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FOURIER MULTIPLIERS FOR LOCAL HARDY SPACES ON CHÉBLI-TRIMÈCHE HYPERGROUPS

WALTER R. BLOOM AND ZENGFU XU

ABSTRACT. In this paper we consider Fourier multipliers on local Hardy spaces $\mathbf{h}^p$ $(0 < p \leq 1)$ for Chébli-Trimèche hypergroups. The molecular characterization is investigated which allows us to prove a version of Hörmander's multiplier theorem.

The theory of Fourier multipliers is well developed on euclidean spaces, with various results having been established to give sufficient conditions for a multiplier operator to be bounded on the Lebesgue spaces L^p $(p > 1)$ or Hardy spaces H^p $(0 < p \leq 1)$. Among these are Hörmander's multiplier theorem and its variants. Over the past twenty years considerable effort has been made to extend the classical Fourier multiplier theory to groups and hypergroups (see [W], [FS], [Al], [FX] on Lie groups, [CS], [ST], [K], [An] on noncompact symmetric spaces and [S] on Bessel-Kingman hypergroups). In the consideration of this problem a dichotomy is emerging, based on the growth of the volume of balls centered at the identity as their radii become large (polynomial or exponential growth). While in the case of polynomial growth the condition on a multiplier is similar to that on euclidean spaces (see [Al], [FX], [FS] and [W]), some holomorphy of the multiplier is necessary for the operator to be bounded on L^p $(p > 1)$ when the volumes of balls grow exponentially. In the latter situation, the L^p Fourier multiplier has to be an analytic function having a holomorphic extension to a prescribed tube, the size of which depends on p (see [CS], [An] and [BX3]). This is a consequence of the "holomorphic extension" property of Fourier transforms of L^p-functions $(1 \leq p < 2)$. It turns out that the holomorphic extension property of the Fourier multiplier corresponds roughly to exponential decay of the kernel.

Most of the work up till now on Fourier multipliers on Lie groups and symmetric spaces has only been concerned with L^p-boundedness for $p > 1$, and the H^p-multiplier results in [FS] were only proved on stratified Lie groups which are of polynomial growth. A version of Hörmander's multiplier theorem was established in [S] for L^p-functions $(p > 1)$ on Bessel-Kingman hypergroups, a particular class of Chébli-Trimèche hypergroups with polynomial growth. For general Chébli-Trimèche hypergroups the L^p-Fourier multipliers were investigated in [BX3].

A natural question that arises is whether we can extend the L^p $(p > 1)$ Fourier multiplier theorem of Hörmander to the case $0 < p \leq 1$ for those hypergroups, Lie groups and symmetric spaces that are of exponential growth. There is indeed a natural candidate for

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such a generalization, using the local Hardy spaces $\mathbf{h^p}$ (see Section 3 for the definition). In [K] there are some results on Fourier multipliers for the local Hardy space $\mathbf{h^1}$ on noncompact symmetric spaces of rank 1 (which is of exponential growth), but the conditions on the multipliers are not as sharp and natural as in the classical Hörmander multiplier theorem. Moreover, the approach in [K] does not work for general $\mathbf{h^p}$ when $p < 1$. In fact the molecules defined in [K] are not appropriate to handle $\mathbf{h^p} - \mathbf{h^p}$ boundedness of a Fourier multiplier operator when $p < 1$. This is because the generalized translation of a polynomial of degree ≥ 1 need not be a polynomial. We have modified the definition of molecule to cater for this new phenomenon (see Definition 2.4 and Remark 2.5). To the best of our knowledge, nobody has examined systematically the local Hardy spaces $\mathbf{h^p}$ and Fourier multipliers for $\mathbf{h^p}$ on Chébli-Trimèche hypergroups and noncompact symmetric spaces (other than in [BX2]).

In this paper we establish a version of Hörmander's multiplier theorem for the local Hardy spaces $\mathbf{h^p}$ $(0 < p \leq 1)$ on Chébli-Trimèche hypergroups with exponential growth. Because of the exponential volume growth and the generalized convolution on the hypergroup, the standard constructions do not apply. Many basic facts relying on the structure of a euclidean space are largely unavailable; the Fourier transform on hypergroups is far less well understood than on euclidean spaces, and furthermore there is no "convenient" dilation structure on hypergroups. Our method is a combination of the techniques for euclidean spaces and for noncompact symmetric spaces, and indeed the approach used here can be easily applied to noncompact symmetric spaces.

The paper is organized as follows. The basic Fourier analysis on hypergroups and some useful estimates for characters are given in Section 1. In Section 2 we give an appropriate definition of (local) molecules and investigate the molecular characterization of local Hardy spaces $\mathbf{h^p}$ for $0 < p \leq 1$. Finally in Section 3 we use this molecular characterization to obtain a version of Hörmander's multiplier theorem.

1. **Preliminaries on Chébli-Trimèche hypergroups.** We begin by recalling some basic facts of Fourier analysis on Chébli-Trimèche hypergroups; for a general reference see [BH].

Throughout the paper we denote by $\big(\mathbf{R}_+, *(A)\big)$ the Chébli-Trimèche hypergroup associated with a function A that is continuous on $\mathbf{R}_+$, twice continuously differentiable on $\mathbf{R}_+^* =]0, \infty[$, and satisfies the following conditions:

(1.1) $A(0) = 0$ and $A(x) > 0$ for $x > 0$;

(1.2) A is increasing and unbounded;

(1.3) $\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x)$ on a neighbourhood of 0 where $\alpha > -\frac{1}{2}$ and B is an odd C^∞-function on $\mathbf{R}$;

(1.4) $\frac{A'(x)}{A(x)}$ is a decreasing C^∞-function on $\mathbf{R}_+^*$, and hence $\rho := \frac{1}{2}\lim_{x\to+\infty} \frac{A'(x)}{A(x)} \geq 0$ exists.

In addition we assume that for each $k \in \mathbf{N}_0$, $(\frac{A'(x)}{A(x)})^{(k)}$ is bounded for large $x \in \mathbf{R}_+$.

The hypergroup $\big(\mathbf{R}_+, *(A)\big)$ is noncompact and commutative with neutral element 0 and the identity mapping as the involution. Haar measure on $\big(\mathbf{R}_+, *(A)\big)$ is given by

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$m := A\lambda_{\mathbf{R}_+}$ where $\lambda_{\mathbf{R}_+}$ is Lebesgue measure on $\mathbf{R}_+$. The growth of the hypergroup is determined by the number ρ in (1.4). If $\rho > 0$ then (1.4) implies that $A(x) \geq A(1)e^{2\rho(x-1)}$ for $x \geq 1$ and so the hypergroup is of exponential growth. Otherwise we say that the hypergroup is of subexponential growth. In this paper we restrict ourselves to Chébli-Trimèche hypergroups of exponential growth.

Let $L = L_A$ be the differential operator defined for $x > 0$ by

$$Lf(x) = -f''(x) - \frac{A'(x)}{A(x)} f'(x) \tag{1.5}$$

for each function f twice differentiable on $\mathbf{R}_+^*$. The multiplicative functions on $\big(\mathbf{R}_+, *(A)\big)$ coincide with all the solutions $\varphi_\lambda (\lambda \in \mathbf{C})$ of the differential equation

$$L\varphi_\lambda(x) = (\lambda^2 + \rho^2)\varphi_\lambda(x), \quad \varphi_\lambda(0) = 1,\ \varphi_\lambda'(0) = 0, \tag{1.6}$$

and the dual space $\mathbf{R}_+^\wedge$ can be identified with the parameter set $\mathbf{R}_+ \cup i[0, \rho]$.

For $0 < p \leq \infty$ the Lebesgue space $L^p(\mathbf{R}_+, Adx)$ is defined as usual, and we denote by $\|f\|_{p,A}$ the L^p-norm of $f \in L^p(\mathbf{R}_+, Adx)$. For $f \in L^1(\mathbf{R}_+, Adx)$ the Fourier transform of f is given by

$$\hat{f}(\lambda) = \int_{\mathbf{R}_+} f(x)\varphi_\lambda(x)A(x)\,dx. \tag{1.7}$$

THEOREM 1.8 (LEVITAN-PLANCHEREL; see [BH, THEOREM 2.2.13]). *There exists a unique nonnegative measure σ on $\mathbf{R}_+^\wedge$ with support $[\rho^2, \infty[$ such that the Fourier transform induces an isometric isomorphism from $L^2(\mathbf{R}_+, Adx)$ onto $L^2(\mathbf{R}_+^\wedge, \sigma)$, and for any $f \in L^1(\mathbf{R}_+, Adx) \cap L^2(\mathbf{R}_+, Adx)$*

$$\int_{\mathbf{R}_+} |f(x)|^2 A(x)\,dx = \int_{\mathbf{R}_+^\wedge} |\hat{f}(\lambda)|^2 \sigma(d\lambda).$$

The inverse is given by

$$f(x) = \int_{\mathbf{R}_+} \hat{f}(\lambda)\varphi_\lambda(x)\sigma(d\lambda).$$

To determine the Plancherel measure σ we must place a further (growth) restriction on A. A function f is said to satisfy condition (H) if for some $a > 0$, f can be expressed as

$$f(x) = \frac{a^2 - \frac{1}{4}}{x^2} + \zeta(x)$$

for all large x where

$$\int_{x_0}^{\infty} x^{\gamma(a)} |\zeta(x)|\,dx < \infty$$

for some $x_0 > 0$ and $\zeta(x)$ is bounded for $x > x_0$; here $\gamma(a) = a + \frac{1}{2}$ if $a \geq \frac{1}{2}$ and $\gamma(a) = 1$ otherwise. For $x > 0$ we put

$$G(x) := \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)}\right)' - \rho^2.$$

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THEOREM 1.9 (see [BX1, PROPOSITION 3.17] or [T2]). *Suppose that G satisfies condition (H) together with one of the following conditions:*

(i) $a > \frac{1}{2}$;

(ii) $a \neq |\alpha|$;

(iii) $a = \alpha \leq \frac{1}{2}$ *and* $\int_0^\infty t^{\frac{1}{2}-\alpha}\zeta(t)\varphi_0(t)A(t)^{\frac{1}{2}}\,dt \neq -2\alpha\sqrt{M_A}$ *or* $\int_0^\infty t^{\alpha+\frac{1}{2}}\zeta(t)\varphi_0(t)A(t)^{\frac{1}{2}}\,dt = 0$ *where* $M_A := \lim_{x\to 0^+} x^{-2\alpha-1}A(x)$ *and* $\zeta(x) = G(x) + \frac{\frac{1}{4}-a^2}{x^2}$.

Then the Plancherel measure σ is absolutely continuous with respect to Lebesgue measure and has density $|c(\lambda)|^{-2}$ where $c(\lambda)$ satisfies the following: There exist positive constants C_1, C_2, K such that for any $\lambda \in \mathbf{C}$ with $\mathrm{Im}(\lambda) \leq 0$

$$C_1|\lambda|^{a+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2|\lambda|^{a+\frac{1}{2}}, \quad |\lambda| \leq K, \quad a > 0$$
$$C_1|\lambda|^{\alpha+\frac{1}{2}} \leq |c(\lambda)|^{-1} \leq C_2|\lambda|^{\alpha+\frac{1}{2}}, \quad |\lambda| > K.$$

In the sequel we assume that A satisfies the conditions in Theorem 1.9. This together with (1.1)–(1.4) with $\rho > 0$ implies the following result (see [BX1, Lemmas 2.5 and 3.8]):

LEMMA 1.10. *We have*

$$A(x) \sim x^{2\alpha+1} \quad (x \longrightarrow 0^+)$$
$$A(x) \sim e^{2\rho x} \quad (x \longrightarrow +\infty).$$

Let ϵ_x be the unit point mass at $x \in \mathbf{R}_+$. For any $x, y \in \mathbf{R}_+$ the probability measure $\epsilon_x * \epsilon_y$ is m-absolutely continuous with

$$\mathrm{supp}(\epsilon_x * \epsilon_y) \subset [|x-y|, x+y]. \tag{1.11}$$

The generalized translation $T_x f$ of a function f by $x \in \mathbf{R}_+$ is defined by

$$T_x f(y) := \int_{\mathbf{R}_+} f(z)\epsilon_x * \epsilon_y(dz). \tag{1.12}$$

The convolution of two functions f and g is defined by

$$f * g(x) = \int_{\mathbf{R}_+} T_x f(y) g(y) A(y)\,dy. \tag{1.13}$$

Let us introduce Schwartz functions and distributions on the hypergroup (see [BX4]). For $0 < p \leq 2$ the generalized Schwartz space $S_p\big(\mathbf{R}_+, *(A)\big)$ consists of the restrictions to $\mathbf{R}_+$ of all functions in $S_p(\mathbf{R})$ where

$$S_p(\mathbf{R}) := \{g \in C^\infty(\mathbf{R}) : g \text{ is even and } \mu_{k,l}^p(g) < \infty, k, l \in \mathbf{N_0}\}$$

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and

$$\mu^p_{k,l}(g) := \sup_{x\in\mathbf{R}_+}(1+x)^l\varphi_0(x)^{-\frac{2}{p}}|g^{(k)}(x)|.$$

For $0 < p \le 2$ set

$$F_\delta := \{z \in \mathbf{C} : |\operatorname{Im}(z)| \le \delta\rho\}$$

where $\delta = \frac{2}{p} - 1$ and $\rho > 0$ as in (1.4). Let $S(F_\delta)$ denote the extended Schwartz space defined by all functions h that are even and holomorphic in the interior of F_δ, and such that h together with all its derivatives extend continuously to F_δ and satisfy $\sup_{\lambda\in F_\delta}|h^{(k)}(\lambda)| < \infty$ for any $k, l \in \mathbf{N}_0$. Also we denote by $S_{\epsilon\rho}(\mathbf{R}_+)$ the space of the restrictions to $\mathbf{R}_+$ of the functions in $S_{\epsilon\rho}(\mathbf{R})$ where

$$S_{\epsilon\rho}(\mathbf{R}) = \{g \in C^\infty(\mathbf{R}) : g \text{ is even and } \nu^{(\epsilon)}_{k,l}(g) < \infty\}$$

with

$$\nu^{(\epsilon)}_{k,l}(g) := \sup_{t\in\mathbf{R}_+}(1+t)^l e^{\epsilon\rho t}|g^{(k)}(t)|.$$

Notice that $S_0 = S(F_0)$ is the usual Schwartz space on $\mathbf{R}_+$ and will be denoted by $S(\mathbf{R}_+)$. In the sequel we use F to denote the Fourier transform on the hypergroup, F_0 the classical Fourier transform and $\mathcal{A}$ the Abel transform (see [T] for the definition).

THEOREM 1.14 (see [BX4]). *Let $\delta = \frac{2}{p} - 1$ with $0 < p \le 2$. Then the Fourier transform F on $\big(\mathbf{R}_+, *(A)\big)$ is an isomorphism from $S_p\big(\mathbf{R}_+, *(A)\big)$ to $S(F_\delta)$, the classical Fourier F_0 an isomorphism from $S_{\epsilon\rho}(\mathbf{R}_+)$ to $S(F_\delta)$, and the Abel transform $\mathcal{A}$ an isomorphism from $S_p\big(\mathbf{R}_+, *(A)\big)$ to $S_{\epsilon\rho}(\mathbf{R}_+)$ satisfying $Ff = F_0(\mathcal{A}f)$.*

A p-distribution on $\mathbf{R}_+$ is a continuous linear functional on $S_p\big(\mathbf{R}_+, *(A)\big)$; the totality of p-distributions on $\mathbf{R}_+$ is denoted by $S'_p\big(\mathbf{R}_+, *(A)\big)$. For $f \in S'_p\big(\mathbf{R}_+, *(A)\big)$ we define the Fourier transform of f by

$$\hat{f}(\phi) := f(F^{-1}\phi), \quad \phi \in S(F_\delta).$$

By Theorem 1.14, $\hat{f}$ is well defined as a distribution in $S'(F_\delta)$ and F is continuous on $S_p\big(\mathbf{R}_+, *(A)\big)$. For $f \in S'_p\big(\mathbf{R}_+, *(A)\big)$ and $\phi \in S_p\big(\mathbf{R}_+, *(A)\big)$ the convolution of f and ϕ is a p-distribution defined by

$$f * \phi(\psi) := f(\phi * \psi), \quad \psi \in S_p\big(\mathbf{R}_+, *(A)\big).$$

We now give some useful estimates for characters and their derivatives.

LEMMA 1.15 (see [C], [AT]). *(i) For each $\lambda \in \mathbf{C}$, φ_λ is an even C^∞-function and $\lambda \longmapsto \varphi_\lambda(x)$ is holomorphic.*

(ii) For each $\lambda \in \mathbf{C}$, φ_λ has an integral representation (i.e. the Laplace representation)

$$\varphi_\lambda(x) = \int_{-x}^{x} e^{(i\lambda-\rho)t}\nu_x(dt), \quad x \in \mathbf{R}_+$$

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where ν_x is a probability measure on $\mathbf{R}$ *supported in* $[-x,x]$.

LEMMA 1.16. *Let* $\lambda = \xi + i\eta \in \mathbf{C}$. *Then*
(i) $|\varphi_\lambda(x)| \le e^{|\eta|x}\varphi_0(x)$,
(ii) $e^{-\rho x} \le \varphi_0(x) \le C(1+x)e^{-\rho x}$.

PROOF. The lemma follows from the Laplace representation of φ_λ in Lemma 1.15 and the following estimate given in [AT]:

$$|\varphi_\lambda(x)| \le C_A(1+x)e^{-\rho x}, \quad x,\lambda \in \mathbf{R}_+. \qquad \blacksquare$$

LEMMA 1.17. *Let* $\lambda = \xi + i\eta \in \mathbf{C}$ *and* $k \in \mathbf{N}_0$. *Then*

$$|\varphi_\lambda^{(k)}(x)| \le \begin{cases} C_A(1+|\lambda|)^k e^{|\eta|x}, & |\lambda|x \le 1,\ x \le 1,\\ C_A x A(x)^{-\frac{1}{2}} e^{|\eta|x}, & |\lambda|x \le 1,\ x > 1,\\ C_A A(x)^{-\frac{1}{2}} |c(\lambda)|(1+|\lambda|)^k e^{|\eta|x}, & |\lambda|x > 1. \end{cases}$$

We also have the following alternative estimate:

$$|\varphi_\lambda^{(k)}(x)| \le C_A A(x)^{-\frac{1}{2}} (|\lambda|x)^{\frac{1}{2}-a} |c(\lambda)|(1+\lambda)^k e^{|\eta|x}, \quad |\lambda|x \le 1,\ x > 1.$$

PROOF. The lemma can be proved similarly to [BX2, Lemma 2.4] using Lemma 1.16. ■

In the sequel we use $[\beta]$ to denote the largest integer not exceeding β.

LEMMA 1.18. *Let* $\lambda = \xi + i\eta \in \mathbf{C}$ *and* $k \in \mathbf{N}_0$.
(i) For all $x \in \mathbf{R}_+$

$$|\varphi_\lambda(x)| \le \begin{cases} C_A x e^{(|\eta|-\rho)x}, & |\eta| < \rho,\\ C_A e^{(|\eta|-\rho)x}, & |\eta| \ge \rho. \end{cases}$$

(ii) For all $x > 1$

$$|\varphi_\lambda^{(k)}(x)| \le \begin{cases} C_A(1+|\lambda|)^{2[\frac{k+1}{2}]} x e^{(|\eta|-\rho)x}, & |\eta| < \rho,\\ C_A(1+|\lambda|)^{2[\frac{k+1}{2}]} e^{(|\eta|-\rho)x}, & |\eta| \ge \rho. \end{cases}$$

PROOF. Part (i) follows readily from Lemma 1.16 and the Laplace representation of φ_λ in Lemma 1.15. Appealing to (1.5) and (1.6) we have

$$\varphi_\lambda'(x) = -\frac{\lambda^2+\rho^2}{A(x)} \int_0^x \varphi_\lambda(t)A(t)\,dt$$

and for $k = 2, 3, \ldots$

$$\varphi_\lambda^{(k)}(x) = -\sum_{j=0}^{k-2} \binom{k-2}{j} \left(\frac{A'(x)}{A(x)}\right)^{(j)} \varphi_\lambda^{(k-1-j)}(x) - (\lambda^2+\rho^2)\varphi_\lambda^{(k-2)}(x).$$

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Therefore (ii) follows by induction using (i) and Lemma 1.10 together with our assumption on the derivatives of $\frac{A'(x)}{A(x)}$. ∎

For an m-measurable subset E we denote by $|E|$ its Haar measure and χ_E its characteristic function. For $x_0 \in \mathbf{R}_+$ and $r > 0$, $B(x_0, r)$ denotes the open interval $]\max\{0, x_0 - r\}, x_0 + r[$. Also in the sequel $\mathbf{N}_0$ will denote the set of all nonnegative integers. Finally we shall use C to denote a positive constant whose value may vary from line to line. Dependence of such constants upon parameters of interest will be indicated through the use of subscripts.

2. **The molecular characterization of local Hardy spaces.** In this section we introduce an appropriate definition of (local) molecules and explore the molecular construction of the local Hardy spaces $\mathbf{h^p}$ (see [TW] for the general theory of molecules on euclidean spaces).

We begin with the definition of the local Hardy space $\mathbf{h^p}$ and detail its characterization by atomic decomposition (see [BX2]). For $f \in S_1'\big(\mathbf{R}_+, *(A)\big)$ the local heat maximal function is defined by

$$H_0^+ f(x) := \sup_{0<t\leq 1} |f * h_t(x)|$$

where h_t is the heat kernel.

DEFINITION 2.1. Let $0 < p < \infty$. The local Hardy space $\mathbf{h^p} = \mathbf{h^p}\big(\mathbf{R}_+, *(A)\big)$ is defined by

$$\mathbf{h^p} := \Big\{ f \in S_1'\big(\mathbf{R}_+, *(A)\big) : H_0^+ f \in L^p(\mathbf{R}_+, A dx) \Big\}.$$

Moreover we introduce the quasi-norm $\|f\|_{\mathbf{h^p}} := \|H_0^+ f\|_{p,A}$ defining the topology on $\mathbf{h^p}$.

We recall that for $1 < p < \infty$, $\mathbf{h^p}$ coincides with $L^p(\mathbf{R}_+, A dx)$. The elementary building blocks of $\mathbf{h^p}$ are the (local) (p, q, s)-atoms. Assume throughout that the exponents p and q are admissible in the sense that $0 < p \leq 1$, $1 \leq q \leq \infty$ and $p < q$, and put $s = [(2\alpha + 2)(\frac{1}{p} - 1)]$.

DEFINITION 2.2. A (local) (p, q, s)-atom is a function $a \in L^q(\mathbf{R}_+, A dx)$ such that for some $x_0 \in \mathbf{R}_+$ and $r > 0$, $\mathrm{supp}(a) \subset B(x_0, r)$ and

$$\|a\|_{q,A} \leq m\big(B(x_0, r)\big)^{\frac{1}{q} - \frac{1}{p}}$$

together with the following (local) moment condition: if r can be chosen not exceeding 1 then

$$\int_0^\infty a(x) x^k A(x)\, dx = 0$$

for all integers k satisfying $0 \leq k \leq s$.

The following result characterizes $\mathbf{h^p}$ in terms of atoms.

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THEOREM 2.3 (see [BX2]). *Let $0 < p \le 1$. Then $f \in \mathbf{h^p}$ if and only if f can be represented as a linear combination of (p,q,s)-atoms for any $1 \le q \le \infty$, $q > p$:*

$$f = \sum_i \lambda_i a_i$$

where the a_i are (local) (p,q,s)-atoms and $\sum_i |\lambda_i|^p < \infty$. Moreover there exist two positive constants C_1 and C_2 depending only on p and A such that

$$C_1\Big\{\sum_i |\lambda_i|^p\Big\}^{1/p} \le \|f\|_{\mathbf{h^p}} \le C_2\Big\{\sum_i |\lambda_i|^p\Big\}^{1/p}.$$

Atoms are very convenient for studying the behaviour of certain operators, like radial maximal operators, on $\mathbf{h^p}$ (see [BX2]). For example, the continuity of an operator T can often be proved by estimating Ta when a is an atom. However when we consider the $\mathbf{h^p} - \mathbf{h^p}$ boundedness of an operator T it is possible that for a general local atom a, Ta may not be an atom itself but has to be decomposed into atoms; indeed in general Ta will not have compact support. As in the case of euclidean spaces we can find a class of functions more general than atoms which still generate $\mathbf{h^p}$. These functions will naturally decompose into atoms, and will be called (local) molecules.

We now introduce the (local) molecules corresponding to the atoms defined above.

DEFINITION 2.4. For admissible components p, q and s and $\epsilon > \max\{\frac{s}{2\alpha+2}, \frac{1}{p} - 1\}$ set $a = 1 - \frac{1}{p} + \epsilon$ and $b = 1 - \frac{1}{q} + \epsilon$. A (local) (p,q,s,ϵ)-molecule centred at $x_0 \in \mathbf{R}_+$ is a function $M \in L^q(\mathbf{R}_+, Adx)$ with $M(x)\big|B(x_0, |x-x_0|)\big|^b \in L^q(\mathbf{R}_+, Adx)$ satisfying the conditions

(i) $\|M\|_{q,A}^{a/b} \Big\|M(x)\big|B(x_0,|x-x_0|)\big|^b\Big\|_{q,A}^{1-a/b} := N_q(M) < \infty$, and

(ii) Let σ be the positive number defined by $|B(x_0,\sigma)|^{\frac{1}{q}-\frac{1}{p}} = \|M\|_{q,A}$. If $\sigma < 1$ then for any R with $\sigma \le R \le 1$,

$$\Big|\int_{B(x_0,R)} M(x)(x-x_0)^l A(x)\,dx\Big| \le C_{A,l}\Big(\frac{\sigma}{R}\Big)^{\beta} R^l |B(x_0,R)|^{1-\frac{1}{p}}$$

for $l = 0, 1, \ldots, s$ where $\beta = \min\{a, s+2-\frac{1}{p}\}$.

REMARK 2.5. The moment condition enjoyed by a typical molecule on a euclidean space is now replaced by (ii) (which is an immediate consequence of the moment condition in the case of euclidean spaces). In contrast to the case for euclidean spaces, the generalized translation of a polynomial on $\big(\mathbf{R}_+, *(A)\big)$ is not necessarily a polynomial. Hence for a local atom a, Ta may not satisfy the moment condition. However (ii) can be satisfied by Ta for the most important convolution operators T if a is a local atom supported in $B(x_0, r)$ with $r < 1$.

For $x_0 \in \mathbf{R}_+$ and $\sigma > 0$ we define the following subsets of $\mathbf{R}_+$:

$$E_0 := \{x \in \mathbf{R}_+ : |x - x_0| \le \sigma\} \quad \text{and}$$
$$E_k := \{x \in \mathbf{R}_+ : 2^{k-1}\sigma < |x-x_0| \le 2^k\sigma\}, \quad k = 1, 2, \ldots$$

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and put

$$J_{1,\sigma} := \{k \in \mathbf{N}_0 : 2^k\sigma < 1\} \quad \text{and} \quad J_{2,\sigma} := \{k \in \mathbf{N}_0 : 2^k\sigma \geq 1\}.$$

In the particular case $0 < x_0 \leq 1$ we associate with x_0 two intervals as follows. First choose the unique integer k_0 such that $2^{k_0-1}\sigma \leq x_0 < 2^{k_0}\sigma$, and then define

$$F_{k_0-1} := \begin{cases} [0, x_0 - 2^{k_0-1}\sigma[, & k_0 \geq 2, \\ [0, x_0 + \frac{\sigma}{2}[, & k_0 = 1, \end{cases}$$
$$F_{k_0} :=]x_0 + 2^{k_0-2}\sigma, x_0 + 2^{k_0}\sigma], \quad k_0 \geq 1.$$

For this k_0 we then use $E_{k_0-1} := F_{k_0-1}$ and $E_{k_0} := F_{k_0}$ in place of the F_{k_0-1} and F_{k_0} defined above. We refer to these intervals E_k with left endpoint 0 as of type I, and the remaining subsets E_k as of type II.

LEMMA 2.6. *Let p, q and s be admissible exponents. Then for each $k \in J_{1,\sigma}$ there exist functions ψ_l^k $(l = 0, 1, \ldots, s)$ on $\mathbf{R}_+$ such that* $\operatorname{supp}(\psi_l^k) \subset E_k$, $l = 0, 1, \ldots, s$,

$$\frac{1}{E_k} \int_{E_k} \psi_l^k(x)(x - x_0)^j A(x)\, dx = \delta_{lj}, \quad l, j = 0, 1, \ldots, s$$

and

$$|\psi_l^k(x)| \leq \begin{cases} C_{A,p}(2^k\sigma)^{-l-1}|E_k|A(x)^{-1}, & \text{if } E_k \text{ is of type I and } p < 1, \\ C_{A,p}(2^k\sigma)^{-l}, & \text{otherwise} \end{cases}$$

where $\delta_{lj} = 1$ if $l = j$ and 0 otherwise.

PROOF. We follow the idea in the proof of [K, Lemma 4.6] and only consider the type I case (the proof for type II intervals is easier and runs similarly). Then E_k has the following form:

$$E_k = \begin{cases} [0, \sigma], & \text{if } x_0 = 0 \text{ and } k = 0, \\ [0, x_0 + \sigma], & \text{if } 0 < x_0 \leq 1, k_0 \leq 0 \text{ and } k = 0, \\ [0, x_0 + \frac{\sigma}{2}], & \text{if } 0 < x_0 \leq 1, k_0 = 1 \text{ and } k = 0, \\ [0, x_0 - 2^{k_0-2}\sigma], & \text{if } 0 < x_0 \leq 1, k_0 \geq 2 \text{ and } k = k_0 - 1. \end{cases}$$

For $0 < p \leq 1$ put $d = [2\alpha + 2]$ if $p = 1$ and $d = s$ otherwise. Denote by $P_i(d; x)$ $(i = 0, 1, \ldots, d)$ the polynomials of degree $\leq d$ on the real line determined uniquely by the conditions

$$\frac{1}{E_k} \int_{E_k} P_i(d; x) x^j\, dx = \delta_{ij}, \quad i, j = 0, 1, \ldots, d.$$

Let R be the right endpoint of the interval E_k and define for $l = 0, 1, \ldots, d$ and $k \in J_{1,\sigma}$

$$\psi_l^k(x) := \begin{cases} |E_k| R^{-1} A(x)^{-1} \chi_{E_k}(x) \sum_{i=l}^d \binom{i}{l} x_0^{i-l} R^{-i} P_i\left(d, \frac{x}{R}\right), & p < 1, \\ |E_k| R^{-d-1} x^d A(x)^{-1} \chi_{E_k}(x) P_d\left(d, \frac{x}{R}\right), & p = 1. \end{cases}$$

Using Lemma 1.10 we can verify that ψ_l^k satisfy the desired conditions. ∎

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LEMMA 2.7. *Let $x_0 \in \mathbf{R}_+$ and $\sigma > 0$. Then*

$$|B(x_0,\sigma)| \le C_A 2^{-k} |B(x_0, 2^k\sigma)|, \quad k \in \mathbf{N}_0$$

and

$$E_k \subset \begin{cases} B(x_0, 2^{k+1}\sigma), & k \in J_{1,\sigma}, \\ B(x_0, 2^k\sigma), & k \in J_{2,\sigma}, \end{cases}$$
$$|E_k| \sim |B(x_0, 2^k\sigma)|, \quad k \in \mathbf{N}_0.$$

PROOF. Appealing to (1.2) and Lemma 1.10 we obtain for any $R > 0$

$$|B(x_0,R)| \sim \begin{cases} R^{2\alpha+2}, & x_0 \le R,\ R \le 1, \\ RA(x_0), & x_0 > R,\ R \le 1, \\ e^{2\rho(x_0+R)}, & R > 1. \end{cases} \tag{2.8}$$

The lemma then follows from (2.8) and the definition of E_k. ∎

The following result shows that molecules are generalization of atoms.

LEMMA 2.9. *Every (p,q,s)-atom a is a (p,q,s,ϵ)-molecule for all $\epsilon > 0$, and $N_q(a) \le 1$.*

PROOF. Condition (i) in Definition 2.4 can be verified in the same way as for euclidean spaces. To prove that a satisfies condition (ii) in Definition 2.4 we can assume $\operatorname{supp}(a) \subset B(x_0, r)$ where $r < 1$. Such an atom satisfies the moment condition

$$\int_0^\infty a(x) x^l A(x)\,dx = 0, \quad l = 0,1,\ldots,s$$

and then the result follows using (2.8). ∎

We now prove the main result of this section: every molecule has an atomic decomposition. From this the molecular characterization of $\mathbf{h^p}$ will be evident.

THEOREM 2.10. *Let M be a (local) (p,q,s,ϵ)-molecule centred at x_0. Then $M \in \mathbf{h^p}$ and*

$$\|M\|_{\mathbf{h^p}} \le C_{A,p} N_q(M)$$

with $C_{A,p}$ independent of M.

PROOF. Consider the sets E_k $(k = 0,1,2,\ldots)$ where $\sigma > 0$ is defined as in 2.4(ii), and abbreviate χ_{E_k} by χ_k. Put $M_k = M\chi_k$. For each integer $k \in J_{1,\sigma}$ define

$$P_k(x) := \sum_{j=0}^{s} m_{kj} \psi_j^k(x) \chi_k(x) \tag{2.11}$$

where

$$m_{kj} = \frac{1}{|E_k|} \int_0^\infty M_k(x)(x-x_0)^j A(x)\,dx.$$

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Then

$$M = \sum_{k=0}^{\infty} M_k = \sum_{k \in J_{1,\sigma}} (M_k - P_k) + \sum_{k \in J_{2,\sigma}} M_k + \sum_{k \in J_{1,\sigma}} P_k. \tag{2.12}$$

The proof will consist of three parts:

1) To show that each $M_k - P_k$ ($k \in J_{1,\sigma}$) is a multiple of a local $(p,1,s)$-atom if E_k is of type I and $p < 1$ and a local (p,q,s)-atom otherwise, and that the coefficients sum appropriately.

2) To show that each M_k ($k \in J_{2,\sigma}$) is a multiple of a local (p,q,s)-atom and that the coefficients sum appropriately.

3) To show that $\sum_{k \in J_{1,\sigma}} P_k$ can be written as a sum of local $(p,1,s)$-atoms and (p,∞,s)-atoms if $p < 1$ and a sum of local $(1,\infty,s)$-atoms if $p = 1$, and that the coefficients sum appropriately.

The theorem will then follow from Theorem 2.3.

Without loss of generality we may assume that $N_q(M) = 1$. For each $k \in \mathbf{N}_0$ applying Lemma 1.10 we obtain from Definition 2.4(ii)

$$\begin{aligned} \|M_k\|_{q,A} &\le C_{A,p} |B(x_0, 2^{k-1}\sigma)|^{-b} \Big\| M(x) \big| B(x_0, |x - x_0|) \big|^b \Big\|_{q,A} \\ &\le C_{A,p} \left(\frac{|B(x_0,\sigma)|}{|B(x_0, 2^{k-1}\sigma)|} \right)^a |B(x_0, 2^{k-1}\sigma)|^{\frac{1}{q} - \frac{1}{p}} \end{aligned}$$

where $a = 1 - \frac{1}{p} + \epsilon$ is as in Definition 2.4. Hence by Lemma 2.7

$$\|M_k\|_{q,A} \le C_{A,p} (2^k)^{-a} |B(x_0, 2^k\sigma)|^{\frac{1}{q} - \frac{1}{p}} \tag{2.13}$$

and similarly

$$\|M_k\|_{1,A} \le C_{A,p} (2^k)^{-a} |B(x_0, 2^k\sigma)|^{1 - \frac{1}{p}}. \tag{2.14}$$

Let us start with Part 1. Clearly $\operatorname{supp}(M_k - P_k) \subset B(x_0, 2^k\sigma)$ and, by Lemma 2.6 and (2.11), $M_k - P_k$ has the right cancellation properties:

$$\int_0^\infty \big(M_k(x) - P_k(x) \big) (x - x_0)^j A(x)\, dx = |E_k| m_{kj} - |E_k| \sum_{i=0}^{s} m_{ki} \delta_{ij} = 0.$$

By Lemma 2.6 and (2.11) we have for $k \in J_{1,\sigma}$

$$|P_k(x)| \le \begin{cases} C_{A,p} (2^k\sigma)^{-1} A(x)^{-1} \int_{E_k} |M_k(u)| A(u)\, du, & \text{if } E_k \text{ is of type I and } p < 1, \\ C_{A,p} |E_k|^{-1} \int_{E_k} |M_k(u)| A(u)\, du, & \text{otherwise.} \end{cases}$$

Consequently for $k \in J_{1,\sigma}$

$$\|P_k\|_{1,A} \le C_{A,p} \|M_k\|_{1,A}, \quad \text{if } E_k \text{ is of type I and } p < 1 \tag{2.15}$$

and

$$\|P_k\|_{q,A} \le C_{A,p} \|M_k\|_{q,A}, \quad \text{if } E_k \text{ is of type II or } p = 1. \tag{2.16}$$

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Therefore appealing to (2.13)–(2.16) we obtain for each $k \in J_{1,\sigma}$

$$\|P_k - M_k\|_{1,A} \le C_{A,p}(2^k)^{-a}|B(x_0, 2^k\sigma)|^{1-\frac{1}{p}}, \quad \text{if } E_k \text{ is of type I and } p < 1$$

and

$$\|P_k - M_k\|_{q,A} \le C_{A,p}(2^k)^{-a}|B(x_0, 2^k\sigma)|^{\frac{1}{q}-\frac{1}{p}}, \quad \text{if } E_k \text{ is of type II or } p = 1$$

and hence $a_k^{(1)} := (\lambda_k^{(1)})^{-1}(M_k - P_k)$ is a local $(p, 1, s)$-atom if E_k is of type I and $p < 1$, and a local (p, q, s)-atom otherwise. Here $\lambda_k^{(1)} = C_{A,p}(2^k)^{-a}$ satisfies $\sum_{k\in J_{1,\sigma}} |\lambda_k^{(1)}|^p \le C_{A,p}$.

For Part 2 we observe that for each $k \in J_{2,\sigma}$, M_k is supported in $B(x_0, 2^k\sigma)$ and $2^k\sigma \ge 1$. From (2.13) we see that $a_k^{(2)} := (\lambda_k^{(2)})^{-1}M_k$ is a local (p, q, s)-atom, and $\lambda_k^{(2)} = C_{A,p}(2^k)^{-a}$ satisfies $\sum_{k\in J_{2,\sigma}} |\lambda_k^{(2)}|^p \le C_{A,p}$.

Finally we turn to Part 3. Let $K \in \mathbf{N}_0$ be the integer such that $2^K\sigma < 1 \le 2^{K+1}\sigma$ and put $\tilde\psi_l^k = |E_k|^{-1}\psi_l^k\chi_k$. Then by (2.11)

$$\sum_{k\in J_{1,\sigma}} P_k(x) = \sum_{l=0}^{s}\sum_{k=0}^{K} m_{kl}|E_k|\tilde\psi_l^k(x) = \sum_{k=0}^{K}\sum_{l=0}^{s} N_l^k\phi_l^k(x)$$

where $N_l^k = \sum_{j=0}^k m_{jl}|E_j|$ and

$$\phi_l^k(x) = \begin{cases} \tilde\psi_l^k(x) - \tilde\psi_l^{k+1}, & k = 0, 1, \dots, K-1, \\ \tilde\psi_l^K(x), & k = K. \end{cases}$$

By the definition of E_k we see that for $k \in J_{1,\sigma}$ and $l = 0, 1, \dots, s$

$$\begin{aligned} N_l^k &= \sum_{j=0}^{k}\int_{E_j} M_j(x)(x-x_0)^l A(x)\,dx \\ &= \begin{cases} \int_{B(x_0,2^{k_0-2}\sigma)} M(x)(x-x_0)^l A(x)\,dx \\ \qquad + \int_{E_{k_0-1}} M(x)(x-x_0)^l A(x)\,dx, & \text{if } 0 < x_0 \le 1, k = k_0 - 1, \\ \int_{B(x_0,2^k\sigma)} M(x)(x-x_0)^l A(x)\,dx, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.17}$$

Applying Definition 2.4(i) and Lemmas 1.10 and 2.7 we obtain for $j \ge 1$

$$\begin{aligned} \int_{E_j} |M(x)|\,|x-x_0|^l A(x)\,dx &\le (2^j\sigma)^l|E_j|\left\{\frac{1}{|E_j|}\int_{E_j}|M(x)|^q A(x)\,dx\right\}^{\frac{1}{q}} \\ &\le C_{A,p}(2^j\sigma)^l|E_j|^{1-\frac{1}{q}}|B(x_0,2^j\sigma)|^{-b}\left\|M(x)\big|B(x_0,|x-x_0|)\big|^b\right\|_{q,A} \\ &\le C_{A,p}(2^j\sigma)^l|B(x_0,2^j\sigma)|^{1-\frac{1}{q}-b}|B(x_0,\sigma)|^a \\ &\le C_{A,p}(2^j)^{-a}(2^j\sigma)^l|B(x_0,2^j\sigma)|^{1-\frac{1}{p}} \end{aligned}$$

and

$$\begin{aligned} \int_{E_0} |M(x)|\,|x-x_0|^l A(x)\,dx &\le \sigma^l|E_0|^{1-\frac{1}{q}}\|M\|_{q,A} \\ &\le C_{A,p}\sigma^l|B(x_0,\sigma)|^{1-\frac{1}{p}}. \end{aligned}$$

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Thus for each integer $j \in \mathbf{N}_0$

$$\int_{E_j} |M(x)|\,|x - x_0|^l A(x)\,dx \le C_{A,p}(2^j)^{-a}(2^j\sigma)^l |B(x_0, 2^j\sigma)|^{1-\frac{1}{p}}. \tag{2.18}$$

Now by (2.17) and (2.18) we apply Definition 2.4(ii) with $R = 2^k\sigma$ for each $k \in J_{1,\sigma}$ to obtain

$$|N_l^k| \le C_{A,p}(2^k)^{-\beta}(2^k\sigma)^l |B(x_0, 2^k\sigma)|^{1-\frac{1}{p}}$$

where $\beta = \min\{a, s+2-\frac{1}{p}\}$. By Lemmas 2.6 and 2.7 we observe $\mathrm{supp}(\phi_l^k) \subset B(x_0, 2^{k+2}\sigma)$ for $k = 0, 1, \ldots, K$ and

$$\int_0^\infty \phi_l^k(x)(x - x_0)^j A(x)\,dx = 0$$

for $j = 0, 1, \ldots, s$ and $k = 0, 1, \ldots, K-1$. Notice that E_k and E_{k+1} cannot simultaneously be of type I. Therefore using Lemmas 1.10, 2.6 and 2.7 we have for $k = 0, 1, \ldots, K$

$$\sum_{l=0}^{s} |N_l^k|\,\|\phi_l^k\|_{1,A} \le C_{A,p}(2^k)^{-\beta}|B(x_0, 2^k\sigma)|^{1-\frac{1}{p}}$$

if $p < 1$ and either E_k or E_{k+1} is of type I, and

$$\sum_{l=0}^{s} |N_l^k \phi_l^k(x)| \le C_{A,p}(2^k)^{-\beta}|B(x_0, 2^k\sigma)|^{-\frac{1}{p}}$$

otherwise. Consequently $a_k^{(3)} := (\lambda_k^{(3)})^{-1}\sum_{l=0}^{s} N_l^k\phi_l^k$ is a local $(p, 1, s)$-atom if $p < 1$ and either E_k or E_{k+1} is of type I, and a local (p, ∞, s)-atom otherwise. Here $\lambda_k^{(3)} := C_{A,p}(2^k)^{-\beta}$ satisfies $\sum_{k\in J_{1,\sigma}} |\lambda_k^{(3)}|^p \le C_{A,p}$ since $\beta = \min\{a, s+2-\frac{1}{p}\} > 0$. The theorem is therefore proved. ∎

We are now in a position to give the following molecular characterization of $\mathbf{h^p}$.

COROLLARY 2.19. *Let $f \in S_1'\big(\mathbf{R}_+, *(A)\big)$. Then $f \in \mathbf{h^p}$ if and only if it has a molecular decomposition:*

$$f = \sum_j M_j$$

where the M_j are (local) (p, q, s, ϵ)-molecules such that

$$\sum_j N_q(M_j)^p < \infty.$$

Moreover if the above decomposition holds then

$$\|f\|_{\mathbf{h^p}} \sim \sum_j N_q(M_j)^p.$$

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3. **Fourier multipliers for $\mathbf{h^p}$.** We now turn to the Fourier multipliers for $\mathbf{h^p}$ on $\big(\mathbf{R}_+, *(A)\big)$. After giving estimates for the Fourier transform of functions in $\mathbf{h^p}$ we introduce the Fourier multipliers for local Hardy spaces and give a necessary condition for a bounded function on the dual space $\mathbf{R}_+^\wedge$ to be a Fourier multiplier for $\mathbf{h^p}$. We then establish a version of Hörmander's multiplier theorem for $\mathbf{h^p}$ using atomic and molecular theory.

LEMMA 3.1. *If a is a local (p,q,s)-atom then the Fourier transform of a is holomorphic in the interior of F_δ, continuous on F_δ and satisfies*

$$|\hat{a}(\lambda)| \le C_{A,p}(1+|\lambda|)^{s+1}, \quad \lambda \in F_\delta$$

where $s = [(2\alpha+2)(\frac{1}{p}-1)]$.

PROOF. Suppose that a is supported in $B(x_0, r)$ for some $x_0 \in \mathbf{R}_+$ and $r > 0$. If $r \ge 1$ then we apply (1.7), Lemma 1.18(i), Definition 2.2 and (2.8) to obtain for $\lambda = \xi + i\eta \in F_\delta$

$$\begin{aligned} |\hat{a}(\lambda)| &\le \int_0^\infty |a(x)\varphi_\lambda(x)|A(x)\,dx \\ &\le \|a\|_{1,A} \le |B(x_0,r)|^{1-\frac{1}{p}} \le C_{A,p} \end{aligned}$$

if $|\eta| \le \rho$, and

$$\begin{aligned} |\hat{a}(\lambda)| &\le \|a\|_{1,A} e^{(|\eta|-\rho)(x_0+r)} \\ &\le |B(x_0,r)|^{1-\frac{1}{p}} e^{(|\eta|-\rho)(x_0+r)} \le C_{A,p} \end{aligned}$$

if $\eta > \rho$.

Now assume $r < 1$. Using (1.7) and the cancellation property of a and the Taylor expansion of φ_λ about x_0 of order s we have

$$\hat{a}(\lambda) = \frac{1}{(s+1)!}\int_0^\infty a(x)(x-x_0)^{s+1}\varphi_\lambda^{(s+1)}(\xi_x)A(x)\,dx \tag{3.2}$$

where $\xi_x \in B(x_0, r)$. First consider $x_0 \le 2$. Then by Lemmas 1.17 and 1.10 and (2.8) we obtain for $\lambda \in F_\delta$

$$\begin{aligned} |\hat{a}(\lambda)| &\le C_{A,p} r^{s+1} \int_{B(x_0,r)} |a(x)|A(x)\,dx \\ &\le C_{A,p} r^{s+1} |B(x_0,r)|^{1-\frac{1}{p}} \le C_{A,p} r^{s+1+n-\frac{n}{p}} \end{aligned}$$

if $|\lambda| \le 1$, and

$$\begin{aligned} |\hat{a}(\lambda)| \le C_{A,p} r^{s+1}(1+|\lambda|)^{s+1}\bigg(&\int_{\max\{x_0-r,0\}}^{1/|\lambda|} |a(x)|e^{|\eta|x}A(x)\,dx \\ &+ \int_{1/|\lambda|}^{x_0+r} |a(x)||\lambda|^{-\alpha-\frac{1}{2}}A(x)^{-\frac{1}{2}}e^{|\eta|x}A(x)\,dx\bigg) \\ \le C_{A,p} r^{s+1}(1+|\lambda|)^{s+1}&|B(x_0,r)|^{1-\frac{1}{p}} \le C_{A,p}(1+|\lambda|)^{s+1} r^{s+1+n-\frac{n}{p}} \end{aligned}$$

if $|\lambda| > 1$.

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If $x_0 > 2$ then for $x \in B(x_0, r)$ we have $x > x_0 - r > 1$. Thus appealing to (3.2), Lemmas 1.18(ii) and 1.10 and (2.8) we obtain for $\lambda \in F_\delta$ and $|\lambda| \leq 2$

$$\begin{aligned}|\hat{a}(\lambda)| &\leq C_{A,p} r^{s+1} \|a\|_{1,A} e^{(\delta-1)\rho(x_0+r)} \\ &\leq C_{A,p} r^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} e^{2\rho(\frac{1}{p}-1)(x_0+r)} \leq C_{A,p} r^{s+2-\frac{1}{p}},\end{aligned}$$

and to Lemma 1.17 in place of Lemma 1.18(ii) we obtain for $\lambda \in F_\delta$ and $|\lambda| > 2$

$$\begin{aligned}|\hat{a}(\lambda)| &\leq C_{A,p} r^{s+1} |\lambda|)^{s+1-\alpha-\frac{1}{2}} \|a\|_{1,A} e^{(|\eta|-\rho)(x_0+r)} \\ &\leq C_{A,p} r^{s+1} |\lambda|^{s+1} |B(x_0, r)|^{1-\frac{1}{p}} e^{2\rho(\frac{1}{p}-1)(x_0+r)} \leq C_{A,p} r^{s+2-\frac{1}{p}} |\lambda|^{s+1}.\end{aligned}$$

Therefore for $\lambda \in F_\delta$

$$|\hat{a}(\lambda)| \leq \begin{cases} C_{A,p}, & r > 1, \\ C_{A,p}(1 + |\lambda|)^{s+1} r^{s+1+n-\frac{n}{p}}, & r \leq 1,\ x_0 \leq 2, \\ C_{A,p} r^{s+2-\frac{1}{p}} (1 + |\lambda|)^{s+1}, & r \leq 1,\ x_0 > 2. \end{cases} \tag{3.3}$$

The result now follows from Lemma 1.12(i), (1.7) and (3.3). ∎

From the definition and Theorem 1.14 we see that the Fourier transform of a tempered distribution is a distribution in $S'(F_1)$. The following result shows that if $f \in \mathbf{h^p}$ then $\hat{f}$ is actually analytic on F_δ.

THEOREM 3.4. *Let $f \in \mathbf{h^p}$, $0 < p \leq 1$. Then the Fourier transform $\hat{f}$ is an even function holomorphic in the interior of F_δ and continuous on F_δ satisfying*

$$|\hat{f}(\lambda)| \leq C_A \|f\|_{\mathbf{h^p}} (1 + |\lambda|)^{[\frac{n}{p}-n]+1}, \quad \lambda \in F_\delta.$$

PROOF. By Theorem 2.3 we have a decomposition

$$f = \sum_j \lambda_j a_j$$

where the a_j are local (p, q, s)-atoms and

$$\sum_j |\lambda_j|^p \leq C_{A,p} \|f\|_{\mathbf{h^p}}.$$

Since the series converges in $S'\big(\mathbf{R}_+, *(A)\big)$ and the Fourier transform is continuous on $S'\big(\mathbf{R}_+, *(A)\big)$ we have

$$\hat{f} = \sum_j \lambda_j \widehat{a_j}.$$

Thus Lemma 3.1 and the fact that

$$\sum_j |\lambda_j| \leq \Big(\sum_j |\lambda_j|^p\Big)^{1/p} \leq C_{A,p} \|f\|_{\mathbf{h^p}}$$

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give the theorem. ∎

For a bounded function m on $\big(\mathbf{R}_+, *(A)\big)$ consider the operator T_m defined by

$$(T_m f)^\wedge(\lambda) = m(\lambda)\hat{f}(\lambda). \tag{3.5}$$

By Theorem 3.4, T_m is a well-defined continuous operator from $\mathbf{h^p}$ $(0 < p \le 1)$ to $S'\big(\mathbf{R}_+, *(A)\big)$ and by Theorem 1.8, T_m is bounded on $L^2(\mathbf{R}_+, A dx)$ whenever m is a bounded function. A bounded function m is said to be a *Fourier multiplier* for $\mathbf{h^p}$ if the operator T_m takes $\mathbf{h^p}$ continuously into $\mathbf{h^p}$.

The following theorem shows that some holomorphy of the function m is necessary for T_m to be bounded on $\mathbf{h^p}$. This new phenomenon, different from the euclidean case, arises from the exponential growth of the hypergroups.

LEMMA 3.6. *Let $0 < p \le 1$. Then every Fourier multiplier m for $\mathbf{h^p}$ extends to an even function holomorphic in the interior of the strip F_δ and continuous on F_δ.*

PROOF. Choose $f(x) = h_1(x)$ where $h_t(x)$ is the heat kernel (see [AT]). Now applying the semigroup property of the heat kernel:

$$h_{t_1} * h_{t_2} = h_{t_1+t_2}$$

and Definition 2.1 we see that $f \in \mathbf{h^p}$ for $0 < p \le 1$. Observe that $\hat{f}(\lambda) = e^{-(\lambda^2+\rho^2)}$ is holomorphic and does not vanish. If m is a Fourier multiplier for $\mathbf{h^p}$ then $T_m f \in \mathbf{h^p}$ and, by (3.5), $m(\lambda) = \frac{(T_m f)^\wedge(\lambda)}{\hat{f}(\lambda)}$. The lemma now follows readily from Theorem 3.4. ∎

We now establish a version of the Hörmander-Mihlin multiplier theorem for $\mathbf{h^p}$ Fourier multipliers on $\big(\mathbf{R}_+, *(A)\big)$, but first we begin with some definitions. The notation K is reserved for the kernel obtained as the Fourier transform of m in the distributional sense. Then $T_m f = f * K$. Choose an even C^∞-function ψ on $\mathbf{R}$ such that $\psi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\psi(x) = 0$ for $|x| \ge 1$, and fix once and for all a kernel decomposition $K = K^0 + K^\infty$ where $K^0 = K\psi$ and $K^\infty = K(1-\psi)$.

DEFINITION 3.7. For a positive integer N we say that a bounded function m satisfies a Hörmander condition of order N (and denote this by $m \in \mathcal{M}(2, N)$) if m extends to an even analytic function inside F_δ and the derivatives $m^{(i)}$ extend continuously to the whole of F_δ and satisfy

$$\sup_{\lambda \in F_\delta} (1 + |\lambda|)^i |m^{(i)}(\lambda)| < \infty, \quad i = 0, 1, \ldots, N.$$

For $m \in \mathcal{M}(2, N)$ set $\|m\|_{\mathcal{M}(2,N)} := \max_{0 \le i \le N} \sup_{\lambda \in F_\delta} (1 + |\lambda|)^i |m^{(i)}(\lambda)|$.

Let ϕ be an even nonnegative C^∞-function supported in $\{x \in \mathbf{R} : \frac{1}{2} < |x| < 2\}$ and satisfying $\sum_{j=-\infty}^{\infty} \phi(2^{-j}x) = 1$ for $x \ne 0$. Put $\phi_j(x) = \phi(2^{-j}x)$ for $j = 1, 2, \ldots$ and $\phi_0(x) = 1 - \sum_{j=1}^{\infty} \phi(2^{-j}x)$. For $m \in \mathcal{M}(2, N)$ we fix once and for all a dyadic decomposition $m(\lambda) = \sum_{k=0}^{\infty} m_k(\lambda)$ where $m_k(\lambda) = m(\lambda)\phi_k(\lambda)$. Thus the corresponding decomposition for the kernel K is $K(x) = \sum_{k=0}^{\infty} K_k(x)$ where $\hat{K}_k(\lambda) = m_k(\lambda)$. Throughout

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the remainder of the paper we shall always assume that m is in fact rapidly decreasing (*i.e.* $m \in \mathcal{S}(F_0)$, the usual Schwartz space) though none of our estimates will depend upon the actual rate of decrease. It suffices to flatten m or, equivalently, to regularize K in the standard way. Thus by Theorem 1.14, $K \in \mathcal{S}_2\big(\mathbf{R}_+, *(A)\big)$.

The proof of the following lemma is similar to that of [An, Proposition 5].

LEMMA 3.8. *Suppose that* $m \in \mathcal{M}(2,N)$ *with* $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$ *and* $0 < p \le 1$. *Then* $K^\infty \in L^1(\mathbf{R}_+, Adx)$ *and*

$$\|K^\infty\|_{1,A} \le C_A \|m\|_{\mathcal{M}(2,N)}.$$

LEMMA 3.9. *For any* $0 < R \le 1$ *we have*

$$\int_0^R y|K^0(y)|A(y)\,dy \le C_A \|m\|_{\mathcal{M}(2,N)} R$$

and

$$\int_R^1 y^{-1}|K^0(y)|A(y)\,dy \le C_A \|m\|_{\mathcal{M}(2,N)} R^{-1}.$$

PROOF. We only give the proof of the first inequality (the second can be handled similarly). Let k_0 be the positive integer such that $1 \le 2^{k_0}R < 2$. Using the dyadic decomposition of m we observe

$$\begin{aligned}\int_0^R y|K^0(y)|A(y)\,dy &\le \sum_{k=0}^{\infty} \int_0^R |K_k^0(y)|yA(y)\,dy \\ &= \sum_{k=0}^{k_0} \int_0^R |K_k^0(y)|yA(y)\,dy + \sum_{k=k_0+1}^{\infty} \int_0^{2^{-k}} |K_k^0(y)|yA(y)\,dy \\ &\quad + \sum_{k=k_0+1}^{\infty} \int_{2^{-k}}^R |K_k^0(y)|yA(y)\,dy \\ &:= \sigma_1 + \sigma_2 + \sigma_3\end{aligned}$$

where $K_k^0 = \psi K$. Now applying Theorems 1.8 and 1.9 and properties of the classical Fourier transform we have

$$\begin{aligned}\sigma_1 &\le C_A R^{\alpha+2} \sum_{k=0}^{k_0} \Big\{ \int_0^\infty |K_k^0(y)|^2 A(y)\,dy \Big\}^{1/2} \\ &\le C_A R^{\alpha+2} \sum_{k=0}^{k_0} \Big\{ \int_0^\infty |m_k(\lambda)|^2 (1+\lambda)^{2\alpha+1}\,d\lambda \Big\}^{1/2} \\ &\le C_A R^{\frac{3}{2}} \sum_{k=0}^{k_0} \Big\{ \int_0^\infty |m_k(\lambda)|^2\,d\lambda \Big\}^{1/2} \\ &\le C_A R^{\frac{3}{2}} \|m\|_{\mathcal{M}(2,N)} \sum_{k=0}^{k_0} 2^{\frac{k}{2}} \le C_A \|m\|_{\mathcal{M}(2,N)} R\end{aligned}$$

and

$$\begin{aligned}\sigma_2 &\le C_A(2^{-k})^{\alpha+2}\sum_{k=k_0}^{\infty}\left\{\int_0^\infty |K_k^0(y)|^2A(y)\,dy\right\}^{1/2}\\ &\le C_A(2^{-k})^{\alpha+2}\sum_{k=k_0}^{\infty}\left\{\int_0^\infty |m_k(\lambda)|^2(1+\lambda)^{2\alpha+1}\,d\lambda\right\}^{1/2}\\ &\le C_A(2^{-k})^{\frac{3}{2}}\sum_{k=k_0}^{\infty}\left\{\int_0^\infty |m_k(\lambda)|^2\,d\lambda\right\}^{1/2}\\ &\le C_A\|m\|_{M(2,N)}R.\end{aligned}$$

To estimate σ_3 we introduce smooth cut-off functions as in [An]. Let ω^0 be an even C^∞-function on $\mathbf{R}$ such that $\omega^0(x) = 1$ for $|x| \le \frac{1}{4}$ and $\omega^0(x) = 0$ for $|x| \ge \frac{1}{2}$, and set $\omega_j^0(x) = \omega^0(2^jx)$ for each $j \in \mathbf{N}_0$. Then $\omega_j^0(x) = 1$ for $|x| \le 2^{-j-2}$, $\omega_j^0(x) = 0$ for $|x| \ge 2^{-j-1}$ and $|\frac{d^i}{dx^i}\omega_j^0(x)| \le C_i2^{ij}$, $i = 0,1,2,\dots$. Denote by l the inverse classical Fourier transform of m. For a dyadic decomposition of m let $l(u) = \sum_{k=0}^\infty l_k(u)$ be the corresponding decomposition where $F_0l_k(\lambda) = m_k(2^{-k}\lambda)$. Put $l_{kj}^0 = (1-\omega_j^0)l_k$ and let $K_{kj}^0 = A^{-1}(l_{kj}^0)$ and $m_{kj}^0 = F_0(l_{kj}^0)$. Then $l_k - l_{kj}^0$ is an even C^∞-function supported in $[-2^{-j-1}, 2^{-j-1}]$, and hence using the property of the Abel transform (see [T, Théorème 6.4]) we see that $K_k - K_{kj}^0 = A^{-1}(l_k - l_{kj}^0)$ is also supported in $[-2^{-j-1}, 2^{-j-1}]$. Consequently

$$K_k(x) = K_{kj}^0(x), \quad x > 2^{-j-1}. \tag{3.10}$$

We now apply (3.10), Theorems 1.8 and 1.9 and the properties of the classical Fourier transform to obtain

$$\begin{aligned}\sigma_3 &\le C_A\sum_{k=k_0+1}^{\infty}\sum_{j=k_0-1}^{k}\int_{2^{-j-1}}^{2^{-j}}|K_k^0(y)|yA(y)\,dy\\ &\le C_A\sum_{k=k_0+1}^{\infty}\sum_{j=k_0-1}^{k}(2^{-j})^{\alpha+2}\left\{\int_{2^{-j-1}}^{2^{-j}}|K_k(y)|^2A(y)\,dy\right\}^{1/2}\\ &= C_A\sum_{k=k_0+1}^{\infty}\sum_{j=k_0-1}^{k}(2^{-j})^{\alpha+2}\left\{\int_{2^{-j-1}}^{2^{-j}}|K_{kj}^0(y)|^2A(y)\,dy\right\}^{1/2}\\ &\le C_A\sum_{k=k_0+1}^{\infty}\sum_{j=k_0-1}^{k}(2^{-j})^{\alpha+2}\left\{\int_{2^{-j-1}}^{2^{-j}}|m_{kj}^0(\lambda)|^2(1+\lambda)^{2\alpha+1}\,d\lambda\right\}^{1/2}.\end{aligned}$$

Put $\Omega_j^0 = 1-\omega_j^0$. Then $m_{kj}^0 = F_0(\Omega_j^0l_k)$ and hence

$$\sigma_3 \le C_A\sum_{k=k_0+1}^{\infty}\sum_{j=k_0-1}^{k}(2^{-j})^{\alpha+2}\left\{\int_{2^{-j-1}}^{2^{-j}}|F_0(\Omega_j^0l_k)(\lambda)(1+\lambda)^{\alpha+\frac{1}{2}}|^2\,d\lambda\right\}^{1/2}.$$

Arguing as in [An, Lemma 15] we have for $\beta_1 < \beta_2$

$$\left(\sum_{j=0}^{k}(2^{-\beta_2j}\|\Omega_j^0l_k\|_{H_2^{\beta_1}})^2\right)^{\frac{1}{2}} \le C_A2^{k(\frac{1}{2}+\beta_1-\beta_2)}\|m\|_{M(2,N)}$$

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where H_2^β is the usual Sobolev space. Therefore by substituting $\beta_1 = \alpha + \frac{1}{2}$ and $\beta_2 = \alpha + 2$ we obtain

$$\sigma_3 \le C_A \|m\|_{M(2,N)} \sum_{k=k_0+1}^{\infty} 2^{-k} \le C_A \|m\|_{M(2,N)} R$$

and this completes the proof of the lemma. ∎

Let ψ be the function defining K^0, and for $R > 0$ and $l \in \mathbf{N}_0$ put $Q_{R,l}(x) = \psi_R(x)(x - x_0)$ where $\psi_R(x) = \psi(\frac{x-x_0}{R})$.

LEMMA 3.11. *For any* $0 < R \le 1$ *and* $k, l \in \mathbf{N}_0$ *we have*
(i) $|\frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \le C_{A,k,l} R^{l-k}$, $|x - x_0| < R$, $y \in \mathbf{R}_+$,
(ii) $|\frac{\partial}{\partial y}\frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \le C_{A,k,l} R^{l-k-1}$, $|x - x_0| < R$, $0 < y \le 1$, *and*
(iii) $|\frac{\partial^k}{\partial x^k} T_y Q_{R,l}(x)| \le C_{A,k,l} R^{l-k+1} y^{-1}$, $|x - x_0| < R$, $R \le y \le 1$.
Here T_y *is the generalized translation defined by (1.12).*

PROOF. By the definition of $Q_{R,l}$ we see that

$$|Q_{R,l}^{(k)}(x)| \le C_{A,k,l} R^{l-k}, \quad k \in \mathbf{N}_0.$$

Thus the lemma can be proved in the same way as in [BX2, Lemma 3.15] using Theorem 1.9 and Lemmas 1.10 and 1.17.

LEMMA 3.12. *Suppose that* $m \in M(2,N)$ *with* $N = [\frac{n}{p} - \frac{n}{2}] + 1$ *and* $0 < p \le 1$, *and* a *is a local* (p,q,s)*-atom supported in* $B(x_0, r)$ *with* $r < 1$. *If* $T_m a$ *is a local* (p,q,s,ϵ)*-molecule then for any* $\sigma \le R \le 1$

$$\left| \int_{B(x_0,R)} T_m a(x)(x - x_0)^l A(x)\, dx \right| \le C_{A,l} R^l |B(x_0,R)|^{1-\frac{1}{p}} \left(\frac{\sigma}{R}\right)^\beta$$

for $l = 0, 1, \ldots, s$, *where* σ *is the positive number defined by* $|B(x_0,\sigma)|^{\frac{1}{q}-\frac{1}{p}} = \|T_m a\|_{q,A}$ *and* $\beta = \min\{1 - \frac{1}{p} + \epsilon, s + 2 - \frac{1}{p}\}$ *as in Definition 2.4.*

PROOF. We first observe that an application of the Hörmander's multiplier theorem for $L^q(\mathbf{R}_+, A dx)$ $(q > 1)$ gives

$$\|T_m a\|_{q,A} \le C_{A,q} \|a\|_{q,A}. \tag{3.13}$$

By Definition 2.2 we see that (3.13) implies that

$$|B(x_0, r)| \le C_{A,q} |B(x_0, \sigma)|. \tag{3.14}$$

If $R < 2r$ then we apply the Cauchy-Schwarz inequality, (3.14) and (2.8) to obtain

$$\begin{aligned}
\left| \int_{B(x_0,R)} T_m a(x)(x - x_0)^l A(x)\, dx \right| &\le C_A R^l \|T_m a\|_{q,A} |B(x_0,R)|^{1-\frac{1}{q}} \\
&= C_A R^l |B(x_0,r)|^{1-\frac{1}{q}} |B(x_0,\sigma)|^{\frac{1}{q}-\frac{1}{p}} \\
&= C_A R^l |B(x_0,R)|^{1-\frac{1}{p}} \left(\frac{|B(x_0,R)|}{|B(x_0,\sigma)|} \right)^{\frac{1}{p}-\frac{1}{q}} \\
&\le C_{A,p} R^l |B(x_0,R)|^{1-\frac{1}{p}} \left(\frac{\sigma}{R}\right)^\beta.
\end{aligned}$$

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We now assume that $R \geq 2r$ and write

$$\begin{aligned}\int_{B(x_0,R)} T_m a(x)(x-x_0)^l A(x)\,dx &= \int_{B(x_0,R)} T_m a(x)\psi_R(x)(x-x_0)^l A(x)\,dx \\ &\quad + \int_{B(x_0,R)} T_m a(x)\big(1-\psi_R(x)\big)(x-x_0)^l A(x)\,dx \\ &:= I_R^{(1)} + I_R^{(2)}.\end{aligned}$$

Let $E_R := \{x \in \mathbf{R}_+ : \frac{R}{2} < |x-x_0| < R\}$. As $T_m a$ is a (p,q,s,ϵ)-molecule by assumption we argue similarly as in showing (2.18) to obtain

$$|I_R^{(2)}| \leq C_{A,p} R^l |B(x_0,R)|^{-\epsilon} |B(x_0,\sigma)|^{1-\frac{1}{p}+\epsilon}.$$

Hence by (2.8)

$$|I_R^{(2)}| \leq C_{A,p} R^l |B(x_0,R)|^{1-\frac{1}{p}} \Big(\frac{\sigma}{R}\Big)^{\beta}.$$

It remains to estimate $I_R^{(1)}$ for $R \geq 2r$. Using the property of the generalized translation T_y and the decomposition of the kernel K we observe

$$\begin{aligned}I_R^{(1)} &= \int_{B(x_0,R)} T_m a(x)\psi_R(x)(x-x_0)^l A(x)\,dx \\ &= \int_0^\infty K(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &= \int_0^\infty K^0(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &\qquad + \int_0^\infty K^\infty(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &= \int_0^R K^0(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &\qquad + \int_R^1 K^0(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &\qquad + \int_0^\infty K^\infty(y)\Big[\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx\Big]A(y)\,dy \\ &:= I_R^{(1,1)} + I_R^{(1,2)} + I_R^{(1,3)}\end{aligned}$$

where $Q_{R,l} = \psi_R(x)(x-x_0)^l$ is as in Lemma 3.11. Note that $\psi_R(x) = 1$ for $|x-x_0| \leq \frac{1}{2}$. Hence using properties of the generalized translation we have

$$T_0 Q_{R,l}(x) = Q_{R,l}(x) = (x-x_0)^l, \quad |x-x_0| < r,\ R \geq 2r.$$

Now using the cancellation properties and the Taylor expansion of $F(x,y) = T_y Q_{R,l}(x)$ we obtain

$$\begin{aligned}\int_0^\infty a(x)T_y Q_{R,l}(x)A(x)\,dx &= \int_{B(x_0,r)} a(x)[T_y Q_{R,l}(x) - Q_{R,l}(x)]A(x)\,dx \\ &= \frac{y}{(s+1)!}\int_0^1\int_0^1 (1-v)^s \Big[\int_{B(x_0,r)} a(x)(x-x_0)^{s+1} \\ &\qquad \times \Big(\frac{\partial^{s+1}}{\partial\eta\partial\xi^{s+1}} T_\eta Q_{R,l}(\xi)\Big)_{\substack{\xi=x_0+v(x-x_0)\\ \eta=uy}} A(x)\,dx\Big]\,du\,dv.\end{aligned}$$

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Thus applying Lemma 3.11(ii) and Definition 2.2 we have for $0 < y \le 1$

$$\Big|\int_0^\infty a(x) T_y Q_{R,l}(x) A(x)\, dx\Big| \le C_{A,p} r^{s+1} R^{l-s-2} y |B(x_0, r)|^{1-\frac{1}{p}}$$

and hence by Lemma 3.9, (2.8) and (3.14)

$$|I_R^{(1,1)}| \le C_{A,l} R^l |B(x_0, R)|^{1-\frac{1}{p}} \Big(\frac{\sigma}{R}\Big)^\beta.$$

Similarly using the Taylor expansion of $T_y Q_{R,l}$ about x_0 and Lemma 3.11(i), (iii) we obtain for $R \le y \le 1$

$$\Big|\int_0^\infty a(x) T_y Q_{R,l}(x) A(x)\, dx\Big| \le C_{A,p} r^{s+1} R^{l-s} y^{-1} |B(x_0, r)|^{1-\frac{1}{p}}$$

and for any $y \in \mathbf{R}_+$

$$\Big|\int_0^\infty a(x) T_y Q_{R,l}(x) A(x)\, dx\Big| \le C_{A,p} r^{s+1} R^{l-s-1} |B(x_0, r)|^{1-\frac{1}{p}}.$$

Therefore applying Lemmas 3.8 and 3.9, (2.8) and (3.14) we obtain

$$|I_R^{(1,i)}| \le C_{A,l} R^l |B(x_0, R)|^{1-\frac{1}{p}} \Big(\frac{\sigma}{R}\Big)^\beta, \quad i = 2, 3$$

and this completes the proof of the lemma. ∎

We also need the following estimates concerning m and its corresponding kernel K.

LEMMA 3.15. *Suppose that $m \in \mathcal{M}(2, N)$ with $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$, $\alpha \ge 0$ and $0 < p \le 1$. Then for $0 < t \le 1$ and $|x - y| \ge 2$*

$$|T_x(K * h_t)(y)| \le C_A \mathcal{M}(2, N) |x - y|^{-N} A(x)^{-\frac{1}{2}} A(y)^{-\frac{1}{2}} e^{-\delta\rho|x-y|}$$

where $\delta = \frac{2}{p} - 1$, h_t is the heat kernel and K is the kernel corresponding to m.

PROOF. We follow [An] and choose $\omega \in C^\infty(\mathbf{R})$ such that $\omega(x) = 0$ for $x \le \frac{1}{2}$ and $\omega(x) = 1$ for $x \ge 1$. For any fixed $x, y \in \mathbf{R}_+$ with $|x - y| \ge 2$ write

$$\omega_{|x-y|}(u) := \omega(|x - y| + u)\, \omega(|x - y| - u).$$

Then $\omega_{|x-y|}$ is an even C^∞-function on $\mathbf{R}$ satisfying $\omega_{|x-y|}(u) = 1$ for $|u| < |x - y| - 1$ and $\omega_{|x-y|}(u) = 0$ for $|u| \ge |x - y| - \frac{1}{2}$. Writing $l := \mathcal{A}(K * h_t)$ and $\tilde{m} := F_0 l$ we see that $\tilde{m}(\lambda) = m(\lambda) e^{-t(\lambda^2+\rho^2)}$. Put $l_{|x-y|} := l(1 - \omega_{|x-y|})$, $K_{|x-y|} := \mathcal{A}^{-1} l_{|x-y|}$ and $m_{|x-y|} := F_0 l_{|x-y|}$. Now $l - l_{|x-y|}$ is supported in $[0, |x - y| - \frac{1}{2}]$. Hence by [T, Théorème 6.4] we have $\operatorname{supp}(K * h_t - K_{|x-y|}) \subset [0, |x - y| - \frac{1}{2}]$ which implies that

$$K * h_t(u) = K_{|x-y|}(u), \quad u > |x - y| - \frac{1}{2}.$$

Thus by (1.12), $T_x(K * h_t)(y) = T_x K_{|x-y|,t}(y)$, and by [BH, Theorem 2.2.36] and [BX1, (2.17) and (2.18)]

$$T_x(K * h_t)(y) = \int_0^\infty m_{|x-y|,t}(\lambda) \varphi_\lambda(x) \varphi_\lambda(y) |c(\lambda)|^{-2}\, d\lambda. \tag{3.16}$$

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We claim now that for any L with $0 \le L < N - \frac{1}{2}$ and $|x-y| \ge 2$

$$\Big\{\int_0^\infty |m_{|x-y|}(\lambda)(1+\lambda)^L|^2\, d\lambda\Big\}^{1/2} \le C_A\|m\|_{M(2,N)}|x-y|^{-N}e^{\delta\rho|x-y|}. \tag{3.17}$$

In fact by interpolation we can restrict ourselves to the case when $l \in \mathbf{N}_0$. In view of the properties of the classical Fourier transform and the analyticity of m we have

$$\begin{aligned}
&\Big\{\int_0^\infty |m_{|x-y|}(\lambda)(1+\lambda)^L|^2\, d\lambda\Big\}^{1/2}\\
&\le C_A\sum_{i=0}^{L}\Big\{\int_0^\infty |l^{(i)}_{|x-y|}(u)|^2\, du\Big\}^{1/2}\\
&\le C_A\sum_{i=0}^{L}\sum_{j=0}^{i} t^{-i+j}\Big\{\int_{|x-y|-1}^\infty |l^{(j)}(u)|^2\, du\Big\}^{1/2}\\
&\le C_A|x-y|^{-N}e^{-\delta\rho|x-y|}\sum_{i=0}^{L}\sum_{j=0}^{i}\Big\{\int_0^\infty |u^N e^{\delta\rho u} l^{(j)}(u)|^2\, du\Big\}^{1/2}\\
&\le C_A|x-y|^{-N}e^{-\delta\rho|x-y|}\sum_{i=0}^{L}\sum_{j=0}^{i}\Big\{\int_0^\infty \Big|\frac{d^N}{d\lambda^N}\big((\lambda+i\delta\rho)^j m(\lambda+i\delta\rho)\big)\Big|^2\, d\lambda\Big\}^{1/2}\\
&\le C_A|x-y|^{-N}e^{-\delta\rho|x-y|}\sum_{i=0}^{L}\Big\{\int_0^\infty \Big|(\lambda+i\delta\rho)^{L-i}m^{(N-i)}(\lambda+i\delta\rho)\Big)\Big|^2\, d\lambda\Big\}^{1/2}\\
&\le C_A|x-y|^{-N}e^{-\delta\rho|x-y|}\|m\|_{M(2,N)}.
\end{aligned}$$

The lemma now follows from (3.16) and (3.17) using Theorem 1.8, Lemmas 1.17 and 1.10 and a straightforward calculation. ∎

Let ω be the function as in the proof of Lemma 3.15. For any integer $j > 1$ we define an even C^∞-function ω_j by

$$\omega_j(u) = \omega\big(2(u+j-1)\big)\omega\big(2(-u+j-1)\big),$$

and denote by l the Abel transform of K. Then by Theorem 1.14, $m = F(K) = F_0(l)$. Put $l_j = (1-\omega_j)l$, $m_j = F_0(l_j)$ and $K_j = A^{-1}(l_j)$. Since $l - l_j$ is supported in $[-j+\frac{5}{4}, j-\frac{5}{4}]$, by the properties of the Abel transform in [T, Théorème 6.4] we see that $K - K_j$ is also supported in $[-j+\frac{5}{4}, j-\frac{5}{4}]$ and hence

$$K(x) = K_j(x), \quad \text{if } x > j - \frac{5}{4}. \tag{3.18}$$

LEMMA 3.19. *Suppose that $m \in M(2,N)$ with $N = [\frac{2\alpha+2}{p} - \frac{n}{2}] + 1$, $\alpha \ge 0$ and $0 < p \le 1$. Then for any L with $0 \le L < N - \frac{1}{2}$*

$$\Big\{\int_0^\infty |m_j(\lambda)(1+\lambda)^L|^2\, d\lambda\Big\}^{1/2} \le C_A\|m\|_{M(2,N)} j^{-N}e^{\delta\rho j}, \quad j = 2, 3, \ldots.$$

PROOF. The proof of the lemma is similar to that of (3.17). ∎

For $m \in \mathcal{M}(2, N)$ fix a dyadic decomposition $m = \sum_{k=0}^{\infty} m_k$ and the corresponding decompositions $K = \sum_{k=0}^{\infty} K_k$ and $l = \sum_{k=0}^{\infty} l_k$ where $F(K_k) = F_0(l_k) = m_k$. Choose an even C^∞-function ω^0 such that

$$\omega^0 = \begin{cases} 1, & |x| \le \frac{1}{4}, \\ 0, & |x| \ge \frac{1}{2}. \end{cases}$$

For any positive integer j and $r > 0$ put $l_{kj} := (1-\tilde{\omega}_j) l_k$, $K_{kj} := \mathcal{A}^{-1}(l_{kj})$ and $m_{kj} := F_0(l_{kj})$ where $\tilde{\omega}_j(x) := \omega^0(\frac{x}{2^j r})$. Observe that $l_k - l_{kj}$ is supported in $\{u : |u| \le 2^j r\}$. Using the properties of the Abel transform in [T] we have

$$K_k(x) = K_{kj}(x), \quad \text{if } x > 2^{j-2} r. \tag{3.20}$$

LEMMA 3.21. *Suppose that $m \in \mathcal{M}(2, N)$ with $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$, $\alpha \ge 0$ and $0 < p \le 1$. Given $j \in \mathbf{N}_0$ and $r > 0$ such that $2^j r \le 1$ we have for any nonnegative numbers L_1 and L_2 with $L_2 \le N$*

$$\Big\{ \int_0^\infty |m_{kj}(\lambda)(1+\lambda)^{L_1}|^2 \, d\lambda \Big\}^{\frac{1}{2}} \le \begin{cases} C_{A,m} (2^j r)^{-L_1} 2^{\frac{k}{2}}, & 2^{j+k} r < 1, \\ C_{A,m} (2^j r)^{-L_2} 2^{k(L_1 - L_2 + \frac{1}{2})}, & \text{otherwise}. \end{cases}$$

where $C_{A,m} = C_A \|m\|_{\mathcal{M}(2,N)}$.

PROOF. We only consider the case when $L_1 \le L_2$ and $2^{j+k} r \ge 1$ (the proof of the other cases is similar), and by interpolation we can restrict ourselves to integers L_1 and L_2. Applying properties of the classical Fourier transform and the classical Plancherel theorem we obtain

$$\begin{aligned} \Big\{ \int_0^\infty |m_{kj}(\lambda)(1+\lambda)^{L_1}|^2 \, d\lambda \Big\}^{1/2} &\le C_A \sum_{i=0}^{L_1} \Big\{ \int_0^\infty |l_{kj}^{(i)}(u)|^2 \, du \Big\}^{1/2} \\ &\le C_A \sum_{i=0}^{L_1} \Big\{ \int_{2^{j-3} r}^{2^{j-2} r} |l_{kj}^{(i)}(u)|^2 \, du \Big\}^{1/2} \\ &\quad + C_A \sum_{i=0}^{L_1} \Big\{ \int_{2^{j-2} r}^{\infty} |l_k^{(i)}(u)|^2 \, du \Big\}^{1/2} \\ &:= I_1 + I_2. \end{aligned}$$

Using the definition of l_{kj} and m_{kj} and properties of the classical Fourier transform we have

$$\begin{aligned} I_1 &\le C_A \sum_{i=0}^{L_1} \sum_{n=0}^{i} (2^j r)^{n-i} \Big\{ \int_{2^{j-3} r}^{2^{j-2} r} |l_k^{(n)}(u)|^2 \, du \Big\}^{1/2} \\ &\le C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \sum_{n=0}^{i} \Big\{ \int_{2^{j-3} r}^{2^{j-2} r} |l_k^{(n)}(u) u^{n-i+L_2}|^2 \, du \Big\}^{1/2} \\ &\le C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \sum_{n=0}^{i} \Big\{ \int_0^\infty \Big| \frac{d^{L_2+n-i}}{d\lambda^{L_2+n-i}} \big(\lambda^n m_k(\lambda) \big) \Big|^2 \, d\lambda \Big\}^{1/2} \\ &\le C_A (2^j r)^{-L_2} \sum_{i=0}^{L_1} \Big\{ \int_0^\infty |\lambda^{L-i} m_k^{(N-i)}(\lambda)|^2 \, d\lambda \Big\}^{1/2}. \end{aligned}$$

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Now recall that

$$m_k(\lambda) = m(\lambda)\phi(2^{-k}\lambda) \text{ for } k = 1, 2, \ldots \text{ and } m_0(\lambda) = m(\lambda)\Big(1 - \sum_{k=1}^{\infty} \phi(2^{-k}\lambda)\Big).$$

Hence

$$|m_k^{(L_2-i)}(\lambda)| \le C_A 2^{k(i-L_2)}$$

and

$$I_1 \le C_A \|m\|_{M(2,N)} (2^j r)^{-L_2} 2^{k(L_1-L_2+\frac{1}{2})}.$$

Similarly

$$I_2 \le C_A \|m\|_{M(2,N)} (2^j r)^{-L_2} 2^{k(L_1-L_2+\frac{1}{2})},$$

and this completes the proof of the lemma. ■

We now give a version of Hörmander's multiplier theorem for local Hardy spaces.

THEOREM 3.22. *Suppose that* $m \in M(2,N)$ *with* $N = [\frac{2\alpha+2}{p} - \alpha - 1] + 1$, $\alpha \ge 0$ *and* $0 < p \le 1$. *Then m is a Fourier multiplier for* $\mathbf{h^p}$.

PROOF. By Definition 2.1 and Theorem 2.3 we are reduced to showing that for any (local) (p, ∞, s)-atom a

$$\|T_m a\|_{\mathbf{h^p}} \le C_{A,p} \tag{3.23}$$

where the constant $C_{A,p}$ is independent of a.

Suppose that a is supported in $B(x_0, r)$ with $x_0 \in \mathbf{R}_+$ and $r > 0$. If $r > 1$ then we write

$$\begin{aligned}\|H_0^+(T_m a)\|_{\mathbf{h^p}}^p &= \int_0^{x_0+r+2} |H_0^+(T_m a)(x)|^p A(x)\,dx \\ &\quad + \int_{x_0+r+2}^{\infty} |H_0^+(T_m a)(x)|^p A(x)\,dx \\ &:= I_1 + I_2.\end{aligned}$$

Note that both H_0^+ and T_m are L^2-bounded (see [BX2]) and any (p, ∞, s)-atom must be a (p, q, s)-atom for all $q > p$, $1 \le q < \infty$. Applying Hölder's inequality, Lemma 1.10 and (2.8) then gives

$$\begin{aligned}I_1 &\le C_{A,p} \|H_0^+(T_m a)\|_{2,A}^p e^{2\rho(x_0+r+2)(1-\frac{p}{2})} \\ &\le C_{A,p} \|a\|_{2,A}^p e^{2\rho(x_0+r+2)(1-\frac{p}{2})} \\ &\le C_{A,p} |B(x_0, r)|^{\frac{p}{2}-1} e^{2\rho(x_0+r+2)(1-\frac{p}{2})} \le C_{A,p}.\end{aligned}$$

To estimate I_2 we first observe

$$T_m a * h_t = a * h_t * K(x) = \int_0^{\infty} a(y) T_x(h_t * K)(y) A(y)\,dy.$$

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For $y \in B(x_0, r)$ and $x > x_0 + r + 2$ we have $x - y > 2$. Hence by Lemmas 3.15 and 1.10 we have for $x > x_0 + r + 2$ and $0 < t \le 1$

$$\begin{aligned}|T_m a * h_t(x)| &\le C_{A,p} A(x)^{-\frac{1}{2}} e^{-\delta\rho x} \int_0^\infty |a(y)|(x-y)^{-N} e^{\delta\rho y} A(y)^{\frac{1}{2}}\, dy \\ &\le C_{A,p} A(x)^{-\frac{1}{p}} (x - x_0 - r)^{-N} \|a\|_{2,A} e^{\delta\rho(x_0+r)} \\ &\le C_{A,p} A(x)^{-\frac{1}{p}} (x - x_0 - r)^{-N} e^{(\delta+1-\frac{2}{p})\rho(x_0+r)} \\ &= C_{A,p} A(x)^{-\frac{1}{p}} (x - x_0 - r)^{-N}.\end{aligned}$$

Consequently by the definition of H_0^+

$$I_2 \le C_{A,p} \int_{x_0+r+2}^\infty (x - x_0 - r)^{-Np}\, dx \le C_{A,p}$$

and (3.23) follows for $r > 1$.

We now assume $r \le 1$. As before let ψ be an even C^∞-function such that $\psi(x) = 1$ if $|x| \le \frac{1}{2}$ and $\psi(x) = 0$ if $|x| \ge 1$, and ϕ an even nonnegative C^∞-function supported in $\{x \in \mathbf{R} : \frac{1}{2} < |x| < 2\}$ and satisfying $\sum_{j=-\infty}^{\infty} \phi(2^{-j}x) = 1$ if $x \ne 0$. Write

$$\begin{aligned}T_m a(x) &:= \sum_{j=-\infty}^{\infty} T_m a(x) \phi_j(x) \tilde{\psi}(x) + T_m a(x)\big(1 - \tilde{\psi}(x)\big) \\ &:= (T_m a)_1(x) + (T_m a)_2(x)\end{aligned}$$

where $\phi_j(x) = \phi(\frac{x-x_0}{2^j r})$ and $\tilde{\psi}(x) = \psi(\frac{x-x_0}{4})$. We first prove that $(T_m a)_2$ has an atomic decomposition and then that $(T_m a)_1$ is a $(p, 2, s, \epsilon)$-molecule.

For each $j = 2, 3, \ldots$ let $Q_j = \{x \in \mathbf{R}_+ : j < |x - x_0| \le j+1\}$. Note that $(T_m a)_2(x) = 0$ if $|x - x_0| \le 2$. Hence

$$(T_m a)_2(x) = \sum_{j=2}^{\infty} (T_m a)_2(x) \chi_{Q_j}(x) := \sum_{j=2}^{\infty} b_j(x).$$

Using (3.18), (1.11) and (1.12) together with the cancellation property of an atom we observe for $x \in Q_j$

$$\begin{aligned}T_m a(x) &= \int_0^\infty a(y) T_x K(y) A(y)\, dy \\ &= \int_0^\infty a(y) T_x K_j(y) A(y)\, dy \\ &= \int_0^\infty a(y)(y - x_0)^s \int_0^1 (1-u)^{s-1} F_{j,y,u,s}(x)\, du A(y)\, dy \\ &= \int_0^1 (1-u)^{s-1} \int_0^\infty a(y)(y - x_0)^s F_{j,y,u,s}(x) A(y)\, dy\, du\end{aligned}$$

if $s > 0$, and

$$T_m a(x) = \int_0^\infty a(y) F_{j,y,u,s}(x) A(y)\, dy$$

if $s = 0$, where

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$$F_{j,y,u,s}(x) = \begin{cases} (T_xK_j)^{(s)}\big(x_0 + u(y - x_0)\big) - (T_xK_j)^{(s)}(x_0), & s > 0, \\ T_xK_j(y) - T_xK_j(x_0), & s = 0. \end{cases}$$

By [BH, Theorem 2.2.36] and [BX1, (2.17) and (2.18)]

$$(T_xK_j)^{(k)}(y) = \int_0^\infty m_j(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k)}(y)|c(\lambda)|^{-2}\,d\lambda, \quad k \in \mathbf{N}_0.$$

Thus

$$\hat{F}_{j,y,x_0,u}(\lambda) = \begin{cases} m_j(\lambda)\big[\varphi_\lambda^{(s)}\big(x_0 + u(y - x_0)\big) - \varphi_\lambda^{(s)}(x_0)\big], & s > 0, \\ m_j(\lambda)[\varphi_\lambda(y) - \varphi_\lambda(x_0)], & s = 0. \end{cases}$$

We only consider $s > 0$ (the case $s = 0$ can be handled similarly). Applying Theorems 1.8 and 1.9 gives

$$\begin{aligned} &\|F_{j,y,u,s}\|_{2,A} \\ &= \Big\{\int_0^\infty \Big|m_j(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0 + u(y - x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda)\Big\}^{\frac12} \\ &\le C_A \sum_{u|y-x_0|2^k \le 1} \Big\{\int_0^\infty \Big|\tilde{m}_{jk}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0 + u(y - x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda)\Big\}^{\frac12} \\ &\quad + C_A \sum_{u|y-x_0|2^k > 1} \Big\{\int_0^\infty \Big|\tilde{m}_{jk}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0 + u(y - x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda)\Big\}^{\frac12} \\ &:= \sum\nolimits_1 + \sum\nolimits_2 \end{aligned}$$

where $\tilde{m}_{jk}(\lambda) = m_j(\lambda)\phi_k(\lambda)$, $\phi_0(\lambda) = 1 - \sum_{i=1}^\infty \phi(2^{-i}\lambda)$ and $\phi_k(\lambda) = \phi(2^{-k}\lambda)$. For $x_0 \le 1$ choose $\gamma > 0$ such that $\gamma < \min(N - \frac{2\alpha+2}{p} + \alpha + 1, N - s - \alpha - 1, s - \frac{2\alpha+2}{p} + 2\alpha + 3)$, and in addition $\gamma < N - s - \alpha - 2$ if $N - s - \alpha - 2 > 0$. Then we use Lagrange's mean-value theorem, Theorem 1.9 and Lemmas 1.17 and 3.19 to obtain

$$\begin{aligned} \sum\nolimits_1 &\le C_A u|y - y_0| \sum_{u|y-x_0|2^k \le 1} \Big\{\int_0^\infty |\tilde{m}_{jk}(\lambda)\varphi_\lambda^{(s+1)}(\xi)|^2 |c(\lambda)|^{-2}\,d\lambda\Big\}^{\frac12} \\ &\le C_A u|y - y_0| \Big\{\int_0^\infty |m_j(\lambda)\phi_0(\lambda)(1 + \lambda)^{s+1}|^2 |c(\lambda)|^{-2}\,d\lambda\Big\}^{\frac12} \\ &\quad + C_A u|y - y_0| \sum_{0 < u|y-x_0|2^k \le 1} \Big\{\int_0^\infty |m_j(\lambda)\phi_k(\lambda)(1 + \lambda)^{s+1}|^2 |c(\lambda)|^{-2}\,d\lambda\Big\}^{\frac12} \\ &\le C_A u|y - y_0| \Big\{\int_0^\infty |m_j(\lambda)|^2\,d\lambda\Big\}^{\frac12} \\ &\quad + C_A u|y - y_0| \sum_{0 < u|y-x_0|2^k \le 1} 2^{k(s+\alpha+3-N+\gamma)} \Big\{\int_0^\infty |m_j(\lambda)(1 + \lambda)^{N-\frac12-\gamma}|^2\,d\lambda\Big\}^{\frac12} \\ &\le \begin{cases} C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} u|y - x_0|, & N - s - \alpha - 2 > 0, \\ C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} (u|y - x_0|)^{N-s-\alpha-1-\gamma}, & N - s - \alpha - 2 < 0, \\ C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} (u|y - x_0|)^{1-\gamma}, & N - s - \alpha - 2 = 0 \end{cases} \end{aligned}$$

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and

$$\begin{aligned}\sum\nolimits_2 &\le C_A \sum_{u|y-x_0|2^k>1} \Big\{\int_0^\infty |\tilde{m}_{jk}(\lambda)(1+\lambda)^s|^2 (1+\lambda)^{2\alpha+1}\,d\lambda\Big\}^{1/2} \\ &\le C_A \sum_{0<u|y-x_0|2^k>1} 2^{k(s+\alpha+1-N+\gamma)} \Big\{\int_0^\infty |m_j(\lambda)\phi_k(\lambda)(1+\lambda)^{N-\frac{1}{2}-\gamma}|^2\,d\lambda\Big\}^{1/2} \\ &\le C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} (u|y-x_0|)^{N-s-\alpha-1-\gamma}.\end{aligned}$$

Thus for $x_0 \le 1$ we have by the definition of an atom and (2.8)

$$\begin{aligned}\|b_j\|_{2,A} &\le \int_0^1 (1-u)^{s-1}\Big[\int_0^\infty a(y)(y-x_0)^s \|F_{j,y,u,s}\|_{2,A} A(y)\,dy\Big]\,du \\ &\le C_A j^{-N} e^{-\delta\rho j} \|m\|_{M(2,N)} \\ &\le j^{-N} \|m\|_{M(2,N)} |B(x_0, j+1)|^{\frac{1}{2}-\frac{1}{p}}.\end{aligned}$$

If $x_0 > 1$ then we argue similarly to obtain

$$\sum\nolimits_1 \le \begin{cases} C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} u|y-x_0|, & N-s-\frac{3}{2} > 0, \\ C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (u|y-x_0|)^{N-s-\frac{1}{2}-\gamma_1}, & N-s-\frac{3}{2} < 0, \\ C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (u|y-x_0|)^{1-\gamma_1}, & N-s-\frac{3}{2} = 0 \end{cases}$$

and

$$\sum\nolimits_2 \le C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} A(x_0)^{-\frac{1}{2}} (u|y-x_0|)^{N-s-\frac{1}{2}-\gamma_1}$$

where $\gamma_1 > 0$ is chosen so that $\gamma_1 < \min(N-\frac{1}{p}+\frac{1}{2}, N-s-\frac{1}{2}, s-\frac{1}{p}+2)$, and in addition $\gamma_1 < N-s-\frac{3}{2}$ if $N-s-\frac{3}{2} > 0$. Hence by (2.8)

$$\begin{aligned}\|b_j\|_{2,A} &\le C_A j^{-N} e^{-\delta\rho j}\|m\|_{M(2,N)} A(x_0)^{\frac{1}{2}-\frac{1}{p}} \\ &\le j^{-N}\|m\|_{M(2,N)} |B(x_0, j+1)|^{\frac{1}{2}-\frac{1}{p}}.\end{aligned}$$

Observe that $\operatorname{supp}(b_j) \subset B(x_0, j+1)$. Therefore $a_j := C_A^{-1}\|m\|_{M(2,N)}^{-1} j^N b_j$ is a (local) $(p,2,s)$-atom for each $j = 2, 3, \ldots$ and

$$\begin{aligned}\|(T_m a)_2\|_{\mathbf{h^p}} &\le C_{A,p}\|m\|_{M(2,N)} \Big\{\sum_{j=2}^\infty j^{-Np}\|a_j\|_{\mathbf{h^p}}^p\Big\}^{1/2} \\ &\le C_{A,p}\|m\|_{M(2,N)}.\end{aligned} \tag{3.24}$$

It remains to show that

$$\|(T_m a)_1\|_{\mathbf{h^p}} \le C_{A,p}\|m\|_{M(2,N)} \tag{3.25}$$

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and then (3.23) for $r \le 1$ will follow from (3.24) and (3.25). Let j_0 be the unique non-negative integer such that $1 \le 2^{j_0} r < 2$. Then

$$(T_m a)_1(x) = \sum_{j=-\infty}^{\infty} T_m a(x)\tilde{\psi}(x)\phi_j(x) = \sum_{j=-\infty}^{j_0+2} T_m a(x)\tilde{\psi}(x)\phi_j(x)$$
$$:= \sum_{j=-\infty}^{j_0+2} \tilde{b}_j(x).$$

Fix a dyadic decomposition $m = \sum_{k=0}^{\infty} m_k$ and the corresponding decomposition $K = \sum_{k=0}^{\infty} K_k$ as before. By (3.20), (1.11) and (1.12) and using the moment condition of an atom we observe for $j = 2, 3, \ldots, j_0 + 2$ and $x \in \operatorname{supp}(\phi_j)$

$$\begin{aligned}
\tilde{b}_j(x) &= \tilde{\psi}(x)\phi_j(x) \sum_{k=0}^{\infty} \int_0^{\infty} a(y) T_x K_k(y) A(y)\, dy \\
&= \tilde{\psi}(x)\phi_j(x) \sum_{k=0}^{\infty} \int_0^{\infty} a(y) T_x K_{kj}(y) A(y)\, dy \\
&= \tilde{\psi}(x)\phi_j(x) \int_0^1 (1-u)^{s-1} \Big\{ \int_0^{\infty} a(y)(y-x_0)^s G_{k,j,y,u,s}(x) A(y)\, dy \Big\}\, du
\end{aligned}$$

where

$$G_{k,j,y,u,s}(x) = \begin{cases} (T_x K_{kj})^{(s)}\big(x_0 + u(y-x_0)\big) - (T_x K_{kj})^{(s)}(x_0), & s > 0, \\ T_x K_{kj}(y) - T_x K_{kj}(x_0), & s = 0. \end{cases}$$

Note that $\hat{G}_{k,j,y,u,s}(\lambda) = m_{kj}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0 + u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)$ if $s > 0$ and $\hat{G}_{k,j,y,u,s}(\lambda) = m_{kj}(\lambda)\big(\varphi_\lambda(y) - \varphi_\lambda(x_0)\big)$ if $s = 0$. Hence by Theorems 1.8 and 1.9

$$\begin{aligned}
&\sum_{k=0}^{\infty} \|\tilde{\psi}\phi_j G_{k,j,y,u,s}\|_{2,A} \\
&\quad\le \|G_{k,j,y,u,s}\|_{2,A} \\
&\quad= \sum_{k=0}^{\infty} \Big\{ \int_0^{\infty} \Big| m_{kj}(\lambda) \Big(\varphi_\lambda^{(s)}\big(x_0 + u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0) \Big) \Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
&\quad\le C_A \sum_{u|y-x_0|2^k \le 1} \Big\{ \int_0^{\infty} \Big| m_{kj}(\lambda) \Big(\varphi_\lambda^{(s)}\big(x_0 + u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0) \Big) \Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
&\quad\quad + C_A \sum_{u|y-x_0|2^k > 1} \Big\{ \int_0^{\infty} \Big| m_{kj}(\lambda) \Big(\varphi_\lambda^{(s)}\big(x_0 + u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0) \Big) \Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
&\quad:= J_1 + J_2.
\end{aligned}$$

Assume that $x_0 \le 2r$. Then by Lagrange's mean-value theorem, Theorem 1.9 and Lemmas 1.17 and 3.21 with $L_1 = s + \alpha + \frac{3}{2}$ and $L_2 = N$ we have

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$$
\begin{aligned}
J_1 &\le C_A u|y-x_0| \sum_{u|y-x_0|2^k \le 1} \Big\{ \int_0^\infty |m_{kj}(\lambda)\varphi_\lambda^{(s+1)}(\xi)|^2 |c(\lambda)|^{-2}\, d\lambda \Big\}^{\frac{1}{2}} \\
&\le C_A u|y-y_0| \sum_{u|y-x_0|2^k \le 1} \Big\{ \int_0^\infty |m_{kj}(\lambda)(1+\lambda)^{s+\alpha+\frac{3}{2}}|^2\, d\lambda \Big\}^{\frac{1}{2}} \\
&\le C_A u|y-x_0| \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r \le 1}} (2^j r)^{-s-\alpha-\frac{3}{2}} 2^{\frac{k}{2}} \\
&\quad + C_A u|y-x_0| \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r > 1}} (2^j r)^{-N} 2^{k(s+\alpha+2-N)} \\
&\le C_A u|y-x_0| (2^j r)^{-s-\alpha-2}
\end{aligned}
$$

if $N-s-\alpha-2>0$, and

$$
\begin{aligned}
J_1 &\le u|y-x_0|(2^j r)^{-N} \sum_{u|y-x_0|\le 1} 2^{k(s+\alpha+2-N)} \\
&\le C_A (u|y-x_0|)^{N-s-\alpha-1} (2^j r)^{-N}
\end{aligned}
$$

if $N-s-\alpha-2<0$. For the particular case when $N-s-\alpha-2=0$ we write

$$
\begin{aligned}
J_1 = {}& C_A \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r \le 1}} \Big\{ \int_0^\infty \Big| m_{kj}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0+u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
&+ C_A \sum_{\substack{u|y-x_0|\le 1 \\ 1<2^{k+j} r \le 2^j}} \Big\{ \int_0^\infty \Big| m_{kj}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0+u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
&+ C_A \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r > 2^j}} \Big\{ \int_0^\infty \Big| m_{kj}(\lambda)\Big(\varphi_\lambda^{(s)}\big(x_0+u(y-x_0)\big) - \varphi_\lambda^{(s)}(x_0)\Big)\Big|^2 \sigma(d\lambda) \Big\}^{\frac{1}{2}} \\
:= {}& J_{1,1} + J_{1,2} + J_{1,3}.
\end{aligned}
$$

Now using Theorem 1.9 and Lemmas 1.17 and 3.21 (with $L_1 = s+\alpha+\frac{3}{2}$ and $L_2 = N$ for $J_{1,1}$ and $L_1 = s+\alpha+\frac{1}{2}$ for $J_{1,3}$) we obtain

$$
\begin{aligned}
J_{1,1} &\le C_A u|y-x_0| \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r \le 1}} \Big\{ \int_0^\infty \Big| m_{kj}(\lambda)(1+\lambda)^{s+\alpha+\frac{3}{2}} \Big|^2 d\lambda \Big\}^{\frac{1}{2}} \\
&\le C_A u|y-x_0|(2^j r)^{-N}
\end{aligned}
$$

and

$$
\begin{aligned}
J_{1,3} &\le C_A \sum_{\substack{u|y-x_0|\le 1 \\ 2^{k+j} r > 2^j}} \Big\{ \int_0^\infty \Big| m_{kj}(\lambda)(1+\lambda)^{s+\alpha+\frac{1}{2}} \Big|^2 d\lambda \Big\}^{\frac{1}{2}} \\
&\le C_A r(2^j r)^{-N}.
\end{aligned}
$$

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For $J_{1,2}$ we choose $\beta > 0$ sufficiently small such that $N - \frac{2\alpha+2}{p} + \alpha + 1 - \beta > 0$ and apply Lemma 3.21 with $L_1 = s + \alpha + \frac{3}{2}$ and $L_2 = N - \beta$ to obtain

$$\begin{aligned} J_{1,2} &\le C_A u|y - x_0|(2^j r)^{-N+\beta} \sum_{\substack{u|y-x_0|\le 1\\ 1<2^{k+j}r\le 2^j}} 2^{k\beta} \\ &\le C_A u|y - x_0|(2^j r)^{-N+\beta} r^{-\beta}. \end{aligned}$$

Hence

$$J_1 \le \begin{cases} C_A r(2^j r)^{-s-\alpha-2}, & N - s - \alpha - 2 > 0, \\ C_A r^{1-\beta}(2^j r)^{-N+\beta}, & N - s - \alpha - 2 = 0, \\ C_A r^{N-s-\alpha-1}(2^j r)^{-N}, & N - s - \alpha - 2 < 0. \end{cases}$$

Similarly applying Theorem 1.9 and Lemmas 1.17 and 3.21 (with $L_1 = s + \alpha + \frac{1}{2}$ and $L_2 = N$) we have

$$\begin{aligned} J_2 &\le C_A \sum_{u|y-x_0|2^k>1} \left\{ \int_0^\infty \left| m_{kj}(\lambda)(1+\lambda)^s \right|^2 \sigma(d\lambda) \right\}^{\frac{1}{2}} \\ &\le r^{N-s-\alpha-1}(2^j r)^{-N}. \end{aligned}$$

Therefore by Definition 2.2, (2.8) and Lemma 1.10

$$\|\tilde{b}_j\|_{2,A} \le C_{A,p} 2^{-j\mu_1} |B(x_0, 2^{j+1} r)|^{\frac{1}{2}-\frac{1}{p}} \tag{3.26}$$

where

$$\mu_1 = \begin{cases} s - \frac{2\alpha+2}{p} + 2\alpha + 3, & \text{if } N - s - \alpha - 2 > 0, \\ N - \frac{2\alpha+2}{p} + \alpha + 1, & \text{if } N - s - \alpha - 2 < 0, \\ N - \frac{2\alpha+2}{p} + \alpha + 1 - \beta, & \text{if } N - s - \alpha - 2 = 0 \end{cases}$$

and $j = 2, 3, \ldots, j_0 + 2$. If $x_0 > 2r$ then a similar argument gives

$$\|\tilde{b}_j\|_{2,A} \le C_{A,p} 2^{-j\mu_2} |B(x_0, 2^{j+1} r)|^{\frac{1}{2}-\frac{1}{p}} \tag{3.27}$$

where

$$\mu_2 = \begin{cases} s - \frac{1}{p} + 2, & \text{if } N - s - \frac{3}{2} > 0, \\ N - \frac{1}{p} + \frac{1}{2}, & \text{if } N - s - \frac{3}{2} < 0, \\ N - \frac{1}{p} + \frac{1}{2} - \beta, & \text{if } N - s - \frac{3}{2} = 0 \end{cases}$$

and $j = 2, 3, \ldots, j_0 + 2$.

We now prove that $(T_m a)_1$ is a (local) $(p, 2, s, \epsilon)$-molecule with ϵ satisfying $\max\{\frac{s}{2\alpha+2}, \frac{1}{p} - 1\} < \epsilon < \min\{\frac{s+1}{2\alpha+2}, \frac{N}{2\alpha+2} - \frac{1}{2} - \frac{\beta}{2\alpha+2}\}$. First by Definition 2.2 and the fact that T_m is L^2-bounded we observe

$$\|(T_m a)_1\|_{2,A} \le \|T_m a\|_{2,A} \le \|a\|_{2,A} \le C_{A,p} |B(x_0, r)|^{(\frac{1}{2}-\frac{1}{p})}.$$

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Put $\mu = \mu_1$ if $x_0 \le 2r$ and $\mu = \mu_2$ otherwise. By (3.26), (3.27) and (2.8) we have with $a = 1 - \frac{1}{p} + \epsilon$ and $b = \frac{1}{2} + \epsilon$ as in Definition 2.4.

$$\begin{aligned}
&\Big\|(T_m a)_1(x)\Big|B(x_0,|x-x_0|)\Big|^b\Big\|_{2,A}^{1-\frac{a}{b}} \\
&\qquad \le C_{A,p}\|\sum_{j=-\infty}^{1}\tilde{b}_j\|_{2,A}^{1-\frac{a}{b}}|B(x_0,r)|^{b-a} \\
&\qquad\qquad + C_{A,p}\sum_{j=2}^{j_0+2}\|\tilde{b}_j\|_{2,A}^{1-\frac{a}{b}}|B(x_0,2^{j+1}r)|^{b-a} \\
&\qquad \le C_{A,p}|B(x_0,r)|^{\frac{1}{p}-\frac{1}{2}}\|T_m a\|_{2,A}^{1-\frac{a}{b}} \\
&\qquad\qquad + C_{A,p}\sum_{j=2}^{j_0+2}2^{-j\mu(1-\frac{a}{b})}|B(x_0,2^{j+1}r)|^{\frac{a}{b}(\frac{1}{p}-\frac{1}{2})} \\
&\qquad \le C_{A,p}\sum_{j=1}^{j_0+2}2^{-j\mu(1-\frac{a}{b})}|B(x_0,2^{j+1}r)|^{\frac{a}{b}(\frac{1}{p}-\frac{1}{2})} \\
&\qquad \le \begin{cases} C_{A,p}r^{\frac{a}{b}(\frac{2\alpha+2}{p}-\alpha-1)}\sum_{j=1}^{j_0+2}(2^{-j})^{(1-\frac{a}{b})(\mu-(2\alpha+2)a)}, & x_0 \le 2r, \\ C_{A,p}\big(rA(x_0)\big)^{\frac{a}{b}(\frac{1}{p}-\frac{1}{2})}\sum_{j=1}^{j_0+2}(2^{-j})^{(1-\frac{a}{b})(\mu-(2\alpha+2)a)}, & x_0 > 2r \end{cases} \\
&\qquad \le C_{A,p}|B(x_0,r)|^{\frac{a}{b}(\frac{1}{p}-\frac{1}{2})}.
\end{aligned}$$

Consequently

$$N_2\big((T_m a)_1\big) \le C_{A,p}. \tag{3.28}$$

For any $0 < R \le 1$ and $l = 0, 1, \ldots, s$ we have

$$\int_{B(x_0,R)}(T_m a)_1(x)(x-x_0)^l A(x)\,dx = \int_{B(x_0,R)}T_m a(x)(x-x_0)^l A(x)\,dx.$$

Thus by Lemma 3.12 and Definition 2.4, $(T_m a)_1$ is a $(p,2,s,\epsilon)$-molecule. The estimate (3.25) now follows from (3.28) and Theorem 2.10, and this completes the proof of the theorem. ∎

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Division of Science
Murdoch University
Perth, WA 6150
Australia
e-mail: bloom@murdoch.edu.au
zengfu@murdoch.edu.au