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ALMOST COMPLEX STRUCTURES ON THE ORTHOGONAL TWISTOR BUNDLE

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We give a construction of 2^s, $s = n(n-1)/2$, many natural almost complex structures on the orthogonal twistor bundle over a $2n$-dimensional Riemannian manifold. The usual almost complex structures are then characterised by the condition that they correspond to integrable invariant complex structures on the standard fibre which is identified with the hermitian symmetric space $SO(2n)/U(n)$.

0. INTRODUCTION

Recently, much attention has been paid to the investigation of the (orthogonal) twistor bundle, $\mathcal{T} \to N$, over an even-dimensional Riemannian manifold N and its application to the theory of harmonic maps. See, for example, [**2, 3**] and many references cited therein. A crucial aspect of the twistor method is the construction of the two almost complex structures, denoted by J_+ and J_- ([**3**] calls them J_1 and J_2), on the total space of the twistor bundle. J_-, which is never integrable, plays an important rôle in the study of conformal minimal immersions from a Riemann surface M into N. When the dimension of the target manifold N happens to be four, the twistor method is particularly effective and [**3**] proved the following result.

A conformal immersion $f\colon M \to N$ is minimal if and only if its twistor lift $T_f\colon M \to \mathcal{T}$ is J_--complex. (The twistor lift of an immersion $f\colon M \to N$ is obtained by composing the Gauss map $M \to G_2(TN)$, the Grassmann bundle of oriented tangent two-planes of N, with the projection $G_2(TN) \to \mathcal{T}$.)

In this paper we give a systematic account of the construction of almost complex structures on $\mathcal{T}$. In particular, we show that there are not two but 2^γ, with $\gamma = n(n-1)/2$ ($\dim N = 2n$), many *natural* almost complex structures on $\mathcal{T}$ (Section 2 (16)). With the possible exception of J_+ none of these almost complex structures are ever integrable. However, these structures are still quite useful in manufacturing new harmonic maps from a Riemann surface into N.

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1. The Hermetian Symmetric Space $SO(2n)/U(n)$

Let $i\colon U(n) \to SO(2n)$ be the Lie group monomorphism induced by the identification

$$\mathbb{R}^{2n} \leftrightarrow \mathbb{C}^n, \qquad (x^\alpha) \leftrightarrow (x^1 + ix^2, \dots, x^{2n-1} + ix^{2n}).$$

We then have

$$(1) \qquad i(U(n)) = \{X \in SO(2n) : {}^tXJ_nX = J_n\},$$

where $J_n = J_1 \oplus \dots \oplus J_1 = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_1 \end{bmatrix}$, $J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Convention. We will identify $U(n)$ with $i(U(n))$ and write $U(n)$ instead of $i(U(n))$, $\mathfrak{u}(n)$ instead of $i_{\star_e}(\mathfrak{u}(n))$, and so on.

At the Lie algebra level we have

$$(2) \qquad i_{\star_e}(\mathfrak{u}(n)) = \mathfrak{u}(n) = \{X \in \mathfrak{o}(2n) : {}^tXJ_n + J_nX = 0\}.$$

Let T be a maximal torus in $SO(2n)$ given by

$$T = \{\operatorname{diag}(D_1, \dots, D_n) : D_i = \begin{pmatrix} c_i & -s_i \\ s_i & c_i \end{pmatrix},\ c_i = \cos 2\pi x_i,\ s_i = \sin 2\pi x_i\} = SO(2)^n.$$

Notation. We identify $SO(2)$ with $U(1)$ and write e^{ix} (or $\exp(ix)$) instead of $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$.

We have the Lie algebra composition

$$\mathfrak{o}(2n) = \mathfrak{t} \oplus \mathfrak{m},$$

where $\mathfrak{t}$ denotes the Lie algebra of T and $\mathfrak{m}$ is the orthogonal complement to $\mathfrak{t}$ relative to the Killing form.

Let $E_{ij} = [e_{mn}]$, $F_{ij} = [f_{mn}]$, $E'_{ij} = [e'_{mn}]$ and $F'_{ij} = [f'_{mn}]$ be $2n \times 2n$ matrices with all entries zero except for

$$\begin{aligned} e_{2i-1,\,2j-1} &= e_{2i,\,2j} = -e_{2j-1,\,2i-1} = -e_{2j,\,2i} = 1, \\ -f_{2i-1,\,2j} &= f_{2i,\,2j-1} = -f_{2j-1,\,2i} = f_{2j,\,2i-1} = 1, \\ e'_{2i-1,\,2j-1} &= -e'_{2i,\,2j} = -e'_{2j-1,\,2i-1} = e'_{2j,\,2i} = 1, \\ f'_{2i-1,\,2j} &= f'_{2i,\,2j-1} = -f'_{2j-1,\,2i} = -f'_{2j,\,2i-1} = 1. \end{aligned}$$

Then

$$\mathfrak{m} = \sum V_{ij} \oplus V'_{ij} \qquad (1 \leqslant i < j \leqslant n), \tag{4}$$

where $V_{ij} = \mathbb{R} - \operatorname{span}\{E_{ij}, F_{ij}\}$ and $V'_{ij} = \mathbb{R} - \operatorname{span}\{E'_{ij}, E'_{ij}\}$.

For $t = \operatorname{diag}(D_1, \dots, D_n) \in T$, $v = xE_{ij} + yF_{ij} \in V_{ij}$, and $v' = xE'_{ij} + yF'_{ij} \in V'_{ij}$ we compute that

$$\begin{aligned} Ad_t \colon v \leftrightarrow x + iy &\mapsto \exp\left(2\pi i(x_i - x_j)\right) \cdot (x + iy), \\ v' \leftrightarrow x + iy &\mapsto \exp\left(2\pi i(x_i + x_j)\right) \cdot (x + iy), \end{aligned} \tag{5}$$

where we use the complex notation to write v, v' relative to their respective bases. (5) means that the root spaces of $SO(2n)$ relative to T are V_{ij} and V'_{ij} and the corresponding roots are

$$\Delta = \{\pm(x_i - x_j),\ \pm(x_i + x_j) : 1 < i < j \leqslant n\}.$$

A routine computation then reveals that

$$\mathfrak{u}(n) = \mathfrak{t} \oplus \sum V_{ij}. \tag{6}$$

It follows that

$$\mathfrak{n} = \oplus \sum V'_{ij} \tag{7}$$

is the orthogonal complement to $\mathfrak{u}(n)$ in $\mathfrak{o}(2n)$ relative to the Killing form.

Via π_{*_e} ($\pi \colon SO(2n) \to SO(2n)/U(n)$) $\mathfrak{n}$ is identified with the tangent space at the identity coset of $SO(2n)/U(n)$.

$$\mathfrak{n} \leftrightarrow T_0(SO(2n)/U(n)), \qquad 0 = U(n).$$

Let $\Omega = \left(\Omega^\alpha_\beta\right)$, $1 \leqslant \alpha,\ \beta \leqslant 2n$, denote the Maurer-Cartan form of $SO(2n)$. The decompositions in (3, 6, 7) induce the decompositions

$$\Omega = \Omega_{\mathfrak{u}(n)} + \Omega_{\mathfrak{n}}, \qquad \Omega_{\mathfrak{n}} = \sum \Omega_{V'_{ij}}.$$

We compute that

$$\Omega_{V'_{ij}} = \frac{1}{2}[\left(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}\right) \otimes E'_{ij} + \left(\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j}\right) \otimes F'_{ij}], \text{ (no sum)}. \tag{8}$$

Put

$$\Theta'^{ij} = \frac{1}{2}[\left(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}\right) + \varepsilon_{ij} i\left(\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j}\right)], \text{ (no sum)} \tag{9}$$

where $\varepsilon_{ij} = \pm 1$ (but fixed).

PROPOSITION. *$SO(2n)/U(n)$ posseses exactly 2^γ, $\gamma = n(n-1)/2$, many invariant almost complex structures.*

PROOF: See [1, p.501] or [8, Chapter II, Section 9]. ∎

The invariant almost complex structures on $SO(2n)/U(n)$ can be described as follows: Choose $\{\varepsilon_{ij} = +1 \text{ or } -1 : 1 \leqslant i < j \leqslant n\}$ and let the pullbacks (by a local section of $SO(2n) \to SO(2n)/U(n)$) of (Θ'^{ij}) span type (1,0) forms on $SO(2n)/U(n)$. The invariant almost complex structure on $SO(2n)/U(n)$ corresponding to the choice $\{\varepsilon_{ij}\}$ will be denoted by

$$\oplus \sum \varepsilon_{ij} J_1, \qquad 1 \leqslant i < j \leqslant n. \tag{10}$$

Letting $\varepsilon_{ij} = 1$ for every i and j we obtain an integrable invariant almost complex structure. $SO(2n)/U(n)$ has exactly one other integrable invariant almost complex structure which is the conjugate structure, corresponding to the choice $\varepsilon_{ij} = -1$ for every i and j.

Any invariant metric on $SO(2n)/U(n)$ is given by the pullback of the symmetric product

$$c \cdot \sum \Theta'^{ij} \cdot \bar{\Theta}'^{ij}, \qquad c > 0. \tag{11}$$

Let $\langle , \rangle$ denote the standard inner product in $\mathbb{R}^{2n}$ and put

$$\mathcal{F} = \{J \in \mathrm{Aut}^+(\mathbb{R}^{2n}) : J^2 = -I, \quad \langle v, w \rangle = \langle Jv, Jw \rangle\}, \tag{12}$$

where $\mathrm{Aut}^+(\mathbb{R}^{2n})$ denotes the set of orientation preserving automorphisms of $\mathbb{R}^{2n}$. $\mathcal{F}$ is the set of all orientation preserving orthogonal complex structures of the vector space $\mathbb{R}^{2n}$.

Let $A = (A_\alpha) = (A_1, \dots, A_{2n}) \in SO(2n)$ and consider the assignment

$$A \mapsto J_A \in \mathcal{F} \tag{13}$$

given by the formula

$$J_A(A_{2i-1}) = A_{2i} \text{ and } J_A(A_{2i}) = -A_{2i-1}, \qquad 1 \leqslant i \leqslant n;$$

that is, the matrix of J_A relative to the basis (A_α) is just J_n. This assignment induces a bijection

$$SO(2n)/U(n) \xrightarrow{\cong} \mathcal{F} \tag{14}$$

as it is easily seen that $J_A = J_B$ if $B = A \cdot U$ for some $U \in U(n)$.

2. The orthogonal twistor bundle

Let N denote a connected oriented $2n$-dimensional Riemannian manifold and also let

$$\pi\colon SO(N) \to N \tag{1}$$

denote the $SO(2)$-principal bundle of oriented orthonormal frames over N.

The $\mathbb{R}^{2n}$-valued canonical form, denoted by $\Theta = (\Theta^\alpha)$, on $SO(N)$ is given by

$$\Theta(X) = e^{-1}(\pi_* X), \qquad X \in T_{(x,e)}SO(N)$$

where e is identified with a linear map $\mathbb{R}^{2n} \to T_x N$.

We have

$$d\Theta^\alpha = -\Omega^\alpha_\beta \wedge \Theta^\beta, \tag{3}$$

where $\Omega = \left(\Omega^\alpha_\beta\right)$, $1 \leqslant \alpha,\ \beta \leqslant 2n$, is the $\mathfrak{o}(2n)$-valued Levi-Civita connection form on $SO(N)$.

Definition. Put $\mathcal{T} = \{\,(x, J) : x \in N,\ J$ is an orientation preserving orthogonal complex structure of the vector space $T_x N\,\}$. $\pi_2\colon \mathcal{T} \to N$, $(x, J) \mapsto x$, is called the *orthogonal twistor bundle over* N. ($\mathcal{T} \to N$ depends only on the conformal structure of N.)

Consider the projection

$$\pi_1\colon SO(N) \to \mathcal{T}, \qquad (x; e_1, \dots, e_{2n}) \mapsto (x; J_e), \tag{4}$$

where $J_e\colon e_{2i-1} \mapsto e_{2i}$, $e_{2i} \mapsto e_{2i-1}$.

$$\begin{array}{ccc} SO(N) & \xrightarrow{\pi_1} & \mathcal{T} \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_2} & \\ & N & \end{array}$$

Take $x \in N$ and fix an oriented orthonormal basis $\delta = (\delta_1, \dots, \delta_{2n})$ of $T_x N$. For any $e = (e_\alpha) \in SO(N)_x = \pi^{-1}(x)$ we write $e_\alpha = e^\beta_\alpha \delta_\beta$ and obtain the identification

$$SO(N)_x \xrightarrow{\cong} SO(2n), \qquad e \mapsto (e^\beta_\alpha). \tag{5}$$

Likewise the isomorphism $\delta\colon T_x N \to \mathbb{R}^{2n}$, $\delta_\alpha \mapsto (\partial/\partial x^\alpha)$, induces an identification

$$\mathcal{T}_x \xrightarrow{\cong} \mathcal{F} = SO(2n)/U(n). \tag{6}$$

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Summing up the preceding discussion we have

$$(7)\qquad \mathcal{T} = SO(N) \times_{SO(2n)} SO(2n)/U(n) = SO(N)U(n),$$

and $\pi_1 : SO(N) \to \mathcal{T}$ is a principal $U(n)$-fibration.

Recall from Section 1 the Lie algebra decomposition

$$(8)\qquad \mathfrak{o}(2n) = \mathfrak{u}(n) \oplus \mathfrak{n}, \qquad \mathfrak{n} = \oplus \sum V'_{ij} \qquad (1 \leqslant i < j \leqslant n).$$

The $\mathfrak{o}(2n)$-valued connection form Ω on $SO(N)$ decomposes accordingly,

$$(9)\qquad \Omega = \Omega_{\mathfrak{u}(n)} \oplus \Omega_{\mathfrak{n}}.$$

So,

$$(10)\qquad \Omega_{\mathfrak{n}} = \sum \Omega_{V'_{ij}} \qquad (1 \leqslant i < j \leqslant n),$$

where

$$\Omega_{V'_{ij}} = \frac{1}{2}[(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}) \otimes E'_{ij} + (\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j}) \otimes F'_{ij}], \text{ (no sum)}.$$

Put

$$(11)\qquad \Theta'^{ij} = \frac{1}{2}[(\Omega^{2i-1}_{2j-1} - \Omega^{2i}_{2j}) + \varepsilon_{ij} i(\Omega^{2i}_{2j-1} + \Omega^{2i-1}_{2j})].$$

Define a symmetric bilinear form on $SO(N)$ by

$$(12)\qquad B = {}^t\Theta \cdot \Theta + c \cdot \sum \Theta'^{ij} \cdot \bar{\Theta}'^{ij}, \qquad c > 0.$$

We then easily verify that: (i) $R_g^* B = B$, where $g \in U(n)$ and R_g denotes the right multiplication by g on $SO(N)$; (ii) $B(v, w) = 0$ if one of v, w is a vertical vector of the fibration $SO(N) \to \mathcal{T}$. Consequently, there exists a unique Riemannian metric, denoted by $ds^2_{\mathcal{T}}$ on $\mathcal{T}$ such that

$$(13)\qquad \pi_1^* ds^2_{\mathcal{T}} = B.$$

Remark. $(SO(N), B) \to (\mathcal{T}, ds^2_{\mathcal{T}})$ is a Riemannian submersion with totally geodesic fibres.

Let $(x, J) \in \mathcal{T}$ and $V_{(x,J)}$ denote the subspace of $T_{(x,J)}\mathcal{T}$ tangential to the fibre $\pi_2^{-1}(x)$. We also let $H_{(x,J)} \subset T_{(x,J)}\mathcal{T}$ denote the orthogonal complement to $V_{(x,J)}$ with

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respect to the metric $ds_{\mathcal{T}}^2$. ($H_{(x,J)}$ is well-defined independent of the choice of c in (12).)

$$T_{(x,J)}\mathcal{T} = H_{(x,J)} \oplus V_{(x,J)}. \tag{14}$$

The distribution defined by $H_{(x,J)}$, $(x, J) \in \mathcal{T}$, will be called the ***horizontal distribution*** of $\mathcal{T} \to N$.

π_{2*} gives an isomorphism $H_{(x,J)} \xrightarrow{\cong} T_xN$. Now J is a complex structure on T_xN, hence via the above isomorphism J defines a complex structure on $H_{(x,j)}$. On the other hand, $\pi_2^{-1}(x) \xrightarrow{\cong} SO(2n)/U(n)$ and an invariant almost complex structure on $SO(2n)/U(n)$ gives rise to a complex structure on $V_{(x,J)}$. More precisely, choose an oriented orthonormal frame at $x \in N$ and use it to identify T_xN with $\mathbf{R}^{2n}$. This identification induces another identification $\pi_2^{-1}(x) \leftrightarrow \mathcal{F} = SO(2n)/U(n)$. Thus $V_{(x,J)}$ is identified with the tangent space at some point of $SO(2n)/U(n)$ and translating this tangent space to the identity coset of $SO(2n)/U(n)$ we obtain the identification

$$V_{(x,J)} \leftrightarrow \mathfrak{n} = \mathfrak{u}(n)^{\perp} \subset \mathfrak{o}(2n).$$

An invariant almost complex structure on $SO(2n)/U(n)$ by restriction defines a complex structure on a vector space $\mathfrak{n}$, and hence one on $V_{(x,J)}$. Taking the direct sum of this "vertical" complex structure with the canonical complex structure defined on $H_{(x,J)}$ above at every $(x, J) \in \mathcal{T}$ we obtain an almost complex structure on $\mathcal{T}$. To put it another way, we obtain an almost complex structure on $\mathcal{T}$ by decreeing that the 1-forms on $SO(N)$ given by

$$\Phi^i = \Theta^{2i-1} + i\Theta^{2i} (1 \leqslant i \leqslant n), \qquad \Theta'^{ij} (1 \leqslant i < j \leqslant n) \tag{15}$$

pull back (by a local section) to give type (1,0) forms on $\mathcal{T}$. This almost complex structure will be denoted by

$$J_H \oplus \sum \varepsilon_{ij} J_1. \tag{16}$$

Summing up, we have

PROPOSITION. *There are 2^γ, $\gamma = n(n-1)/2$, many **natural** almost complex structures on $\mathcal{T} \to N$, where $\dim N = 2n$.*

Example. Let $N = S^4$. Then $SO(N) = SO(5)$ and

$$\mathcal{T} = SO(5)/U(2) \xrightarrow{\cong} Sp(2)/U(1) \times Sp(1) \xrightarrow{\cong} \mathbf{C}P^3.$$

The usual complex structure on $\mathbf{C}P^3$ corresponds to $J_H \oplus \sum \varepsilon_{12} J_1$ with $\varepsilon_{12} = 1$.

Notation. $J_+ = J_h \oplus \varepsilon_{ij} J_1$ with every $\varepsilon_{ij} = 1$; $J_- = J_h \oplus \sum \varepsilon_{ij} J_1$ with every $\varepsilon_{ij} = -1$.

Remark. The almost complex structures J_+ and J_- have been studied extensively by various authors; for example, see [**2, 3, 4, 5, 6, 7**]. However, the remaining almost complex structures on $\mathcal{T}$, to the author's knowledge, have not been explored.

PROPOSITION. *With the possible exception of J_+ the above almost complex structures on $\mathcal{T}$ are never integrable.*

PROOF: Exterior differentiate the forms Φ^i and Ω'^{ij}. ∎

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