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DIFFERENTIAL COMPLETIONS AND DIFFERENTIALLY SIMPLE ALGEBRAS

BY

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ABSTRACT. Differentially simple local noetherian $\mathcal{Q}$-algebras are shown to be always (a certain type of) subrings of formal power series rings. The result is established as an illustration of a general theory of differential filtrations and differential completions.

Introduction. The present paper takes up a theme which appears first in a paper of R. Hart: Are differentially simple local noetherian $\mathcal{Q}$-algebras always subrings of formal power series rings; and what sort of subrings do thus occur? The answer to the first question is affirmative, and a first-step characterization of the relevant type of subrings is given. As a natural way towards the result we choose the approach via differential filtrations and differential completions, which we first discuss in full (that is characteristic-free) generality.

1. **Differential filtrations and differential completions.** Recall first the basic facts about differential filtrations (cf. [3]). Let R be an arbitrary unital commutative ring, and fix a set $\mathbf{D}$ of derivations on R. $(R, \mathbf{D})$, or simply R, is called a differential ring. Every localization $S^{-1}R$ of R will be tacitly considered as a differential ring, namely $(S^{-1}R, S^{-1}\mathbf{D})$, where $S^{-1}\mathbf{D}$ is the set of extensions of elements of $\mathbf{D}$ to $S^{-1}R$. We shall write (R, d) for $(R, \{d\})$. For an ideal I of R define $D(I) = \{f \in I : df \in I \text{ for all } d \in \mathbf{D}\}$. Then $D(I)$ is an ideal of R such that, for every $n \geqq 1, I^{n+1} \subseteq D(I^n) \subseteq I^n$. Furthermore, the operation D commutes with arbitrary intersections of ideals. Note that we can reduce certain considerations to the case of one single derivation: Let $\mathbf{D} = \cup \mathbf{D}_\nu$ and set $\mathbf{D}_\nu(I) = \{f \in I : df \in I \text{ for all } d \in \mathbf{D}_\nu\}$. Then $D(I) = \cap \mathbf{D}_\nu(I)$. For $f \in R, \mathbf{D}$ as above, and $k \geqq 1$ we set $\mathbf{D}^k f = \{(d_1 \circ \cdots \circ d_k)f : d_i \in \mathbf{D}, 1 \leqq i \leqq k\}$. We define $D^0 I = I, D^n I = D(D^{n-1}I), n \geqq 1$. Then $D^n I = \{f \in I : \mathbf{D}^k f \subseteq I \text{ for } 1 \leqq k \leqq n\}$, as is easily seen by induction on n.

DEFINITION 1.1. *Let $(R, \mathbf{D})$ be a differential ring, I an ideal of R. Define* $I_{(0)} = R, I_{(n)} = D^{n-1}I, n \geqq 1$.

PROPOSITION 1.2. *$(I_{(n)})_{n \geqq 0}$ is a multiplicative filtration of R. More precisely, we have $I_{(n)} I_{(m)} \subseteq I_{(n+m)}$ for all $n, m \geqq 0$.*

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PROOF. First observe that for $f, g \in R$, and derivations $d_1, \ldots, d_r$ *of* $R, r \geqq 2$, the following formula holds: $(*) (d_1 \circ \ldots \circ d_r)(fg) = f(d_1 \circ \ldots \circ d_r)(g) + (d_1 \circ \ldots \circ d_r)(f)g$

$$+\sum_{k=1}^{r-1} \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_{r-k}}} (d_{i_1} \circ \ldots \circ d_{i_k})(f)(d_{j_1} \circ \ldots \circ d_{j_{r-k}})(g)$$

(where the j-indexing is complementary to the i-indexing). Let now $n \geqq 0$ be fixed. We have to show, by induction on $m \geqq 0$, that $D^n(I)D^m(I) \subseteq D^{n+m+1}(I)$. Look first at $m = 0$: Choose $f \in D^n(I), g \in D^0(I) = I$. We have to show that $fg \in D^{n+1}(I)$, that is that $fg \in I, \mathbf{D}(fg) \subseteq I, \ldots, \mathbf{D}^{n+1}(fg) \subseteq I$. First, since $f, g \in I$, we get $\mathbf{D}(fg) \subseteq I$, by the derivation property, and in the case when $n = 0$ the proof is complete. Let us pick up now $d_1, \ldots, d_r, 2 \leqq r \leqq n+1$. Then our formula (*) shows that $(d_1 \circ \ldots \circ d_r)(fg) \in I$, by hypothesis on f and g. This gives finally what we want: $fg \in D^{n+1}(I)$. As to the inductive step, suppose that $D^n(I)D^m(I) \subseteq D^{n+m+1}(I)$. We have to make sure that $D^n(I)D^{m+1}(I) \subseteq D^{n+m+2}(I)$. Take $f \in D^n(I), g \in D^{m+1}(I)$. By the inductive hypothesis we get immediately $fg \in D^{n+m+1}(I)$. We need only show that $\mathbf{D}^{n+m+2}(fg) \subseteq I$. Look once more at (*), with $d_1, \ldots, d_{n+m+2} \in \mathbf{D}$, that is with $r = n + m + 2$. For $k \leqq n$ we have $(d_{i_1} \circ \ldots \circ d_{i_k})(f) \in I$, and for $k > n$ we have $n + m + 2 - k \leqq m + 1$, that is $(d_{j_1} \circ \ldots \circ d_{j_{r-k}})(g) \in I$, which shows finally our claim.

Define $\Delta(I) = \cap_{n \geqq 1} D^n(I)$. Then $\Delta\ (I)$ is obviously the greatest $\mathbf{D}$-stable ideal contained in I, and the operation Δ commutes with arbitrary intersections of ideals. The most interesting elementary observation (see [3]) is that for a primary ideal Q of $R, D(Q)$ is also primary. Hence, for a prime ideal P of R, the filtration $(P_{(n)})_{n \geqq 0}$ consists of P-primary ideals (for $n \geqq 1$).

REMARK 1.3. Let P be a prime ideal of R. Then for all $n \geqq 1$ we have $P^{(n)} \subseteq P_{(n)}$.

PROOF. It is easily seen that for every localization $R \longrightarrow S^{-1}R$ we have $D(S^{-1}I) = S^{-1}DS(I)$ (where $S(I)$ means S-saturation). In particular, if Q is primary, we get $D(S^{-1}Q) = S^{-1}D(Q)$. An easy induction shows that if $Q \cap S = \phi$, we obtain (with $\varphi : R \longrightarrow S^{-1}R$ the localizing homomorphism) $D^nQ = \varphi^{-1}D^nS^{-1}Q$ for all $n \geqq 0$. Now take $S = R \backslash P, \varphi : R \longrightarrow R_p$, and put $M = S^{-1}P = PR_p$. Since $M^n \subseteq D^{n-1}M$ for all $n \geqq 1$, we get $P^{(n)} = \varphi^{-1}M^n \subseteq \varphi^{-1}D^{n-1}M = D^{n-1}P = P_{(n)}$, as claimed. □

For a prime ideal P of R, and any localization $R \longrightarrow S^{-1}R$ such that $P \cap S = \phi$, inspection of the proof 1.3 shows that the $P_{(n)}-$ filtration on R is the trace of the $(S^{-1}P)_{(n)}-$ filtration on $S^{-1}R$. Furthermore, $P_{(n)} = P^{(n)}$ if and only if $(S^{-1}P)_{(n)} = (S^{-1}P)^{(n)}$. As another complement, we see that for a primary ideal Q of R and for every localization $\varphi : R \longrightarrow S^{-1}R$ such that $Q \cap S = \emptyset$, we have $\Delta(Q) = \varphi^{-1}\Delta(S^{-1}Q)$. Thus Q is $\mathbf{D}$-stable if and only if $S^{-1}Q$ is $S^{-1}\mathbf{D}$-stable.

DEFINITION 1.4. *Let* $(R, \mathbf{D})$ *be a differential ring, and let* $(I_n)_{n>0}$ *be a decreasing sequence of ideals of* R. *We call the corresponding filtration* $\mathbf{D}$*-good whenever all* $d \in \mathbf{D}$ *are (uniformly) continuous in the uniform structure defined by* $(I_n)_{n>0}$.

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EXAMPLES 1.5. (1) Let $I \subseteq R$ be a fixed ideal, and consider $(I_n)_{n>0} = (I^n)_{n>0}$, that is the I-adic filtration on R. Since every derivation d of R satisfies $d(I^{n+1}) \subseteq I^n, n \geqq 0$, an I-adic filtration on R is $\mathbf{D}$-good for any set $\mathbf{D}$ of derivations on R.

(2) Let $I \subseteq R$ be a fixed ideal, as before, $\mathbf{D}$ a set of derivations on R. Let $(I^n)_{n>0} = (I_{(n)})_{n>0}$ be the differential filtration associated with $\mathbf{D}$ (and I); we shall call such a filtration a $\mathbf{D}$-adic filtration. Then $(I_{(n)})_{n>0}$ is $\mathbf{D}^\circ$-good for every $\mathbf{D}^\circ \subseteq \mathbf{D}$. We have only to observe that for $d \in \mathbf{D}$ we have $dI_{(n+1)} \subseteq I_{(n)}, n \geqq 0$. In order to see this, take $f \in I_{(n+1)}$; since $\mathbf{D}f \subseteq I, \mathbf{D}^2 f \subseteq I, \ldots, \mathbf{D}^n f \subseteq I$, we get in particular $df \in I, \mathbf{D}\, df \subseteq I, \ldots, \mathbf{D}^{n-1} df \subseteq I$, which means precisely that $df \in I_{(n)}$.

REMARK 1.6. Let $(I_n)_{n>0}$ be a $\mathbf{D}$-good filtration on R. $I_\infty = \cap_{n>0} I_n$ *is* $\mathbf{D}$-stable. Thus, in the given situation, we may pass to $R^1 = R/I_\infty$, with the differential structure defined by the set of induced derivations $\mathbf{D}^1$, say. We shall henceforth assume that all our filtrations are separated (that is $\cap_{n>0} I_n = 0$).

PROPOSITION 1.7. *Let $(I_n)_{n>0}$ be a $\mathbf{D}$-good separated filtration on R, and let R^* be the completion of R relative to this filtration. (1) Every $d \in \mathbf{D}$ has a unique prolongation d^* on R^* which is a derivation of R^*. Let $\mathbf{D}^*$ be the set of these prolongations. (2) If $\mathbf{D}$ is finite, or if the topology on R is such that for every open ideal I of R, I^2 is also open, then the extension $(R, \mathbf{D}) \longrightarrow (R^*, \mathbf{D}^*)$ of differential rings has the following property: For every open ideal I of R we have $D^*(I^*) = (D(I))^*$. $(( )^*$ means closure in R^*, D^* has the obvious meaning relative to $\mathbf{D}^*$).*

PROOF. (1) is immediate by the elementary properties of completions of rings. (2): Recall that the set of open ideals I of R and the set of open ideals J of R^* are in bijection via $J \longrightarrow I = J \cap R$ and $I \longrightarrow J = I^*$ (closure in R^*). Let I be an open ideal of R. Then, by our assumptions, $D(I), I^*$ and $D^*(I^*)$ must also be open, since $I^2 \subseteq D(I) = I \cap_{d \in D} d^{-1} I \subseteq I$, and $I^2 \subseteq I^{*2} \subseteq D^*(I^*) = I^* \cap_{d \in D} d^{*-1} I^* \cap I^*$. Note that $D^*(I^*)$ is closed, and thus contains $(I^2)^*$; if I^2 is open, $(I^2)^*$ is also open. We need only show that $D^*(I^*) \cap R = D(I)$. But this follows from the definitions.

COROLLARY 1.8. *Under the conditions above, we have for every open ideal I of R, and all $n \geqq 0, (I^*)_{(n)} = (I_{(n)})^*$, and thus $I_{(n)} = (I^*)_{(n)} \cap R$.*

PROPOSITION 1.9. *Let R be a noetherian ring, m an ideal of R such that R is a Zariski ring relative to its m-adic topology, and let $\hat{R}$ be its m-adic completion. If $\mathbf{D}$ is a finite set of derivations on R, then for every ideal I of R we have $(D(I))\hat{} = \hat{D}(\hat{I})$, and thus $\hat{I}_{(n)} = (I_{(n)})\hat{}$ for all $n \geqq 0$.*

PROOF. Note that now closure equals extension, that is we may write $\hat{I} = I\hat{R}$ for every ideal I of R. Let us first consider the case of one single derivation, that is $\mathbf{D} = \{d\}$. Let $E(R,R)$ be the idealization of R, that is $E(R,R) = R \oplus R$, with multiplication: $(x,x')(y,y') = (xy, xy' + x'y)$. Let $\delta : R \longrightarrow E(R,R)$ be the ring homomorphism given by $\delta(x) = (x, dx), x \in R$. Look first at $E(R,R)$, considered as an $R =$ module via

δ. We have $r.(x,y) = \delta(r)(x,y) = (r,dr)(x,y) = (rx, ry + dr.x)$. Note that $E(R,R)$ is generated by $(1,0)$ and $(0,1)$, also for its δ-structure: $(x,y) = x.(1,0) + (y - dx).(0,1), x, y \in R$. Consider now the $(m \oplus R)$-adic filtration on $E(R,R)$, which is given by the decreasing sequence of ideals $(E(m^n, m^{n-1}))_{n>0}$. We obtain the uniform structure of the direct m-adic sum, and for the δ-structure we get $m^k.E(m^n, m^{n-1}) \subseteq E(m^{n+k}, m^{n+k-1}), k, n \geqq 1$.

Now, $\delta : R \longrightarrow E(R,R)$ is a homomorphism of filtered rings, which prolongs to the completions. More precisely, $\hat{\delta} : \hat{R} \longrightarrow E(R,R)\hat{} = E(\hat{R},\hat{R})$ is given by $\hat{\delta}(\xi) = (\xi, \hat{d}\xi)$, where $\hat{d}$ is the prolongation of d to $\hat{R}$.

For every ideal I of $R, E(I,I)$ is an ideal of $E(R,R)$, hence an R-submodule for the δ-structure. We have $\hat{R}.E(I,I) = E(\hat{I},\hat{I})$, since $\xi.(x,y) = (\xi x, \xi y + \hat{d}\xi.x)$ for $\xi \in \hat{R}$ and $x, y \in I$, which gives, by [5, p. 266, Cor. 3], $(D(I))\hat{} = \hat{R}(I \cap d^{-1}I) = \hat{R}\delta^{-1}E(I,I) = \hat{\delta}^{-1}E(\hat{I},\hat{I}) = \hat{I} \cap \hat{d}^{-1}\hat{I} = \hat{D}(\hat{I})$. Now, by [5, p. 266, Cor. 2], we have for $\mathbf{D} = \{d_1, \ldots, d_r\}$ the following equalities: $\hat{D}(\hat{I}) = \cap_{1\leqq i\leqq r}\hat{D}_i(\hat{I}) = \cap_{1\leqq i\leqq r}(D_i(I))\hat{} = (\cap_{1\leqq i\leqq r}D_i(I))\hat{} = (D(I))\hat{}$.

This completes the proof. □

We now look more closely at the relation between I-adic and $\mathbf{D}$-adic completion. Let $(R, \mathbf{D})$ be a differential ring, I an ideal of $R, \hat{R}$ the I-adic completion of R, and R^* the $\mathbf{D}$-adic completion relative to the filtration $(I_{(n)})_{n>0}$, where $I_{(n+1)} = \{ f \in I : \mathbf{D}f \subseteq I, \ldots, \mathbf{D}^n f \subseteq I\}, n \geqq 1$. We suppose that $\cap_{n>0} I_{(n)} = 0$, hence a fortiori that $\cap_{n>0} I^n = 0$. We write $\hat{\mathbf{D}}$ for the set of prolongations of the elements of $\mathbf{D}$ to $\hat{R}$, and $\mathbf{D}^*$ for the corresponding set of prolongations on R^*.

Theorem 1.10. *In the above situation we have a surjective ring homomorphism* $\varphi : \hat{R} \longrightarrow R^*$, *which prolongs the identity on* R. *(1) Let* I^* *be the closure of* I *in* R^*; *then the* $\mathbf{D}^*$*-filtration associated with* I^* *is separated. (2) Let* $\hat{I}$ *be the closure of* I *in* $\hat{R}$, *and let* $(I_{(n)})_{n>0}$ *be the* $\hat{\mathbf{D}}$*-filtration associated with* $\hat{I}$ *in* $\hat{R}$. *Then* $\varphi^{-1}I^*_{(n)} = \hat{I}_{(n)}$ *for all* $n \geqq 0$. *Thus Ker* φ *equals* $\hat{\Delta}(\hat{I})$, *the biggest* $\hat{\mathbf{D}}$*-invariant ideal of* $\hat{R}$ *contained in* $\hat{I}$. *(3)* $\mathbf{D}^*$ *is the set of derivations induced by* $\hat{\mathbf{D}}$ *on* $R^* = \hat{R}/\hat{\Delta}(\hat{I})$. *(4)* R^* *is* I*-adically complete; hence, if* R *is noetherian,* R^* *is also* I^**-adically complete.*

Proof. First, it is easy to see that $I^n \subseteq I_{(n)}$ for all $n \geqq 0$. Hence the I-adic structure on R is finer than the $\mathbf{D}$-adic structure (relative to I). Thus we obtain a prolongation of the identity on $R, \varphi : \hat{R} \longrightarrow R^*$, say. R^* is separated, and $\varphi(\hat{R})$ is dense and complete in R^*, which gives the surjectivity of φ. (1) By definition of R^* we know that the filtration $((I_{(n)})^*)_{n>0}$ satisfies $\cap_{n>0}(I_{(n)})^* = 0$. We must verify that $(I_{(n)})^* = (I^*)_{(n)}$ for all $n \geqq 0$. Note that this is not a consequence of 1.8. First, the equality is trivial for $n = 0, 1$. Assume that $(I_{(n)})^* = (I^*)_{(n)}$. We have to show that $(I_{(n+1)})^* = (D(I_{(n)}))^* = (I^*)_{(n+1)}$. By the inductive hypothesis this amounts to showing that $(D(I_{(n)}))^* = D^*(I_{(n)})^*$. Comparing with the proof of 1.7, this equality is true provided all the ideals in question are open. Only for $D^*(I_{(n)})^*$ this is not trivial by definition. But $(D(I_{(n)}))^* \subseteq D^*(I_{(n)})^*$, which yields the result. (2) The equality

$(I_{(n)})\,\hat{} = \hat{I}_{(n)}, n \geqq 0$, follows from 1.8, since now we are dealing with an I-adic filtration. The continuity of φ gives immediately $\hat{I}_{(n)} \subseteq \varphi^{-1} I^*_{(n)}$ for all $n \geqq 0$. Now, these are open ideals in $\hat{R}$; we need only observe that $\varphi^{-1}(I^*_{(n)}) \cap R = \hat{I}_{(n)} \cap R = I_{(n)}$ for all $n \geqq 0$, which follows from the fact that φ prolongs the identity on R. (3) For every $d \in \mathbf{D}$ we have that d^*, the prolongation of d on R^*, and $\hat{d}'$, the derivation induced by $\hat{d}$ on R^*, coincide with d on R. This yields immediately the assertion. (4) R^* is I-adically complete, as a homomorphic image of $\hat{R}$. Suppose now R to be noetherian. Then the I-adic and the $\hat{I}$-adic structures on $\hat{R}$ coincide, and we have $(I^n)\,\hat{=}\,\hat{I}^n$ for all $n \geqq 0$. But $\varphi(\hat{I}^n) = (I^*)^n, n \geqq 0$, hence the I-adic and the I^*-adic structures on R^* are equal. This finishes the proof of our theorem. □

Example 1.11. The topological situation, as described by 1.10, is the following: Let R be noetherian. R^* is I^*-adically complete, but whenever $\hat{\Delta}(\hat{I}) \neq 0$, the induced I-adic topology on $R \subseteq R^*$ is not the given one (which, in this case, is strictly finer). R is I^*-adically dense in R^*. We should give an easy example in order to make the situation clear. Consider $R = k[x, y]_{(x,y)}$, the local ring of the affine k-plane at the origin, and assume char $k = 0$. Let $d = \partial/\partial x + (y-1)\partial/\partial y$ be the k-derivation of R which maps x onto 1, and keeps $(y-1)$ fixed. R does not contain any nontrivial d-invariant ideal (see [4, (2.10)]); this is equivalent to the fact that $\Delta(m) = 0$, where $m = (x, y)_{(x,y)}$ is the maximal ideal of R. Consider now $(\hat{R}, \hat{d})$, where $\hat{R} = k[[x, y]]$ is the formal power series ring in x and y over k. We have $\hat{\Delta}(\hat{m}) = \hat{R}f$, with $f = e^x - 1 + y$ (note that $\hat{d}f = f$, hence $\hat{R}f$ is $\hat{d}$-invariant; on the other hand, $\hat{R}f$ is a prime ideal of height one in $\hat{R}, \hat{m}$ is not $\hat{d}$-invariant, and $\hat{\Delta}(\hat{m})$ must be a prime ideal, since in characteristic zero all associated prime ideals of a differential ideal need also be differential). We get $R^* = k[[x, 1-e^x]] = k[[x]]$, with $d^* = \partial/\partial x$, the derivative relative to x. The embedding $R = k[x, y]_{(x,y)} \longrightarrow R^* = k[[x]]$ is given by substitution of $1 - e^x$ for y. We have $m^* = R^*x$, and $m^{*n+1} \cap R = (\hat{m}^{n+1} + \hat{R}f) \cap R = (y + x + \ldots + 1/n!x^n) + m^{n+1}$, which shows that the m-adic structure on R is strictly finer than the induced m^*-adic structure.

2. **Differentially simple local noetherian $\mathcal{Q}$-algebras.** In order to derive nontrivial consequences of our somehow too general (since characteristic-free) theory, we have to impose the standard $\mathcal{Q}$-algebra condition (we are not working with higher rank derivations), together with noetherian assumptions.

Lemma 2.1. *Let (S, m, K) be a regular local m-adically complete $\mathcal{Q}$-algebra (hence a formal power series ring in a finite number of variables over K), and let* $\mathbf{D}$ *be a set of $\mathcal{Q}$-derivations on S. Then S is* $\mathbf{D}$*-simple (that is $\Delta(m) = 0$) if and only if there is a $k \geqq 1$ with $m_{(k+1)} = D^k(m) \subseteq m^2$.*

Proof. Assume first S to be $\mathbf{D}$-simple; S is thus separated relative to the filtration $(m_{(n)})_{n>0}$. By a well-known theorem of Chevalley ([5, p. 270, theorem 13]) there is a function $\sigma : \mathcal{N} \longrightarrow \mathcal{N}, \lim \sigma(n) = \infty$, such that $m_{(\sigma(n))} \subseteq m^n$ for all $n \geqq 1$. In

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particular, there is a $k \geqq 1$ such that $m_{(k+1)} = D^k(m) \subseteq m^2$. Conversely, assume that $m_{(k+1)} = D^k(m) \subseteq m^2$ for some $k \geqq 1$. This condition means explicitly that for every regular parameter $t \in m \backslash m^2$ there are $d_1, \ldots, d_2 \in \mathbf{D}, j \leqq k$, such that $(d_1 \circ \ldots \circ d_j)(t) \notin m$. Consider now $P = \Delta(m) = \bigcap_{n>0} m_{(n)}$, the maximal $\mathbf{D}$-invariant (prime) ideal of S. We have to show that $P = 0$. Now, S is excellent, hence $S' = S/P$ is also excellent. But S' is $\mathbf{D}'$-simple (where $\mathbf{D}'$ is the set of derivations on S' induced by the elements of $\mathbf{D}$). Thus, by [1, Corollary to theorem 1], S' is regular. We get $P = (t_1, \ldots, t_i)$ for some regular system of parameters $(t_1, \ldots, t_r)$ of S. For $i \geqq 1$ the $\mathbf{D}$-invariance of P is in contradiction to the above explicit formulation of our assumption. Thus $P = 0$, and we have finished our proof.

DEFINITION 2.2. *Let (R, m) be a local ring, $\mathbf{D}$ a set of derivations on R. We call $\mathbf{D}$ exhaustive if and only if there is a $k \geqq 1$ such that for every $t \in m \backslash m^2$ there are $d_1, \ldots, d_j \in \mathbf{D}, 1 \leqq k$, with $(d_1 \circ \ldots \circ d_j)(t) \notin m$ (every $t \in m \backslash m^2$ can be made invertible by iterated application of appropriate elements of $\mathbf{D}$, in at most k steps).*

THEOREM 2.3. *A local noetherian $\mathbb{Q}$-algebra (R, m, K) is differentially simple (for some set of $\mathbb{Q}$-derivations on R) if and only if* (1) *R is a dense subalgebra of some power series ring $R^* = K[[T_1, \ldots, T_r]]$ (for its $(T_1, \ldots, T_r)$-adic topology).* (2) *There is an exhaustive set $\mathbf{D}^*$ of $\mathbb{Q}$-derivations on R^* which leaves R invariant (that is we have $\mathbf{D}^* R \subseteq R$).*

PROOF. One implication is an immediate consequence of 1.10, since R^*, the $\mathbf{D}$-adic completion of R, is an excellent local $\mathbf{D}^*$-simple $\mathbb{Q}$- algebra, hence regular (by corollary to theorem 1 in [1]). The other implication follows from 2.1.

COMPLEMENT 2.4. There is a natural question arising in the context of 2.3: Let (R, m, K) be a noetherian local $\mathbb{Q}$-algebra which is $\mathbf{D}$-simple for some set $\mathbf{D}$ of $\mathbb{Q}$-derivations on R. Is the following assertion true: R is regular if and only if R is excellent? One implication is a well-known result of R. Hart, the other implication would be in the spirit of a theorem of Mizutani (see [2, Theorem 10]).

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