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Criteria for Simultaneous Solutions of $X^2 - DY^2 = c$ and $x^2 - Dy^2 = -c$

R. A. Mollin

Abstract. The purpose of this article is to provide criteria for the simultaneous solvability of the Diophantine equations $X^2 - DY^2 = c$ and $x^2 - Dy^2 = -c$ when $c \in \mathbb{Z}$, and $D \in \mathbb{N}$ is not a perfect square. This continues work in [6]–[8].

1 Introduction

The norm form equations in the title have long borne the designation *Pell's equations* due to Euler's misapprehension that John Pell (1611–1685) had developed the method (for $c = 1$). This confusion arose from the method of a solution given by John Wallis (1616–1703) in his book *Algebra*, which Euler misinterpreted as having been originally given by Pell. Most historians agree that the honour actually goes to Lord Brouncker (1620–1684), the first president of the Royal Society. However, as noted by E. E. Whitford [9]: "to attempt to rename it would be like trying to give another name to North America because Vespucius was not its discoverer."

Instances of the Pell equation can be traced back to Archimedes in his book *Liber Assumptorum* or *Book of the Lemmas*, where we find the *Cattle Problem*, which involves the equation $x^2 - 4729494y^2 = 1$. Also, Brahmagupta, considered to be the greatest of the Hindu mathematicians, is also credited with first studying the equation $x^2 - py^2 = 1$ for a prime p. He wrote his masterpiece (ca. 628 A.D.) on astronomy *Brahma-sphuta-siddhanta* or *The revised system of Brahma*, which had two chapters devoted to mathematics.

Lagrange used continued fractions to give direct techniques for solving the Diophantine equation $x^2 - Dy^2 = c$. It is in this vein that we are interested in this paper for determining simultaneous solutions to the Diophantine equations in the title. For a detailed history surrounding the developments of research into the Pell equation, the reader may consult Dickson [1].

2 Notation and Preliminaries

We will be studying solutions of quadratic Diophantine equations of the general shape

$$x^2 - Dy^2 = c, \tag{2.1}$$

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where D is not a perfect square and $c \in \mathbb{Z}$. If $x, y \in \mathbb{Z}$ is a solution of (2.1), then it is called *positive* if $x, y \in \mathbb{N}$ and it is called *primitive* if $\gcd(x, y) = 1$. Among the primitive solutions of (2.1), if such a solutions exists, there is one in which both x and y have their least values. Such a solution is called the *fundamental solution*. We will use the notation

$$\alpha = x + y\sqrt{D}$$

to denote a solution of (2.1), and we let

$$N(\alpha) = x^2 - Dy^2$$

denote the *norm* of α. We will be linking such solutions to simple continued fraction expansions that we now define.

Recall that a *quadratic irrational* is a number of the form

$$(P + \sqrt{D})/Q$$

where $P, Q, D \in \mathbb{Z}$ with $D > 1$ not a perfect square, $P^2 \equiv D \pmod{Q}$, and $Q \neq 0$. Now we set:

$$P_0 = P, \quad Q_0 = Q, \quad \text{and recursively for } j \geq 0,$$

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{2.2}$$

$$P_{j+1} = q_j Q_j - P_j, \tag{2.3}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{2.4}$$

Hence, we have the simple continued fraction expansion:

$$\alpha = \frac{P + \sqrt{D}}{Q} = \frac{P_0 + \sqrt{D}}{Q_0} = \langle q_0\,; q_1, \dots, q_j, \dots \rangle,$$

where the q_j for $j \geq 0$ are called the *partial quotients* of α.

To further develop the link with continued fractions, we first note that it is well-known that a real number has a periodic continued fraction expansion if and only if it is a quadratic irrational (see [4, Theorem 5.3.1, p. 240]). Furthermore a quadratic irrational *may* have a *purely* periodic continued fraction expansion which we denote by

$$\alpha = \langle \overline{q_0\,; q_1, q_2, \dots, q_{\ell-1}} \rangle$$

meaning that $q_n = q_{n+\ell}$ or all $n \geq 0$, where $\ell = \ell(\alpha)$ is the period length of the simple continued fraction expansion. It is known that a quadratic irrational α *has* such a purely periodic expansion if and only if $\alpha > 1$ and $-1 < \alpha' < 0$. Any quadratic irrational which satisfies these two conditions is called *reduced* (see [4, Theorem 5.3.2, p. 241]). If α *is* a reduced quadratic irrational, then

$$(2.5)\qquad 0 < Q_j < 2\sqrt{D}, \quad 0 < P_j < \sqrt{D}, \quad \text{and} \quad q_j \le \lfloor \sqrt{D} \rfloor.$$

Finally, we need an important result which links the solutions of quadratic Diophantine equations with the Q_j defined above. We first need the following notation.

Let $D_0 > 1$ be a square-free positive integer and set:

$$\sigma_0 = \begin{cases} 2 & \text{if } D_0 \equiv 1 \pmod 4, \\ 1 & \text{otherwise.} \end{cases}$$

Define:

$$\omega_0 = (\sigma_0 - 1 + \sqrt{D_0})/\sigma_0, \quad \text{and} \quad \Delta_0 = (\omega_0 - \omega_0')^2 = 4D_0/\sigma_0^2,$$

where ω_0' is the *algebraic conjugate* of ω_0, namely

$$\omega_0' = (\sigma_0 - 1 - \sqrt{D_0})/\sigma_0.$$

The value Δ_0 is called a *fundamental discriminant* or *field discriminant* with associated *radicand* D_0, and ω_0 is called the *principal fundamental surd associated with* Δ_0. Let $\Delta = f_\Delta^2 \Delta_0$ for some $f_\Delta \in \mathbb{N}$. If we set

$$g = \gcd(f_\Delta, \sigma_0), \quad \sigma = \sigma_0/g, \quad D = (f_\Delta/g)^2 D_0, \quad \text{and} \quad \Delta = 4D/\sigma^2,$$

then Δ is called a *discriminant* with associated *radicand* D. Furthermore, if we let

$$\omega_\Delta = (\sigma - 1 + \sqrt{D})/\sigma = f_\Delta \omega_0 + h$$

for some $h \in \mathbb{Z}$, then ω_Δ is called the *principal surd* associated with the discriminant

$$\Delta = (\omega_\Delta - \omega_\Delta')^2.$$

This will provide the canonical basis element for certain rings that we now define.

Let $[\alpha, \beta] = \alpha\mathbb{Z} + \beta\mathbb{Z}$ be a $\mathbb{Z}$-module. Then $\mathcal{O}_\Delta = [1, \omega_\Delta]$, is an *order* in $K = \mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{D_0})$ with conductor f_Δ. If $f_\Delta = 1$, then $\mathcal{O}_\Delta$ is called the *maximal order in* K. The units of $\mathcal{O}_\Delta$ form a group which we denote by U_Δ. The positive units in U_Δ have a generator which is the smallest unit that exceeds 1. This selection is unique and is called the *fundamental unit of* K, denoted by ε_Δ. Moreover, we will have need of the following, which may be traced back to Lagrange.

Theorem 2.1 *Let $\alpha = (P + \sqrt{D})/Q$ be a quadratic irrational. If P_j and Q_j for $j = 1, 2, \ldots, \ell(\alpha) = \ell$ are defined by Equations (2.2)–(2.4) in the simple continued fraction expansion of α, then*

$$\varepsilon_\Delta = \prod_{i=1}^{\ell} (P_i + \sqrt{D})/Q_i$$

and

$$N(\varepsilon_\Delta) = (-1)^\ell.$$

Proof See [3, Theorems 2.1.3–2.1.4, pp. 51–53]. ∎

3 Results

In what follows, the symbol $p^t \parallel b$ means that the prime power p^t *properly divides* $b \in \mathbb{Z}$, namely $p^t \mid b$, but $p^{t+1} \nmid b$.

Theorem 3.1 *Let $c \in \mathbb{Z}$, $D \in \mathbb{N}$ where D is not a perfect square, and* $\gcd(c, D) = 1$. *If*

$$x^2 - Dy^2 = -c \tag{3.6}$$

has a primitive solution α, then

$$X^2 - DY^2 = c \tag{3.7}$$

has a primitive solution if and only if either

(a) $\ell(\sqrt{D})$ *is odd,*

or

(b) *Each of the following holds:*

- (i) *There exists a proper divisor $d \in \mathbb{N}$ of c, with* $\gcd(d, c/d) \mid 2$, *such that $x^2 - Dy^2 = -d^2$ has a (not necessarily primitive) solution β and $x^2 - Dy^2 = -c^2/d^2$ has a (not necessarily primitive) solution γ.*
- (ii) *$\alpha^2 = \beta\gamma$ and $\alpha\beta'/d$ a primitive element of $\mathbb{Z}[\sqrt{D}]$.*

Proof If Equations (3.6)–(3.7) have primitive solutions $\alpha_0 = x_0 - y_0\sqrt{D}$ and $\alpha_1 = x_1 + y_1\sqrt{D}$ respectively, then $N(\alpha_0/\alpha_1) = -1$, where,

$$\begin{aligned}\frac{\alpha_0}{\alpha_1} = \frac{x_0 + y_0\sqrt{D}}{x_1 + y_1\sqrt{D}} &= \frac{(x_0 + y_0\sqrt{D})(x_1 - y_1\sqrt{D})}{x_1^2 - y_1^2 D}\\ &= \frac{(x_0x_1 - y_0y_1D) + (y_0x_1 - x_0y_1)\sqrt{D}}{-c}.\end{aligned}$$

Thus,

$$(x_0x_1 - y_0y_1D)^2 - (y_0x_1 - x_0y_1)^2 D = -c^2. \tag{3.8}$$

Multiplying x_1^2 times $x_0^2 - Dy_0^2 = -c$ and subtracting x_0^2 times $x_1^2 - Dy_1^2 = c$, we get $D(x_0^2y_1^2 - x_1^2y_0^2) = -c(x_0^2 + x_1^2)$. Since $\gcd(c, D) = 1$, then any prime p dividing c must divide $x_0^2y_1^2 - x_1^2y_0^2$.

Claim 3.1 *If $p \mid c$, then if $p > 2$, either $p \nmid Y_1 = (x_0y_1 - x_1y_0)$, or $p \nmid Y_2 = (x_0y_1 + x_1y_0)$, and if $p = 2$, then $4 \nmid$* $\gcd(Y_1, Y_2)$.

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If p divides Y_j for $j = 1, 2$, then $p \mid 2x_0y_1$. If $p \mid y_1$, then $p \mid x_1$ since $p \mid c$ and $x_1^2 - Dy_1^2 = c$. However, this contradicts the primitivity of $x_1 + y_1\sqrt{D}$. Similarly, if $p \mid x_0$, then $p \mid y_0$, contradicting the primitivity of $x_0 + y_0\sqrt{D}$. Hence, $p = 2$. If $2^t \parallel \gcd(Y_1, Y_2)$ for $t \in \mathbb{N}$, then both $x_0y_1 \equiv x_1y_0 \pmod{2^t}$ and $x_0y_1 \equiv -x_1y_0 \pmod{2^t}$, so $x_1y_0 \equiv -x_1y_0 \pmod{2^t}$. Since x_1y_0 is odd in this case, then we may take multiplicative inverses to get $-1 \equiv 1 \pmod{2^t}$. Thus, $t = 1$. This establishes Claim 3.1.

Now set

$$X_1 = (x_0x_1 - y_0y_1D) \quad \text{and} \quad X_2 = (x_0x_1 + y_0y_1D).$$

If $p \mid Y_1$, then by Equation (3.8), $p \mid X_1$. Thus, if $p^t \parallel c$ for $p > 2$, then by Claim 3.1, $p^t \mid Y_1$ and $p^t \mid X_1$. Let d be the product of all prime powers dividing both c and $\gcd(X_1, Y_1)$. Thus, by Equation (3.8),

$$N\big(X_1/d + (Y_1/d)\sqrt{D}\big) = -(c/d)^2. \tag{3.9}$$

If $c = d$, this shows that $N(\varepsilon_D) = -1$, so by Theorem 2.1, $\ell(\sqrt{D})$ is odd. If $c \neq d$, then by Claim 3.1, all the odd prime powers dividing c/d also divide Y_2, together with the remaining power of 2 dividing c/d. However, $N(\alpha_0/\alpha_1') = -1$, where

$$\frac{\alpha_0}{\alpha_1'} = \frac{x_0 + y_0\sqrt{D}}{x_1 - y_1\sqrt{D}} = \frac{(x_0 + y_0\sqrt{D})(x_1 + y_1\sqrt{D})}{-c}$$
$$= \frac{(x_0x_1 + y_0y_1D) + (x_1y_0 + x_0y_1)\sqrt{D}}{-c} = \frac{X_2 + Y_2\sqrt{D}}{-c},$$

so $N(X_2 + Y_2\sqrt{D}) = -c^2$. Thus, those odd prime powers dividing c/d and Y_2, together with the remaining power of 2 dividing c/d, also divide X_2. Therefore,

$$N\left(\frac{X_2}{c/d} + \frac{Y_2}{c/d}\sqrt{D}\right) = -d^2. \tag{3.10}$$

Note that in the case where c is even, by Claim 3.1, either $2 \parallel \gcd(X_1, Y_1)$ or $2 \parallel \gcd(X_2, Y_2)$. Therefore, $\gcd(c, c/d) \mid 2$. Also, $\alpha = x_0 - y_0\sqrt{D}$ is a primitive solution of Equation (3.6), $\beta = (X_2 - Y_2\sqrt{D})/(c/d)$ is a solution of Equation (3.10), and $\gamma = (X_1 + Y_1\sqrt{D})/d$ is a solution of Equation (3.9), such that $\alpha^2 = \beta\gamma$. Also, $\alpha\beta'/d = -\alpha_1$ is a primitive element of $\mathbb{Z}[\sqrt{D}]$.

Conversely, if $N(\varepsilon_D) = -1$, namely if $\ell(\sqrt{D})$ is odd by Theorem 2.1, then clearly both Equations (3.6)–(3.7) have primitive solutions if one of them has. On the other hand, if there exists a d as in the hypothesis, then $N(d\gamma/\alpha) = N(-\alpha\beta'/d) = c$, where $\alpha_1 = -\alpha\beta'/d$ is primitive in $\mathbb{Z}[\sqrt{D}]$, by hypothesis. Hence, α_1 is a primitive solution of Equation (3.7). ■

Remark 3.1 When $\alpha\beta'/d$ is a primitive element of $\mathbb{Z}[\sqrt{D}]$ in Theorem 3.1, then this is a primitive solution of Equation (3.7). Hence, the theorem provides a mechanism for finding such solutions. See the examples below for illustrations.

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Corollary 3.1 **(Lagrange)** *The Pell equation $x^2 - Dy^2 = -1$ has a solution if and only if $\ell(\sqrt{D})$ is odd.*

Proof Since $c = 1$ has no proper divisors, then $x^2 - Dy^2 = -1$ if and only if $\ell(\sqrt{D})$ is odd. ■

Corollary 3.2 **([7, Theorem 3.3])** *Suppose that $D \in \mathbb{N}$ is not a perfect square and p is a prime not dividing D. Then both $x^2 - Dy^2 = -p$ and $X^2 - DY^2 = p$ have primitive solutions if and only if $\ell(\sqrt{D})$ is odd.*

Proof Since the only proper divisor of $c = p$ is $d = 1$, then the result follows. ■

Example 3.1 The Diophantine equation $x^2 - 27y^2 = 13$ has the solution $11 + 2\sqrt{27}$, but $x^2 - 27y^2 = -13$ has no solutions $x, y \in \mathbb{Z}$. Here $\ell(\sqrt{27}) = 2$.

Corollary 3.3 *Suppose that $D \in \mathbb{N}$ is not a perfect square, $c = pq$ is a product of two primes such that $\gcd(c, D) = 1$, and α is a primitive solution of $x^2 - Dy^2 = -pq$. Then $X^2 - DY^2 = pq$ has a primitive solution if and only if either $\ell(\sqrt{D})$ is odd, or $x^2 - Dy^2 = -p^2$ has a primitive solution β and $X^2 - DY^2 = -q^2$ has a primitive solution γ with $\alpha^2 = \beta\gamma$ and $\alpha\beta'/p$ is a primitive element of $\mathbb{Z}[\sqrt{D}]$.*

Proof Since the only proper divisors of $c = pq$ are p, q, and 1, then the result follows. ■

Example 3.2 To illustrate the method of proof in Theorem 3.1, let $c = 33 = pq$ and $D = 34$, for which $\ell(\sqrt{34}) = 4$. By setting $p = 3$, we see that $N(5 + \sqrt{34}) = -3^2 = -p^2$ and $N(27 + 5\sqrt{34}) = -11^2 = -q^2$. The reader may see the process developed in the proof of Theorem 3.1 by setting $X_1 = -55$, $Y_1 = 11$, $X_2 = 81$ and $Y_2 = 15$. Set $\alpha = 1 + \sqrt{34}$, $\beta = -5 + \sqrt{34}$ and $\gamma = 27 + 5\sqrt{34}$. Then $\alpha^2 = \beta\gamma$. Thus, $x^2 - Dy^2 = -33$ and $X^2 - DY^2 = 33$ have primitive solutions $\alpha = 1 + \sqrt{34}$ and $\alpha_1 = 13 + 2\sqrt{34}$, respectively. Notice as well that that $\alpha_1 = -\alpha\beta'/p$.

Corollary 3.4 **(Eisenstein—see [3, Footnote 2.1.10, p. 60])** *If $D \in \mathbb{N}$ is not a perfect square and is odd, then both*

$$x^2 - Dy^2 = -4 \tag{3.11}$$

and

$$X^2 - DY^2 = 4 \tag{3.12}$$

have primitive solutions if and only if $\varepsilon_D \notin \mathbb{Z}[\sqrt{D}]$ and $N(\varepsilon_D) = -1$.

Proof If Equations (3.11)–(3.12) have primitive solutions, then by Theorem 3.1, $\ell(\sqrt{D})$ is odd. Therefore, by Theorem 2.1, $N(\varepsilon_D) = -1$. Since $x^2 - Dy^2 = -4$ has a primitive solution, then $\varepsilon_D \notin \mathbb{Z}[\sqrt{D}]$.

Conversely, if $N(\varepsilon_D) = -1$ and $\varepsilon_D \notin \mathbb{Z}[\sqrt{D}]$, then clearly both Equations (3.11)–(3.12) have primitive solutions. ■

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Remark 3.2 As noted by Dickson [1, p. 400], to solve $x^2 - Dy^2 = -4$, set $D = a^2 + b^2$, $y = z^2 + t^2$, and solve the simultaneous equations,

$$(bz - at)^2 - Dt^2 = \pm 2b,$$

and

$$(bt + az)^2 - Dz^2 = \mp 2b.$$

Dickson gives $D = 3^2 + 10^2$ with minimum solution $t = 3$, $z = 4$, as an example, so that $261^2 - 25^2 \cdot 109 = -4$. Note that

$$\varepsilon_{109} = \frac{261 + 25\sqrt{109}}{2}.$$

Example 3.3 If $D = 65$ and $c = 29$, then $\ell(\sqrt{65}) = 1$, and $x^2 - 65y^2 = -29$ has primitive solution $(x, y) = (6, 1)$, while $X^2 - DY^2 = 29$ has primitive solution $(X, Y) = (17, 2)$.

Example 3.4 If $D = 845$ and $c = 29$, then $\ell(\sqrt{845}) = 5$. Primitive solutions of $x^2 - 845y^2 = -29$ and $X^2 - 845Y^2 = 29$ are $(x, y) = (436, 15)$ and $(X, Y) = (407, 14)$.

The following illustrates the case where c is even and *both* conditions (a)–(b) in Theorem 3.1 are satisfied.

Example 3.5 Let $c = 64$ and $D = 145$, where $\ell(\sqrt{145}) = 1$. A primitive solution of $x^2 - 145y^2 = -64$ is $\alpha = 9 - \sqrt{145}$. Moreover, if we set $d = 2$, $\beta = -24 + 2\sqrt{145}$, and $\gamma = 51 + 5\sqrt{145}$, then $N(\beta) = 24^2 - 2^2 \cdot 145 = -4 = -d^2$, $N(\gamma) = 51^2 - 5^2 \cdot 145 = -32^2 = -(c/d)^2$, and $\alpha^2 = \beta\gamma$. Also, $\alpha\beta'/d = 37 + 3\sqrt{145} = \alpha_1$ is a primitive element of $\mathbb{Z}[\sqrt{D}]$. Thus both conditions (a)–(b) in Theorem 3.1 are satisfied, and α_1 is a primitive solution of $X^2 - 145Y^2 = 64$.

The following illustrates the case in Theorem 3.1 where neither condition (a)–(b) in Theorem 3.1 is satisfied.

Example 3.6 Let $c = 100$, and $D = 221$, for which $\ell(\sqrt{D}) = 6$. Thus, condition (a) fails in Theorem 3.1. Also, condition (b) fails since there are no divisors d of c satisfying the conditions. To see this, note that the only possible proper divisors of $c = 100$ for which $\gcd(d, c/d) \mid 2$ are $d = 1$, $d = 25$, or $d = 50$. However, if $d = 1$, then $x^2 - Dy^2 = -1 = -d^2$ has no solutions by Theorem 2.1 since $N(\varepsilon_D) = 1$. If $d = 50$, then although there is a solution $140 + 10\sqrt{221}$ to $x^2 - 221y^2 = -50^2 = -d^2$, there is no solution to $x^2 - dy^2 = -4 = -(c/d)^2$ by Corollary 3.4. Similar considerations apply to the divisor $d = 25$. Hence, although $x^2 - 221y^2 = -100$ has the primitive solution $11 + \sqrt{221}$, the equation $x^2 - 221y^2 = 100$ has no primitive solutions. It *does* have non-primitive solutions such as $75 + 5\sqrt{221}$, however.

We conclude with an observation that there is some ideal theory and related phenomena underlying what we have presented here. For instance, underlying Example 3.2 is the following quadratic irrational and its simple continued fraction expansion:

$$\delta = \frac{-1+\sqrt{34}}{13-2\sqrt{34}} = \frac{(-1+\sqrt{34})(13+2\sqrt{34})}{33} = \frac{55+11\sqrt{34}}{33}$$
$$= \frac{5+\sqrt{34}}{3} = \langle \overline{3\,;1,1,1,1,3} \rangle.$$

This is an example of a reduced quadratic irrational with *pure symmetric period*, namely $\delta = \langle \overline{q_0\,; q_1, \dots, q_{\ell-1}} \rangle$ with $q_0 q_1 \cdots q_{\ell-1}$ being a palindrome.[1] In [3, Theorem 6.1.5, p. 194], we proved that the existence of a reduced quadratic irrational δ with pure symmetric period representing an ideal in a cycle of reduced ideals is tantamount to the existence of a reduced quadratic irrational δ with $N(\delta) = \delta\delta' = -1$ representing an ideal in that cycle. Moreover, we proved that these are in turn equivalent to that cycle being an ambiguous cycle containing at most one ambiguous ideal. (All of these ideals are in the ring of integers of the underlying real quadratic field having discriminant given by the quadratic irrational.) For the interested reader, we devoted an entire chapter to the study of these interrelated phenomena in [3]. Also, see [5].

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Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
T2N 1N4.
website: http://www.math.ucalgary.ca/˜ramollin/
email: ramollin@math.ucalgary.ca

[1] A palindrome is "never even", indeed it is "never odd or even". It is "a toyota".