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ON THE MONOTONE NATURE OF BOUNDARY VALUE FUNCTIONS FOR nth-ORDER DIFFERENTIAL EQUATIONS

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1. **Introduction.** We are concerned with the nth ($n \geq 3$) order linear differential equation

$$y^{(n)}+\sum_{k=0}^{n-1} p_{n-k-1}(x)y^{(k)} = 0 \tag{1}$$

where the coefficients are continuous on $(-\infty, \infty)$. Our main result is to give conditions under which the two-point boundary value function $r_{ij}(t)$ (see Definition 2) are strictly increasing continuously differentiable functions of t. Levin [1] states without proof a similar theorem concerning just the monotone nature of the $r_{ij}(t)$ but assumes that the coefficients in (1) satisfy the standard differentiability conditions when one works with the formal adjoint of (1). Bogar [2] looks at the same problem for an nth-order quasi-differential equation where he makes no assumption concerning the differentiability of the coefficients in the quasi differential equation that he considers. Bogar gives conditions under which the $r_{ij}(t)$ are strictly increasing and continuous. The different approach of the author to this problem also enables the author to establish the continuous differentiability of the $r_{ij}(t)$ and to express the derivatives $r'_{ij}(t)$ in terms of the principal solutions $u_j(x, t)$, $j=0, 1, \ldots, n-1$ (see Definition 4).

2. **Definitions and main result.** Before we define the two-point boundary value functions $r_{ij}(t)$, we give the following definition.

DEFINITION 1. A solution y of (1) is said to have an (i, j)-pair of zeros, $1 \leq i$, $j \leq n$, on $[t, b]$ provided there are numbers α, β such that $t \leq \alpha < \beta \leq b$ and y has a zero of order at least i at α and at least j at β.

DEFINITION 2. Let $R=\{r>t$: there is a nontrivial solution of (1) having an (i, j)-pair, $1 \leq i$, $j \leq n$, $i+j=n$, of zeros on $[t, r]\}$. If $R \neq \phi$, set $r_{ij}(t)=\inf R$. If $R=\phi$, set $r_{ij}(t)=\infty$.

REMARK 1. If $R \neq \phi$, then $r_{ij}(t)=\min R$.

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REMARK 2. If $t \leq \alpha < \beta < r_{ij}(t) \leq \infty$, then there is a unique solution of (1) satisfying

$$y^{(p)}(\alpha) = A_p, \qquad y^{(q)}(\beta) = B_q$$

$p=0,\ldots,i-1$, $q=0,\ldots,j-1$, where the A_p and B_q are constants.

For the convenience of the statement of Theorem 1 we define $r_{n0}(t)=r_{0n}(t)=\infty$. In light of the above remark one could think of $r_{0n}(t)=r_{n0}(t)=\infty$ just meaning that all initial value problems of (1) have unique solutions.

In the following definition we use notation introduced by Dolan [3], and used by Barrett [4] and the author [5].

DEFINITION 3. Let $Z=\{z>t$: there is a nontrivial solution of (1) having a zero of order at least i at t and a zero of order at least j at z, $1 \leq i, j \leq n$, $i+j=n\}$. If $Z \neq \phi$, set $z_{ij}(t)=\inf Z$. If $Z=\phi$, set $z_{ij}(t)=\infty$.

REMARK 3. If $Z \neq \phi$, then $z_{ij}(t)=\min Z$.

DEFINITION 4. A fundamental set $\{u_j(x,t): j=0,1,\ldots,n-1\}$ of solutions of (1) is defined by the initial conditions at $x=t$,

$$u_j^{(n-i-1)}(t,t) = \delta_{ij}, \quad i,j = 0,\ldots,n-1.$$

In the following lemma we use the notation

$$W[u_{i_0}(x,t),\ldots,u_{i_k}(x,t)] = \det\left(u_{i_p}^{(q)}(x,t)\right)$$

$q=0,\ldots,k;\quad p=0,\ldots,k.$

LEMMA 1. *If $0 \leq i_0 < i_1 < \cdots < i_k \leq n-1$, then in a right hand deleted neighborhood of $x=t$*

$$\operatorname{sgn} W[u_{i_0},\ldots,u_{i_k}] = (-1)^{k(k+1)/2}$$

Proof. We prove this theorem by mathematical induction. The case $k=0$ is trivial. By considering the Taylor's formula with remainder for $u_{i_0}(x,t),\ldots,u_{i_k}(x,t)$ at $x=t$ it is not difficult to see that

$$\operatorname{sgn} W[u_{i_0},\ldots,u_{i_k}] = \operatorname{sgn} W[(x-t)^{n-i_0-1},\ldots,(x-t)^{n-i_k-1}]$$

for $x>t$ but sufficiently close to t. It follows that it suffices to show that

$$\operatorname{sgn} W[x^{n-i_0-1},\ldots,x^{n-i_k-1}] = (-1)^{k(k+1)/2}$$

for $x>0$ but sufficiently small. But for $x>0$, $v_p(x)=x^{n-i_p-1}$, $p=0,\ldots,k$ are $k+1$ linearly independent solutions of an Euler equation of order $k+1$ and hence $W[x^{n-i_0-1},\ldots,x^{n-i_k-1}]$ is of one sign for $x>0$. Letting $x=1$ we see that it suffices to show that $\operatorname{sgn} f(n)=(-1)^{k(k+1)/2}$ where

$$f(n) = \begin{vmatrix} 1 & \cdots & 1 \\ n-i_0-1 & \cdots & n-i_k-1 \\ \vdots & & \vdots \\ (n-i_0-1)(n-i_0-2)\ldots(n-i_0-k) & \cdots & (n-i_k-1)\ldots(n-i_k-k) \end{vmatrix}$$

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Now replace n by the real variable τ, then by using elementary properties of determinants one can show that $f'(\tau)=0$. Therefore $f(\tau)$ is a constant. To find the sign of this constant let $\tau=a$, where $a=i_k+1$. By expanding along the last column of $f(a)$ we obtain

$$f(a) = (-1)^k \begin{vmatrix} a-i_0-1 & \cdots & a-i_{k-1}-1 \\ \vdots & & \vdots \\ (a-i_0-1)\ldots(a-i_0-k) & \cdots & (a-i_{k-1})\ldots(a-i_{k-1}-k) \end{vmatrix}$$

$$= (-1)^k A \begin{vmatrix} 1 & \cdots & 1 \\ b-i_0-1 & \cdots & b-i_{k-1}-1 \\ \vdots & & \vdots \\ [b-i_0-1]\ldots\ldots[b-i_0-(k-1)] & \cdots & [b-i_{k-1}-1]\ldots\ldots[b-i_{k-1}-(k-1)] \end{vmatrix}$$

where $A=\prod_{m=0}^{k-1}(a-i_m-1)>0$ and $b=a-1$. By arguments similar to those above the sign of this last determinant is the same as the sign of $W[u_{i_0},\ldots,u_{i_{k-1}}]$. Hence, by the induction hypothesis,

$$\operatorname{sgn} f(n) = \operatorname{sgn} f(a) = (-1)^k(-1)^{[(k-1)k]/2} = (-1)^{k(k+1)/2}$$

and the proof is complete.

The above lemma for the case $i_p=p$, $p=0,\ldots,k$, was stated without proof in [6]. The next lemma follows immediately from [7, Theorem V-3.1].

LEMMA 2.

$$\frac{\partial u_k^{(l)}(x,t)}{\partial t} = -u_{k+1}^{(l)}(x,t)+p_k(t)u_0^{(l)}(x,t)$$

$$\frac{\partial u_{n-1}^{(l)}(x,t)}{\partial t} = p_{n-1}(t)u_0^{(l)}(x,t)$$

$l=0,1,\ldots,n$; $k=0,\ldots,n-2$.

We now state our main result.

THEOREM 1. *For those values of t for which*

$$r_{n-k,k}(t) < \min[r_{n-k+1}(t), r_{n-k-1,k+1}(t)], \quad k=1,\ldots,n-1,$$

$r_{n-k,k}(t)$ *is a continuously differentiable strictly increasing function of t. In particular*

$$r'_{n-k,k}(t) = \frac{W[u_0,\ldots,u_{k-2},u_k]}{W'[u_0,\ldots,u_{k-1}]}(r_{n-k,k}(t),t).$$

Proof. Let $\omega(x, t) = W[u_0(x, t), \ldots, u_{k-1}(x, t)]$, $1 \le k \le n-1$. The reader can easily verify Theorem 1 for $k=1$ with slight modifications of the following proof for $2 \le k \le n-1$.

Let

$$D = \{t: z_{n-k,k}(t) < \min [r_{n-k+1,k-1}(t), r_{n-k-1,k+1}(t)]\}.$$

If $D = \phi$, there is nothing to prove. Assume $D \ne \phi$ and set $\beta(t) = z_{n-k,k}(t)$ for $t \in D$. Since $\omega(\beta(t), t) = 0$ is equivalent to the existence of a nontrivial solution having t and $\beta(t)$ as an $(n-k, k)$-pair of zeros we have that $\omega(\beta(t), t) = 0$ for all $t \in D$. Let a_j, $j=0, \ldots, k-1$, be constants, not all zero, such that

$$y_1(x) = \sum_{j=0}^{k-1} a_j u_j(x, t)$$

has a $(n-k, k)$-pair of zeros at t and $\beta(t)$. Assume that $(\partial/\partial x)\omega(\beta(t), t) = 0$, then there are constants b_j, $j=0, \ldots, k-1$, not all zero, such that

$$y_2(x) = \sum_{j=0}^{k-1} b_j u_j(x, t)$$

has a $(n-k, k-1)$-pair of zeros at t and $\beta(t)$, and $y_2^{(k)}(\beta(t)) = 0$. If $y_2^{(k-1)}(\beta(t)) = 0$ we contradict $\beta(t) < r_{n-k-1,k+1}(t)$. Therefore $y_1(x)$ and $y_2(x)$ are linearly independent. But then there is a nontrivial linear combination of $y_1(x)$ and $y_2(x)$ with a $(n-k+1, k-1)$-pair of zeros at t and $\beta(t)$ which contradicts $\beta(t) < r_{n-k+1,k-1}(t)$. Hence $\omega(\beta(t), t) = 0$ and $(\partial/\partial x)\omega(\beta(t), t) \ne 0$ for all t in the domain D of $\beta(t)$. The principal solutions $u_j(x, t)$, $j=0, \ldots, n-1$, depend continuously on t and hence $\omega(x, t)$ depends continuously on t. Since $\omega(x, t)$ has a simple zero at $\beta(t)$ it follows from the continuous dependence of $\omega(x, t)$ on t that β is a continuous function of t and its domain is of the form $(-\infty, a)$. For more details on these last two statements see [2]. By use of the implicit function theorem and Lemma 2 we get that $\beta(t)$ is continuously differentiable and, when we differentiate both sides of $\omega(\beta(t), t) = 0$ implicitly with respect to t, that

$$\sum_{j=1}^{k} A_j + \beta'(t) W'[u_0, \ldots, u_{k-1}](\beta(t), t) = 0 \tag{2}$$

where A_j, $j=1, \ldots, k$ is the determinant

$$\omega(\beta(t), t) = W[u_0, \ldots, u_{k-1}](\beta(t), t)$$

with its jth row replaced by the row vector

$$(-u_1^{(j-1)}(\beta(t), t) + p_0(t)u_0^{(j-1)}(\beta(t), t), \ldots, -u_k^{(j-1)}(\beta(t), t) + p_{k-1}(t)u_0^{(j-1)}(\beta(t), t)).$$

Note that

$$\sum_{j=1}^{k} A_j = \sum_{l=1}^{k} B_l \tag{3}$$

where

$$B_l = [-u_l(\beta(t), t)+p_{l-1}(t)u_0(\beta(t), t)]M_{1l}+\dots$$
$$+[-u_l^{(k-1)}(\beta(t), t)+p_{l-1}(t)u_0^{(k-1)}(\beta(t), t)]M_{kl}$$

where M_{pq}, $1 \le p, q \le k$, is the cofactor of the (p, q) element in the determinant A_p. Also

$$B_l = -[u_l(\beta(t), t)M_{1l}+\cdots+u^{(k-1)}(\beta(t), t)M_{kl}]$$
$$+p_{l-1}(t)[u_0(\beta(t), t)M_{1l}+\cdots+u_0^{(k-1)}(\beta(t), t)M_{kl}].$$

Now make the important observation that M_{pq} is also the cofactor of the (p, q) element in the determinant $W[u_0, \dots, u_{k-1}](\beta(t), t)$. Hence

$$B_l = -C_l+p_{l-1}(t)D_l, \quad 1 \le l \le k,$$

where C_l is the determinant $\omega(\beta(t), t)$ with its lth column replaced by the column vector

$$(u_l(\beta(t), t), \dots, u_l^{(k-1)}(\beta(t), t)),$$

and D_l is the determinant $\omega(\beta(t), t)$ with its lth column replaced by the column vector

$$(u_0(\beta(t), t), \dots, u_0^{(k-1)}(\beta(t), t)).$$

It is easy to see that

$$B_l = 0, \quad l = 0, \dots, k-1,$$

and

$$B_k = -W[u_0, \dots, u_{k-2}, u_k](\beta(t), t).$$

It follows from (2) and (3) that

$$z'_{n-k,k}(t) = \frac{W[u_0, \dots, u_{k-2}, u_k]}{W'[u_0, \dots, u_{k-1}]} \quad (z_{n-k,k}(t), t).$$

From Lemma 1 we have that $W[u_0, \dots, u_{k-2}, u_k]$ and $W[u_0, \dots, u_{k-1}]$ are of the same sign in a right-hand deleted neighborhood of t. Since

$$\beta(t) < \min [r_{n-k+1,k-1}(t), r_{n-k-1,k+1}(t)],$$

$$\left\{\frac{W[u_0, \dots, u_{k-2}, u_k]}{W[u_0, \dots, u_{k-1}]}\right\}' = \frac{W[u_0, \dots, u_{k-2}]W[u_0, \dots, u_k]}{W^2[u_0, \dots, u_{k-1}]} \neq 0 \quad \text{for } t < x < \beta(t).$$

[7, pp. 51–54].

It follows from Rolle's theorem that $W[u_0, \dots, u_{k-2}, u_k]$ has at most one zero in $(t, \beta(t))$. If both $W[u_0, \dots, u_{k-1}]$ and $W[u_0, \dots, u_{k-2}, u_k]$ are zero at $(\beta(t), t)$ one can show that this implies the existence of a nontrivial solution of (1) with either a $(n-k+1, k-1)$-pair or $(n-k, k+1)$-pair of zeros at t and $\beta(t)$, which is a contradiction. Hence,

$$\frac{W[u_0, \dots, u_{k-1}]}{W[u_0, \dots, u_{k-2}, u_k]} \quad (\beta(t), t) = 0.$$

By considering the Taylor's formula with remainder at $x=t$ for each of the elements of $W[u_0, \ldots, u_{k-2}, u_k]$ and $W[u_0, \ldots, u_{k-1}]$ it is easy to see that

$$\frac{W[u_0, \ldots, u_{k-1}]}{W[u_0, \ldots, u_{k-2}, u_k]}(t+0, t) = 0.$$

Assume $W[u_0, \ldots, u_{k-2}, u_k] \neq 0$ for $t < x < \beta(t)$, then by Rolle's Theorem

$$\left\{\frac{W[u_0, \ldots, u_{k-1}]}{W[u_0, \ldots, u_{k-2}, u_k]}\right\}' = -\frac{W[u_0, \ldots, u_{k-2}]W[u_0, \ldots, u_k]}{W^2[u_0, \ldots, u_{k-2}, u_k]}$$

has a zero in $(t, \beta(t))$, which is a contradiction. Hence $W[u_0, \ldots, u_{k-2}, u_k]$ has exactly one zero in $(t, \beta(t))$. It follows from Lemma 1 and the fact that $W[u_0, \ldots, u_{k-2}, u_k]$ has exactly one simple zero in $(t, \beta(t))$ that $W'[u_0, \ldots, u_{k-1}]$ and $W[u_0, \ldots, u_{k-2}, u_k]$ have the same sign at $(\beta(t), t)$. Hence $z'_{n-k,k}(t) > 0$. Therefore, for $t \in D$, $z_{n-k,k}(t)$ is a strictly increasing continuously differentiable function of t and consequently

$$r_{n-k,k}(t) = z_{n-k,k}(t), \quad t \in D.$$

Of course we now know that

$$D = \{t : r_{n-k,k}(t) < \min[r_{n-k+1,k-1}(t), r_{n-k-1,k+1}(t)]\}$$

and the proof is complete.

For numerous examples of differential equations satisfying the hypotheses of Theorem 1 see ([1], [2], [5]).

References

1. A. Ju. Levin, *Distribution of the zeros of solutions of a linear differential equation*, Soviet Math. Dokl. (1964), 818–821.

2. G. A. Bogar, *Properties of two-point boundary value functions*, Proc. Amer. Math. Soc. **23** (1969), 335–339.

3. J. M. Dolan, *Oscillating behavior of solutions of linear differential equations of third order*, Unpublished doctoral dissertation, Univ. of Tennessee, Knoxville, Tenn., 1967.

4. J. H. Barrett, *Oscillatory theory of ordinary linear differential equations*, Advances in Math. **3** (1969), 415–509.

5. A. C. Peterson, *Distribution of zeros of solutions of a fourth order differential equation*, Pac. J. Math. **30** (1969), 751–764.

6. G. A. Jutkin, *On the question of the distribution of zeros of solutions of linear differential equations*, (Russian) Diff. Equations, **15** (1969), 1821–1829.

7. P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.

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