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MAZUR'S INTERSECTION PROPERTY OF BALLS FOR COMPACT CONVEX SETS

J.H.M. WHITFIELD AND V. ZIZLER

We show that every compact convex set in a Banach space X is an intersection of balls provided the cone generated by the set of all extreme points of the dual unit ball B_1^* of X^* is dense in X^* in the topology of uniform convergence on compact sets in X. This allows us to renorm every Banach space with transfinite Schauder basis by a norm which shares the mentioned intersection property.

It was proved by Phelps in [5] that for a finite dimensional Banach space X the set of all extreme points of the dual unit ball B_1^* is dense in the unit sphere $S_1^* \subset X^*$ if and only if X has the following property called here property *(CI)* :

every compact convex set G in X is an intersection of closed balls.

We extend the necessity part of this result to general Banach spaces (Theorem 1), by using significantly ideas of Giles, Gregory and Sims in [3]. We then prove that every Banach space with a transfinite Schauder

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basis can be equivalently renormed to have the *(CI)* property (Theorem 2). This shows that property *(CI)* is quite a weak condition on X.

It should be pointed out that the research in this area originated with Mazur [4].

In this note Banach spaces will be considered to be real spaces and balls will be assumed closed. If G is a compact convex symmetric subset of a Banach space X, then $||f||_G$ will denote the seminorm on X^* defined by $||f||_G = \sup f(G)$. The C-topology on X^* will mean the topology of uniform convergence on compact sets in X. For a set $A \subset X^*$, the closure of A in the C-topology will be denoted by $C\text{-}c\ell A$. If $A \subset X$, then $\overline{cs}A$ means the closed convex symmetric hull of A in X. A slice of the unit ball B_1 is a nonempty intersection of B_1 with an open halfspace. If x is an element of the unit sphere $S_1 \subset X$, then $D(x) = \{f \in B_1^* ; f(x) = 1\}$, where B_1^* stands for the dual unit ball of X^*. If $A \subset S_1$, then $D(A) = \bigcup_{x \in A} D(x)$. The elements in $D(x)$ will be denoted by f_x, The set of all positive integers is denoted by $\mathbb{N}$. If G is a compact convex symmetric set in X and $A \subset X^*$, then the G-diam A means $\sup\{||f-g||_G, f,g \in A\}$. If $A \subset S_1^* \subset X^*$, then the cone generated by A is the set $\{ta, t > 0, a \in A\}$.

We will use the following "compact" version of a Definition in [3], [8]:

DEFINITION 1. If G is a compact convex symmetric set in a Banach space X and $\varepsilon > 0$, we say that a point $x \in S_1 \subset X$ belongs to the set $M_{G,\varepsilon}$ if there is a $\delta > 0$ such that

$$\sup_{\substack{y \in G \\ 0 < t < \delta}} \frac{||x+ty|| + ||x-ty|| - 2}{t} < \varepsilon .$$

With this definition we have, similarly to [3],

LEMMA 1. *Let G be a compact convex symmetric set in a Banach space X, $x \in S_1$, $\varepsilon > 0$. Then the following statements are equivalent*

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(i) $x \in M_{G,\varepsilon}$

(ii) *there is a* $\delta > 0$ *such that*

$$G\text{-}diam\{f \in B_1^* \, , \, f(x) > 1-\delta\} < \varepsilon$$

(iii) *there is a* $\delta > 0$ *such that*

$$G\text{-}diam\{\cup D(z), \, z \in S_1 \, , \, \|z-x\| < \delta\} < \varepsilon$$

Proof. An easy adjustment of that for Lemma 2.1 in [3]. We omit it.

LEMMA 2. *Let* X *be a Banach space,* G *be a compact convex symmetric subset of* X*,* f *be an extreme point of* $B_1^* \subset X_1^*$, $\varepsilon > 0$. *Then*

$$f \in C - c\ell D(M_{G,\varepsilon}) \, .$$

Proof. Since B_1^* is w^*-compact and the restricted C-topology on B_1^* coincides with the restricted w^*-topology, the Theorem on page 107 in [2] asserts that slices determined by functionals from X form a neighbourhood base of f in the restricted C-topology on B_1^* . It means that if $\eta \in (0,\varepsilon)$ and G_1 is a compact set in X , then there is an $x \in S \subset X$ and $\delta > 0$ such that if

$$S = \{g \in B_1^* \, ; \, g(x) > 1-\delta\} \quad \text{and} \quad G_0 = \overline{cs}(G \cup G_1) \, ,$$

then

(i) $f \in S$

(ii) G_0-diam $S < \eta$.

Then $x \in M_{G_0,\eta}$ by Lemma 1. Plainly, $M_{G_0,\eta} \subset M_{G,\varepsilon}$. Furthermore, $\|f_x - f\|_{G_0} < \eta$ for any $f_x \in D(x)$ since such an $f_x \in S$.

Therefore

$$\sup(|(f-f_x)(y)|, \, y \in G_1) < \eta$$

and it follows that $f \in C\text{-}c\ell \, D(M_{G,\varepsilon})$. □

THEOREM 1. *Let* X *be a Banach space. Suppose that the cone* K *generated by the set* E *of all extreme points of the dual unit ball* $B_1^* \subset X^*$ *is dense in* X^* *in the topology of uniform convergence on compact sets in* X . *Then* X *has property (CI)* .

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Proof. An adjustment of that of Lemma 2.2 in [3] ((i) => ii)). We are to show that if for some $f \in S_1^*$ and some compact convex set $G \subset X$ we have $\inf f(G) > 0$, then there is a ball $B \subset X$ such that $B \supset G$ and $0 \notin B$.

Let $\varepsilon = \frac{1}{5} \inf f(G)$ and $G_0 = \overline{cs}\, G$.

Since $C - c\ell\, K = X^*$, there is an $h \in E$ and $t > 0$ such that

$$||f - th||_{G_0} < \varepsilon .$$

By using Lemma 1, we have that there is an $x \in M_{G_0, \varepsilon/t}$ and $f_x \in D(X)$ such that

$$||h - f_x||_{G_0} < \frac{\varepsilon}{t} .$$

Consider the sequence of balls B_n : B_n is centred at $\frac{n\varepsilon}{t} x$ and has radius $\frac{n-1}{t} \varepsilon$; $n = 2,3,\ldots$.

Since no B_n contains 0, it is enough to show that for some $n \in N$, $B_n \subset G$.

Suppose otherwise and choose $x_n \in G \setminus B_n$, $n = 2,3,\ldots$. Let

$$t_n = \frac{t}{n\varepsilon} .$$

Then

$$\frac{\|x + t_n x_n\| + \|x - t_n x_n\| - 2}{t_n} = \frac{\|x + t_n x_n\| - 1}{t_n} + \|x_n - \frac{1}{t_n} x\| - \frac{1}{t_n}$$

$$\geq f_x(x_n) + \frac{n-1}{t} \varepsilon - \frac{n\varepsilon}{t}$$

$$\geq \frac{1}{t} f(x_n) - \|h - \frac{1}{t} f\|_{G_0} - \|h - f_x\|_{G_0} - \frac{\varepsilon}{t}$$

$$\geq \frac{5\varepsilon}{t} - \frac{3\varepsilon}{t} = \frac{2\varepsilon}{t} .$$

Since $\lim t_n = 0$ and $x_n \in G \subset G_0$, we have a contradiction with

$$x \in M_{G_0, \varepsilon/t} .$$

Thus Theorem 1 is proved. □

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DEFINITION 2. (see for example [1]). Let X be a Banach space. Let us call a system S_α, where the α are ordinals, $1 \le \alpha \le \gamma$, of continuous projections of X a transfinite Schauder-Bessaga basis if

(i) $S_1 = 0$, $S_\gamma =$ Identity ;

(ii) $S_\alpha S_\beta = S_\beta S_\alpha = S_\alpha$ if $\alpha \le \beta$;

(iii) for every $x \in X$, the function $\alpha \to S_\alpha x$ is continuous on ordinals (we use the norm topology on X) ;

(iv) $\dim(S_{\alpha+1} - S_\alpha)X = 1$ for $1 \le \alpha < \gamma$.

Before proceeding, let us notice that it follows from (iii) in Definition 2, from the compactness of the segment $[1,\gamma]$ of ordinals and from the Banach Steinhauss uniform boundedness principle, that $\sup_\alpha \|S_\alpha\| < \infty$.

LEMMA 3. *Let X be a Banach space with a transfinite Schauder-Bessaga basis $\{S_\alpha\}$, $1 \le \alpha \le \gamma$. Let H be the norm closed linear hull of $\bigcup_{1\le\alpha<\gamma} (S^*_{\alpha+1} - S^*_\alpha)X^*$. Then $C\text{-}\ell H = X^*$.*

Proof. Given $f \in X^*$, we prove by transfinite induction that $S^*_\alpha f \in C\text{-}c\ell H$ for every $1 \le \alpha \le \gamma$.

$S^*_1 f = 0 \in C\text{-}c\ell H$. If $S^*_\beta f \in C\text{-}c\ell H$ for all $\beta < \alpha$ and if $\alpha = \beta + 1$ for some $\beta < \alpha$, then

$$S^*_\alpha f = S^*_\beta f + (S^*_{\beta+1} - S^*_\beta)f \in C\text{-}c\ell H$$

since both summands do. If α is a limiting ordinal, then it follows from (iii) in Definition 2 that

$$S^*_\alpha f = \lim_{\substack{\beta\to\alpha \\ \beta<\alpha}} S^*_\beta f$$

in the w^* topology and since $\sup \|S^*_\alpha\| < \beta$, also in the C-topology,

$$S^*_\alpha f \in C\text{-}c\ell\ H .$$

THEOREM 2. *Let X be a Banach space with a transfinite Schauder-Bessaga basis $\{S_\alpha\}$, $1 \le \alpha \le \gamma$. Then there is an equivalent norm on X*

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which has property (CI).

Proof. For $1 \le \alpha \le \gamma$, choose $e_\alpha \in (S_{\alpha+1} - S_\alpha)X$, $\|e_\alpha\| = 1$. For simplicity, denote the set of ordinals $1 \le \alpha < \gamma$ by Γ. Consider the map T of X^* into $\ell_\infty(\Gamma)$ defined by

$$Tf(\alpha) = f(e_\alpha) \quad \text{for} \quad \alpha \in \Gamma .$$

Then T is bounded, linear and continuous with respect to w^*-topologies of X^* and $\ell_\infty(\Gamma)$. If for some $f \in X^*$, $f(e_\alpha) = 0$ for every $\alpha \in \Gamma$, then it follows easily by transfinite induction that $S_\alpha^* f = 0$ for every $0 \le \alpha \le \gamma$. Therefore T is a 1-1 map.

Furthermore, if H is the norm closed linear hull of $\bigcup_{0 \le \alpha < \gamma} (S_{\alpha+1}^* - S_\alpha^*)X^*$, then T maps H into $c_0(\Gamma)$. This follows from the orthogonality of $(S_{\alpha+1} - S_\alpha)$ and $(S_{\alpha'+1} - S_{\alpha'})$ for $\alpha \ne \alpha'$ (see iii) in Definition 2) and from the boundedness of T.

Let us introduce a dual equivalent norm on X^* by

$$\|f\|_1^2 = \|f\|^2 + \|Tf\|_D^2 ,$$

where $\|f\|$ is the original dual norm on X^* and $\|\cdot\|_D$ is Day's norm on $\ell_\infty(\Gamma)$ (see for example [6],[7]).

It is well known (see [7] or examine the proof in[6]) that Day's norm is an equivalent norm on $\ell_\infty(\Gamma)$ which is locally uniformly convex at every point $x \in c_0(\Gamma)$ in the following sense:

Whenever $x \in c_0(\Gamma)$ and $x_n \in \ell_\infty(\Gamma)$ are such that

$$\lim_n 2\|x\|_D^2 + 2\|x_n\|_D^2 - \|x + x_n\|_D^2 = 0 ,$$

then $\lim \|x - x_n\|_D = 0$.

From this property of $\|\cdot\|_D$ and from standard convexity arguments it follows that if $f \in H$, $f_n \in X^*$ are such that

$$\lim_n 2\|f\|_1^2 + 2\|f_n\|_1^2 - \|f + f_n\|_1^2 = 0 ,$$

then

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$$\lim_n \|Tf - Tf_n\|_D = 0 .$$

This implies that $\lim_n (f_n - f)e_\alpha = 0$ for every $\alpha \in \Gamma$.

This in particular means that if B_1^* and S_1^* denote the unit ball and the unit sphere of the new dual equivalent norm $\|f\|_1$ on X^*, then every point of $H \cap S_1^*$ is an extreme point of B_1^*. Therefore the cone K generated by the set of all extreme points of this new B_1^* contains H. By using Lemma 3,

$$C\text{-}c\ell\ K \supset C\text{-}c\ell\ H = X^*$$

and Theorem 1 may be used to finish the proof of Theorem 2. □

Let us finish the paper by noticing that as in [4], [5], [3], property *(CI)* has an interesting application stated here as the following

PROPOSITION 1. *Let S be a Banach space with property (CI). Then a sequence $\{x_n\} \subset X$ is norm convergent to $x \in X$ if and only if*

(i) $\{x_n\}$ is relatively norm compact and

(ii) every closed ball in X which contains infinitely many points of $\{x_n\}$ also contains x.

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Department of Mathematical Sciences
Lakehead University
Thunder Bay, Ontario
Canada P7B 5E1

Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

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