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ON ALMOST-N-CONTINUOUS FUNCTIONS

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Abstract

Recently the class of almost-N-continuous functions between topological spaces has been defined. This paper continues the study of such functions, especially from the point of view of changing the topology on the codomain.

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1. Introduction

In a recent paper, Malghan and Hanchinamani [16] have considered the class of almost-N-continuous functions between topological spaces. A subset B of a topological space (X, τ) is called N*-closed* (relative to τ) if for any cover $\mathscr{U}$ of B by τ- open sets there is a finite subcollection $\mathscr{V}$ of $\mathscr{U}$ such that $B \subset \bigcup\{\tau\mathrm{int}(\tau\mathrm{cl}V) : V \in \mathscr{V}\}$. The concept of N-closed subsets was first considered by Carnahan [1]. The space (X, τ) is nearly compact if and only if X is N-closed relative to τ. A function $f : X \to Y$ is called *almost-*N*-continuous* if for each point $x \in X$ and each regular open set V containing $f(x)$ and having N-closed complement there is an open set U containing x such that $f(U) \subset V$. The basic properties of such mappings are studied in [16].

One purpose of this paper is to emphasize the fact that if the codomain of an almost-N-continuous function f is retopologized in an obvious way then f is simply a continuous function or an almost-c-continuous function [6,9,19]. This puts the notion of almost-N-continuity in a more natural setting, and indicates that the distinction made in [16] between the classes of continuous mappings and almost-N-continuous mappings must be interpreted very strictly.

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In Section 2 we define and study the almost co$N$-closed topology $p(\tau)$ of a topological space (Y, τ). We relate $p(\tau)$ to the co$N$-closed topology $n(\tau)$ of (Y, τ) considered by Mršević and Reilly [17], to $n(\tau_s)$ the co$N$-closed topology of τ_s, where τ_s is the semi-regularization of τ, to the cocompact topology $c(\tau)$ of (Y, τ) considered by Gauld [5], to $c(\tau_s)$ the cocompact topology of τ_s, and to the topology $e(\tau)$ considered by Gauld [6]. Recall that a set B in (Y, τ) is called *regular open* if $B = \tau\operatorname{int}(\tau\operatorname{cl} B)$, and that the family of all regular open sets in (Y, τ), which is denoted by $RO(Y, \tau)$, forms a base for a smaller topology τ_s on Y, called the *semiregularization* of τ. The space (Y, τ) is said to be *semiregular* if $\tau_s = \tau$. A detailed study of the relationship between τ and τ_s is made in Janković [10] and Mršević, Reilly and Vamanamurthy [18]. Section 3 discusses some properties of almost-N-continuous functions. Section 4 is a short collection of results dealing with strongly closed graphs and Section 5 is concerned with the behaviour of almost-N-continuous functions in product spaces.

2. Almost co$N$-closed topologies

Let (Y, τ) be a topological space and consider the collection $p'(\tau)$ of subsets of Y defined by $p'(\tau) = \{U \in RO(Y, \tau) : Y - U \text{ is } N\text{-closed relative to } \tau\}$. Since the intersection of two regular open sets is regular open and the union of two N-closed sets is N-closed, $p'(\tau)$ is a base for a topology $p(\tau)$ on Y, called the *almost* co$N$*-closed topology*. The basic relationship between the topology $p(\tau)$ and the concept of almost-N-continuity is given by the following result. The topology on X is unchanged, so it is not specified. The proof is immediate from the definitions.

LEMMA 1. *The function $f : X \to (Y, \tau)$ is almost-N-continuous if and only if $f : X \to (Y, p(\tau))$ is continuous.*

Thus almost-N-continuity is a $\emptyset$-continuous property in the sense of [7]. Hence all general remarks for $\emptyset$-continuity properties can be applied to almost-N-continuous functions. So, for example, [16, Theorem 2.4 and 2.5] are direct corollaries of [7, Lemma 1 and §6].

Lemma 1 can be used to give elegant proofs of results about almost-N-continuous functions (see Propositions 3 and 4 for example), and this approach can be taken to give alternative proofs of [16, 2.1 and 2.2], for example.

Since $p(\tau) \subset \tau$, the identity function $i : (Y, \tau) \to (Y, p(\tau))$ is continuous and also $i^{-1} : (Y, p(\tau)) \to (Y, \tau)$ is almost-N-continuous. Mršević and Reilly [17] have considered co$N$-closed topologies and their relationship to the N-continuous functions introduced by Malghan and Hanchinamani [15]. For a topological space (Y, τ), the co$N$*-closed topology* of τ on Y is denoted by $n(\tau)$ and has as a base the collection $n'(\tau) = \{U \in \tau : Y - U$ is N-closed relative to $\tau\}$. Gauld [5] has considered cocompact topologies and their relationship to the c-continuous functions introduced by Gentry and Hoyle [8]. For a topological space (Y, τ) the *cocompact topology* of τ on Y is denoted by $c(\tau)$ and defined by $c(\tau) = \{\emptyset\} \cup \{U \in \tau : Y - U$ is τ-compact$\}$. The function $f : X \to Y$ is *c-continuous* if whenever $U \subset Y$ is an open set with compact complement, $f^{-1}(U)$ is open in X. Theorem 1 of Gauld [5] corresponds to our Lemma 1 above.

It is obvious from the definitions that every N-continuous function is almost-N-continuous. The converse does not hold in general (see Example 1).

For a topological space (Y, τ), the topology $e(\tau)$ considered by Gauld [6], has as base the collection $e'(\tau) = \{U \in RO(Y, \tau) : Y - U$ is τ-compact$\}$, and the function $f : X \to Y$ is *almost-c-continuous* if whenever $U \subset Y$ is a regular open set with compact complement, $f^{-1}(U)$ is open in X. For every topological space (Y, τ) each compact set is N-closed, so from the previous definitions we have that every almost-N-continuous function is almost-c-continuous.

For any topological space we have in general that $e(\tau) \subset p(\tau) \subset n(\tau) \subset \tau$.

We observe that $e'(\tau_s) = \{U \in RO(Y, \tau_s) : Y - U$ is τ_s-compact$\} = \{U \in RO(Y, \tau) : Y - U$ is N-closed relative to $\tau\}$, since a subset A of (Y, τ) is N-closed relative to τ if and only if A is compact in (Y, τ_s) [20, Theorem 3.1], and [18, Remark preceding Lemma 5] the family of all regular open subsets of (Y, τ_s) coincides with the collection of all regular open subsets of (Y, τ). Hence $e'(\tau_s) = p(\tau)$ and thus $e(\tau_s) = p(\tau)$. So we have

LEMMA 2. *The function $f : X \to (Y, \tau)$ is almost-N-continuous if and only if the function $f : X \to (Y, \tau_s)$ is almost-c-continuous.*

Obviously from Lemma 2, if the space (Y, τ) is semiregular, then the two notions of generalised continuity are equivalent.

From the above definitions we also have the following implications. The function $f : X \to (Y, \tau)$ is N-continuous implies $f : X \to (Y, \tau)$ is c-continuous which implies $f : X \to (Y, \tau)$ is almost-N-continuous. The following examples show that these implications are not reversible in general.

The diagram below indicates obvious inclusion relations between these topologies.

$$\begin{array}{ccccc} \tau & \longleftarrow & n(\tau) & \longleftarrow & c(\tau) \\ \uparrow & & \uparrow & & \uparrow \\ \tau_s & \longleftarrow & p(\tau) & \longleftarrow & e(\tau) \end{array}$$

We shall see that in general there are no other relations, and will consider conditions under which the inclusions are reversible.

EXAMPLE 1. [9] Let $X = \mathbb{R}$ have the usual topology and let $Y = [0, \infty) \subset \mathbb{R}$ whose topology τ has the sets [0,1], {1}, (r, ∞), with $r > 1$, as its basic open sets.

$$\text{Define } f : X \to Y \text{ to give } \begin{cases} 2, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x < 0. \end{cases}$$

The open subsets of Y containing $f(0)$ are $[1, \infty)$, $\{1\}$, $[0,1]$ and Y. Among them only the sets Y and $[1, \infty)$ have N-closed complements. Since $[1, \infty) \in n(\tau)$, $f^{-1}([1, \infty)) = [0, \infty)$ which is not open in X. Thus, by Theorem 1 of [15], f is not N-continuous. The only regular open subsets of Y with N-closed complement are Y and $[1, \infty)$. The complements of these two sets are also compact in τ_s. So $[1, \infty) \in c(\tau_s)$ and $f^{-1}([1, \infty)) = [0, \infty)$ which is not open in X. Therefore $f : X \to (Y, \tau_s)$ is not c-continuous. But, since $\operatorname{int}(\operatorname{cl}[1, \infty)) = Y$, we have $f(X) \subset \operatorname{int}(\operatorname{cl}[1, \infty))$, and thus f is almost-N-continuous at $x = 0$.

EXAMPLE 2. Let $X = \mathbb{R}$ whose topology τ has all the points in $\mathbb{R}$ isolated except 0, and the only neighbourhood of 0 is $\mathbb{R}$. Let $Y = \mathbb{R}$ have the cofinite topology $\mathcal{U}$. Then $\mathcal{U}_s$ is the indiscrete topology on Y. If $f : (X, \tau) \to (Y, \mathcal{U})$ is a bijection, then for every closed, compact set K in $\mathcal{U}_s$, $f^{-1}(K)$ is closed in τ, and so $f : (X, \tau) \to (Y, \mathcal{U}_s)$ is c-continuous. Let C be an $n(\mathcal{U})$-closed set. Then C is finite and $\mathcal{U}_s$-compact, but $f^{-1}(C)$ is not closed because of the neighbourhood of 0. Thus $f : (X, \tau) \to (Y, \mathcal{U})$ is not N-continuous.

LEMMA 3. *If the space $(Y, p(\tau))$ is Hausdorff, then (Y, τ) is nearly compact and $p(\tau) = \tau_s$.*

PROOF. Let x and y be a pair of distinct points of Y. There are $p(\tau)$-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Hence $Y = U \cup (Y - U) = (Y - V) \cup (Y - U)$, so Y is N-closed as the union of two N-closed sets. Thus (Y, τ) is nearly compact and hence by Theorem 4.1 of [1], (Y, τ_s) is compact. Since $p(\tau) \subset \tau_s$ and $(Y, p(\tau))$ is Hausdorff, (Y, τ_s) is Hausdorff. Hence (Y, τ_s) is minimal Hausdorff. So, we obtain $p(\tau) = \tau_s$.

Note that in Lemma 3 we can obtain (Y, τ) is nearly compact if the space $(Y, p(\tau))$ is either KC or hyperconnected (instead of being Hausdorff). But then we do not obtain the equality between the topologies $p(\tau)$ and τ_s.

COROLLARY 1. *If the space $(Y, p(\tau))$ is Hausdorff, then $c(\tau_s) = \tau_s$.*

PROOF. Since $(Y, p(\tau))$ is Hausdorff, by Lemma 3 (Y, τ) is nearly compact and so (Y, τ_s) is compact. Hence [5, Corollary 3] $c(\tau_s) = \tau_s$.

We recall that a function $f : X \to (Y, \tau)$ is called *almost-continuous* if $f : X \to (Y, \tau_s)$ is continuous [18, Definition 5 and Proposition 12], [24].

COROLLARY 2. *If the space $(Y, p(\tau))$ is Hausdorff, then the function $f : X \to (Y, \tau)$ is almost-N-continuous if and only if f is almost-continuous, if and only if $f : X \to (Y, \tau_s)$ is c-continuous.*

LEMMA 4. *If the space (Y, τ) is Hausdorff, then $p(\tau) \subset n(\tau_s) = c(\tau_s)$.*

PROOF. We know from [18, Lemma 5] that $(\tau_s)_s = \tau_s$, and from [17, Lemma 3] that $n(\tau_s) = c(\tau_s)$. So, we have $p(\tau) \subset n(\tau_s) = c(\tau_s)$.

COROLLARY 3. *If the space (Y, τ) is Hausdorff, then $f : X \to (Y, \tau_s)$ being N-continuous (respectively c-continuous) implies that $f : X \to (Y, \tau)$ is almost-N-continuous.*

LEMMA 5. *If the space $(Y, p(\tau))$ is Hausdorff, then $p(\tau) \subset c(\tau_s) = n(\tau)$.*

PROOF. Let the space $(Y, p(\tau))$ be Hausdorff. Since $p(\tau) \subset n(\tau) \subset \tau$, the space (Y, τ) is Hausdorff and therefore, by Lemma 4 and [17, Lemma 3], we have that $p(\tau) \subset n(\tau_s) = c(\tau_s) = n(\tau)$.

THEOREM 1. *For any topological space (Y, τ), the space $(Y, p(\tau))$ is nearly compact.*

PROOF. Let $\{U_i : i \in I\}$ be any $p(\tau)$-open cover of Y. Let $y \in Y$. Then, there exist an $i_0 \in I$ and a $V \in p'(\tau)$ such that $y \in V \subset U_{i_0}$, since $p'(\tau)$ is a base for the topology $p(\tau)$. But the set $Y - V$ is N-closed relative to τ and so there exists a finite subset K of I such that $Y - V \subset \bigcup\{\tau\mathrm{int}(\tau\mathrm{cl}U_{i_k}) : k \in K\}$. Hence we have $Y = V \cup (Y - V) = U_{i_0} \cup (\bigcup\{\tau\mathrm{int}(\tau\mathrm{cl}U_{i_k}) : k \in K\})$. Since $U_{i_0} \in p(\tau)$, Y is N-closed and hence the space $(Y, p(\tau))$ is nearly compact.

THEOREM 2. *If the space (Y, τ) is nearly compact Hausdorff, then the space $(Y, p(\tau))$ is Hausdorff and $p(\tau) = \tau_s$.*

PROOF. Let y_1 and y_2 be a pair of distinct points in Y. Since (Y, τ) is Hausdorff, there exist disjoint τ-open sets V_1 and V_2 containing y_1 and y_2 respectively. Therefore, we have $(\tau\mathrm{int}(\tau\mathrm{cl}V_1)) \cap (\tau\mathrm{int}(\tau\mathrm{cl}V_2)) = \emptyset$ and $y_i \in \tau\mathrm{int}(\tau\mathrm{cl}V_i)$, $i = 1, 2$. Since (Y, τ) is nearly compact, the set $Y - (\tau\mathrm{int}(\tau\mathrm{cl}V_i))$ is N-closed in Y and so $\tau\mathrm{int}(\tau\mathrm{cl}V_i)$ $(i = 1, 2)$ belongs to $p(\tau)$. Therefore the space $(Y, p(\tau))$ is Hausdorff. Since (Y, τ) is nearly compact Hausdorff, (Y, τ_s) is compact Hausdorff and so (Y, τ_s) is minimal Hausdorff. Hence $\tau_s \subset p(\tau)$ and thus $p(\tau) = \tau_s$.

So, from Theorem 1 and Lemma 3 we have

COROLLARY 4. *The space* (Y, τ) *is nearly compact Hausdorff if and only if the space* $(Y, p(\tau))$ *is Hausdorff and* $p(\tau) = \tau_s$.

COROLLARY 5. *If a function* $f : X \to (Y, \tau)$ *is almost-N-continuous, the space* (Y, τ) *is semi-regular and* $(Y, p(\tau))$ *is Hausdorff, then* f *is continuous.*

PROOF. Let $(Y, p(\tau))$ be Hausdorff. Then, by Lemma 3, (Y, τ) is nearly compact and $p(\tau) = \tau_s$. Since (Y, τ) is semi-regular, $p(\tau) = \tau$ and thus the function $f : X \to (Y, \tau)$ is continuous.

PROPOSITION 1. *If the space* (Y, τ) *is*

(i) *semi-regular, then* $p(\tau) = e(\tau)$,
(ii) *nearly compact Hausdorff, then* $p(\tau) = \tau_s$,
(iii) *locally nearly compact Hausdorff, then* $p(\tau) = c(\tau_s)$.

PROOF. (i) From the definitions and Lemma 2 we have that $p(\tau) = e(\tau_s)$. Since (Y, τ) is semi-regular, $p(\tau) = e(\tau)$.

(ii) See Theorem 2.

(iii) Let (Y, τ) be locally nearly compact Hausdorff. Then, by [18, Theorem 6], (Y, τ_s) is locally compact Hausdorff and thus [6, Proposition 12] $c(\tau_s) = e(\tau_s) = p(\tau)$.

Proposition 1 enables us to obtain conditions on the codomain of a function under which almost-N-continuity can be related to existing variations of continuity.

COROLLARY 6. *Let* $f : X \to (Y, \tau)$ *be a function.*

(i) *If* Y *is semi-regular, then* f *is almost-N-continuous if and only if* f *is almost-c-continuous.*
(ii) *If* Y *is nearly compact Hausdorff, then* f *is almost-N-continuous if and only if* f *is almost-continuous.*
(iii) *If* Y *is locally nearly compact Hausdorff, then* f *is almost-N-continuous if and only if* $f : X \to (Y, \tau_s)$ *is c-continuous.*

PROPOSITION 2. *If the space* (Y, τ) *is nearly compact, then*

$$p(\tau) \subset c(\tau_s) = \tau_s \subset n(\tau).$$

PROOF. By [17, Proposition 1(ii)], we have that $p(\tau) \subset c(\tau_s) = \tau_s \subset n(\tau)$.

COROLLARY 7. *If the space* (Y, τ) *is nearly compact, then* $f : X \to (Y, \tau_s)$ *is c-continuous (almost-continuous) implies that* $f : X \to (Y, \tau)$ *is almost-N-continuous.*

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3. Properties of almost-N-continuous functions

PROPOSITION 3. *If $f : X \to (Y, \tau_f)$ is a quotient map and $g : (Y, \tau_f) \to (Z, \mathscr{U})$ is an almost-N-continuous function, then $g \circ f$ is an almost-N-continuous function.*

PROOF. Let τ_f be the quotient topology on Y. Then $f : X \to (Y, \tau_f)$ is continuous. Since $g : Y \to (Z, \mathscr{U})$ is almost-N-continuous, $g : Y \to (Z, p(\mathscr{U}))$ is continuous and so $g \circ f : X \to (Z, p(\mathscr{U}))$ is continuous. Thus $g \circ f : X \to (Z, \mathscr{U})$ is almost-N-continuous.

Recall that a set A in (X, τ) is called *locally dense* if $A \subset \text{int}(\text{cl}A)$ [2].

PROPOSITION 4. *If $f : X \to (Y, \tau)$ is an almost-N-continuous function and $A \subset X$ is such that $f(A)$ is locally dense in Y with N-closed complement, then $f/A : A \to f(A)$ is almost-N-continuous.*

PROOF. Let $f(A)$ be locally dense in Y and U be a regular open set in $f(A)$ with N-closed complement in $f(A)$. Then by [18, Lemma 4] and [10, Lemma 2], $U = f(A) \cap \text{int}(\text{cl}U)$. Since $Y - f(A)$ is N-closed in τ, $Y - f(A)$ is compact in τ_s[20] and so $f(A) - U$ is compact in $(\tau/f(A))_s$. By [18, Lemma 4], $(\tau/f(A))_s = \tau_s/f(A)$ and thus $f(A) - U$ is compact in τ_s, that is, N-closed in τ. Then $Y - \text{int}(\text{cl}U)$ is N-closed in τ. But, by [16, Theorem 2.2], $f^{-1}(Y - \text{int}(\text{cl}U))$ is closed in X, and since $f^{-1}(Y - \text{int}(\text{cl}U)) = X - f^{-1}(\text{int}(\text{cl}U))$, $f^{-1}(\text{int}(\text{cl}U))$ is open in X. But $(f/A)^{-1}(U) = A \cap f^{-1}(\text{int}(\text{cl}U))$ and thus $(f/A)^{-1}(U)$ is an open set in A.

THEOREM 3. *If the function $f : X \to (Y, \tau)$ is almost-N-continuous and the subset A of X is compact in X, then $f(A)$ is a compact set in the space $(Y, p(\tau))$.*

PROOF. Since $f : X \to (Y, \tau)$ is almost-N-continuous, $f : X \to (Y, p(\tau))$ is continuous and so $f(A)$ is compact in the space $(Y, p(\tau))$.

COROLLARY 8. *If the function $f : X \to (Y, \tau)$ is almost-N-continuous and the subset A of X is N-closed in X, then $f(A)$ is N-closed in $(Y, p(\tau))$.*

We recall that a function $f : (X, \mathscr{U}) \to (Y, \tau)$ is called *almost-open* if $f : (X, \mathscr{U}_s) \to (Y, \tau)$ is open [24], [18, Definition 6].

THEOREM 4. *If $f : X \to (Y, \tau)$ is an almost-continuous, almost-open bijection and (Y, τ) is Hausdorff, then f^{-1} is almost-N-continuous.*

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PROOF. Let C be a regular closed, N-closed subset of X. Then by [1, Theorem 2.9], $f(C)$ is N-closed in (Y, τ) and so $f(C)$ is compact in (Y, τ_s). Since (Y, τ) is Hausdorff, (Y, τ_s) is Hausdorff and $f(C)$ is closed in (Y, τ_s). Hence $f(C)$ is closed in (Y, τ). But $f(C) = (f^{-1})^{-1}(C)$ and thus [16, Theorem 2.2], f^{-1} is almost-N-continuous.

A space (X, τ) is called *δ-compact* [23], if every countable cover consisting of τ_s-open sets admits a finite subcover. It is known [23, Theorem 2], that a space (X, τ) is δ-compact if and only if (X, τ_s) is countably compact. So we can give the following:

THEOREM 5. *If f is an almost-N-continuous function from a first countable space X to a locally nearly compact, δ-compact, Hausdorff space Y, then f is almost-continuous.*

PROOF. Let $f : X \to (Y, \tau)$ be an almost-N-continuous function, let X be a first countable space and let Y be a locally nearly compact, δ-compact, Hausdorff space. Then by Lemma 2, $f : X \to (Y, \tau_s)$ is almost-c-continuous and [18, Theorem 6], (Y, τ_s) is locally compact, countably compact, Hausdorff and hence, by Theorem 10 of Hwang [9], $f : X \to (Y, \tau_s)$ is continuous, and so $f : X \to (Y, \tau)$ is almost-continuous.

It is known from Kuratowski [11, page 103] that the set of all points of X at which the function $f : X \to (Y, \tau)$ is not continuous is the set $D(f) = \bigcup\{\mathrm{cl}(f^{-1}(V)) - \mathrm{int}(f^{-1}(V)) : V \in B_y\}$, where B_y is a base for the topology τ. By our Lemma 1, this result can be extended to almost-N-continuous functions, as follows:

THEOREM 6. *The set of all points of X at which $f : X \to (Y, \tau)$ is not almost-N-continuous is the set $D_{al\text{-}N}(f) = \bigcup\{\mathrm{cl}(f^{-1}(V)) - \mathrm{int}(f^{-1}(V)) : V \in p'(\tau)\}$.*

A space (X, τ) is said to be *almost-regular* [22] if for each point $x \in X$ and each regular open set V containing x, there exists a regular open set U such that $x \in U \subset \mathrm{cl}U \subset V$.

A space (X, τ) is defined to be *saturated* [12] if any intersection of open sets is open.

The next result improves Theorem 7 of Gentry and Hoyle [8].

THEOREM 7. *Let X be a saturated space and let Y be a locally nearly compact, almost regular space. If $f : X \to (Y, \tau)$ is almost-N-continuous, then f is almost-continuous.*

PROOF. Let $x \in X$ and let O be a regular open subset of Y containing $f(x)$. Since Y is almost regular, there exists a regular open set U such that $f(x) \in U \subset \mathrm{cl}U \subset$

$O = \text{int}(\text{cl}O)$. Let $y \in Y - \text{cl}U$. Since Y is almost regular $\text{cl}U$ is regular closed, and so there exists an open set V_y containing y such that $V_y \cap \text{cl}U = \emptyset$. Also, Y is locally nearly compact and so there exists an open set C_y containing y such that $\text{cl}C_y$ is N-closed relative to τ and $\text{cl}C_y \cap U = \emptyset$. Since $\text{cl}C_y = \text{cl}(\text{int}(\text{cl}C_y))$, the set $Y - \text{cl}C_y$ is regular open, contains $f(x)$ and its complement is N-closed. Hence, by the almost-N-continuity of the function f, there exists an open set N_y containing x such that $f(N_y) \subset Y - \text{cl}C_y = \text{int}(\text{cl}(Y - \text{cl}C_y))$. Suppose $N = \bigcap\{N_y : y \in Y - \text{cl}U\}$. Since the space X is saturated, N is open and $x \in N$. Thus $f(x) \in f(N) \subset \text{cl}U \subset O$. Therefore, f is almost-continuous.

Lemma 2 enables us to improve Theorem 3.1(d) of Singh and Prasad [25] as the following result shows.

THEOREM 8. *Let X be a saturated space and let Y be a locally nearly compact Hausdorff space. If $f : X \to (Y, \tau)$ is almost-N-continuous, then f is almost-continuous.*

PROOF. Let (Y, τ) be a locally nearly compact Hausdorff space. Then by [18, Theorem 6], (Y, τ_s) is locally compact Hausdorff. Since from Lemma 2, $f : X \to (Y, \tau_s)$ is almost-c-continuous, then by [25, Theorem 3.1(d)], $f : X \to (Y, \tau_s)$ is continuous and hence $f : X \to (Y, \tau)$ is almost-continuous.

4. Strongly-closed graphs

Let $f : X \to Y$ be a function. The subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$ is called the *graph* of f and usually denoted by $G(f)$.

DEFINITION 1 [14]. The graph $G(f)$ is said to be *strongly-closed*, if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $[U \times \text{cl}(V)] \cap G(f) = \emptyset$.

The following lemma is a useful characterization of functions with strongly-closed graphs.

LEMMA 6 [14]. *The graph $G(f)$ is strongly-closed if and only if for each $(x, y) \notin G(f)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y respectively, such that $f(U) \cap \text{cl}(V) = \emptyset$.*

Of course, a function with a strongly closed graph has a closed graph. We recall that if a function $f : X \to Y$ has a closed graph, then the inverse image of every compact set is closed [4, Theorem 3.6]. Now, we have a variation of [15, Theorem 15].

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PROPOSITION 5. *If the function* $f : X \to (Y, \tau_s)$ *has closed graph, then* $f : X \to (Y, \tau)$ *is* N*-continuous.*

PROOF. Let $G(f)$ be closed and let K be an $n(\tau)$-closed subset of Y. Then K is closed, N-closed relative to τ. So K is closed in τ and compact in τ_s. Hence by [4, Theorem 3.6], $f^{-1}(K)$ is closed in X and thus [15, Theorem 1] $f : X \to (Y, \tau)$ is N-continuous.

THEOREM 9. *If the function* $f : X \to (Y, \tau)$ *has strongly closed graph, then* $f : X \to (Y, \tau)$ *is almost-*N*-continuous.*

PROOF. Let $G(f)$ be strongly-closed and let K be a regular closed, N-closed subset of Y. Suppose $x \notin f^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(f)$, and so, by Lemma 6, there exist open sets $Uy(x) \subset X$ and $V(y) \subset Y$ such that $x \in Uy(x)$, $y \in V(y)$ and $f(Uy(x)) \subset \mathrm{cl}V(y) = \emptyset$. But the collection $\{V(y) : y \in K\}$ is an open cover of K and, since K is N-closed, there exists a finite subset K_0 of K such that $K \subset \bigcup\{\mathrm{int}(\mathrm{cl}V(y)) : y \in K_0\} \subset \bigcup\{\mathrm{cl}V(y) : y \in K_0\}$. Let $U = \bigcap\{Uy(x) : y \in K_0\}$. Then U is an open set in X containing x and $U \cap f^{-1}(K) = \emptyset$. This shows that $f^{-1}(K)$ is a closed set of X and so, from [16, Theorem 2.2] f is almost-N-continuous.

The converse of the above theorem does not hold as we can see from the following example.

EXAMPLE 3. Suppose that $f : (X, \mathscr{U}) \to (Y, \tau)$ are the function and the spaces of Example 2. We proved in Example 2 that f is almost-N-continuous at $x = 0$. One can easily see that the graph $G(f)$ is not strongly closed.

Using our Lemma 2 and [18, Theorem 6], we can obtain [16, Theorem 3.2 and Corollaries 3.3 and 3.5] as corollaries of the corresponding results for almost-c-continuous functions [19, Theorem 3.4, Corollaries 3.5 and 3.7]. Also, we can improve the result of Hwang [9, Theorem 8] and Singh and Prasad [25, Theorem 1.5(*d*)] and provide a partial converse of Proposition 5 at the same time.

THEOREM 10. *Let* $f : X \to (Y, \tau)$ *be an almost-*N*-continuous function and let* Y *be a locally nearly compact Hausdorff space. Then* $f : X \to (Y, \tau_s)$ *has closed graph.*

5. Product Spaces

We recall that a subset S of a space X is said to be *quasi-*H*-closed* [21], if for every cover $\{V_a : a \in \Delta\}$ of open sets of X, there exists a finite subfamily Δ_0 of Δ

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such that $S \subset \bigcup\{\mathrm{cl}(V_a) : a \in \Delta_0\}$. A function $f : X \to Y$ is called *H-continuous* [13] if for each $x \in X$ and each open neighbourhood V of $f(x)$ such that $Y - V$ is quasi-H-closed, there exists an open neighbourhood U of x such that $f(U) \subset V$. Let $\{Y_a : a \in \mathscr{A}\}$ be any family of topological spaces and $Y = \prod\{Y_a : a \in \mathscr{A}\}$ denote the product space. It is known [19, Theorem 4.1] and [15, Theorem 19] that if Y_a is a locally compact Hausdorff (respectively locally nearly compact Hausdorff) space and $f_a : X \to Y_a$ is an almost-c-continuous (respectively N-continuous) function for each $a \in \mathscr{A}$, then the function $f : X \to Y$, defined by $f(x) = \{f_a(x)\}$ for each $x \in X$, is H-continuous (respectively N-continuous).

The following result generalizes the theorems stated above.

THEOREM 11. *If Y_a is a locally nearly compact Hausdorff space and $f_a : X \to Y_a$ is an almost-N-continuous function for each $a \in \mathscr{A}$, then the function $f : X \to Y$ defined by $f(x) = \{f_a(x)\}$ for each $x \in X$, is almost-N-continuous.*

PROOF. For each $b \in \mathscr{A}$ we have that Y_b is locally nearly compact Hausdorff and $f_b : X \to Y_b$ is almost-N-continuous, so by [16, Theorem 3.2] $G(f_b)$ is strongly closed. Hence, by Lemma 20 of [15], $G(f)$ is strongly closed and so, from Theorem 9, f is almost-N-continuous.

COROLLARY 9. *If X is Hausdorff, Y is locally nearly compact Hausdorff and $f : X \to Y$ is almost-N-continuous, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is almost-N-continuous.*

PROOF. The identity function $i_x : X \to X$ is continuous and X is Hausdorff. Then $G(i_x)$ is strongly closed [14]. Since f is almost-N-continuous and Y is locally nearly compact Hausdorff, $G(f)$ is strongly closed [16, Theorem 3.2]. Hence, from [15, Lemma 20], the graph function g has strongly closed graph and so, from Theorem 9, g is almost-N-continuous.

Also, it is known [19, Theorem 4.3] that if Y_a is a locally compact Hausdorff space and $f_a : X_a \to Y_a$ is an almost-c-continuous function for each $a \in \mathscr{A}$, then the function $f : \prod X_a \to \prod Y_a$, defined by $f(\{x_a\}) = \{f_a(x_a)\}$ for each $\{x_a\} \in \prod X_a$, is H-continuous.

Our final result generalizes the above theorem.

THEOREM 12. *If Y_a is a locally nearly compact Hausdorff space and $f_a : X_a \to Y_a$ is an almost-N-continuous function for each a in $\mathscr{A}$, then the function $f : \prod X_a \to \prod Y_a$, defined by $f(\{x_a\}) = \{f_a(x_a)\}$ for each $\{x_a\} \in \prod X_a$, is almost-N-continuous.*

PROOF. Let $(x, y) \notin G(f)$. Then $y \neq f(x)$ and there exists $b \in \mathscr{A}$ such that $y_b \neq f_b(x)$. Since Y_b is locally nearly compact Hausdorff and f_b is almost-N-continuous, from Theorem 3.2 of [16], $G(f_b)$ is strongly-closed. Thus, by Lemma 6, there exist open sets $U_b \subset X_b$ and $V_b \subset Y_b$ containing x_b and y_b, respectively, such that $f_b(U_b) \cap \mathrm{cl}(V_b) = \emptyset$. Let $U = U_b \times \prod_{a \neq b} X_a$ and $V = V_b \times \prod_{a \neq b} Y_a$. Then U and V are open sets containing x and y, respectively, such that $f(U) \cap \mathrm{cl}(V) = \emptyset$. Therefore $G(f)$ is strongly-closed and so, from Theorem 9, f is almost-N-continuous.

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