom, 'Quelques propriétés globales des variétés différentiables', *Comment. Math. Helv.* **28** (1954), 17–86.
- [44] O. Toeplitz, 'Ueber einige aufgaben der analysis situs', *Verhandlugen Der Schweizerischen Naturafoschenden Gesellschaft in Solothurn*, **4** (1922), 197.
- [45] S. T. Vrećica and R. T. Živaljević, 'Fulton-MacPherson compactification, cyclohedra, and the polygonal pegs problem', *Israel J. Math.* **184** (2011), 221–249.