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Orbital Integrals on p-Adic Lie Algebras

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Abstract. Let G be a connected reductive p-adic group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{O}$ be any G-orbit in $\mathfrak{g}$. Then the orbital integral $\mu_{\mathcal{O}}$ corresponding to $\mathcal{O}$ is an invariant distribution on $\mathfrak{g}$, and Harish-Chandra proved that its Fourier transform $\hat{\mu}_{\mathcal{O}}$ is a locally constant function on the set $\mathfrak{g}'$ of regular semisimple elements of $\mathfrak{g}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and ω is a compact subset of $\mathfrak{h} \cap \mathfrak{g}'$, we give a formula for $\hat{\mu}_{\mathcal{O}}(tH)$ for $H \in \omega$ and $t \in F^\times$ sufficiently large. In the case that $\mathcal{O}$ is a regular semisimple orbit, the formula is already known by work of Waldspurger. In the case that $\mathcal{O}$ is a nilpotent orbit, the behavior of $\hat{\mu}_{\mathcal{O}}$ at infinity is already known because of its homogeneity properties. The general case combines aspects of these two extreme cases. The formula for $\hat{\mu}_{\mathcal{O}}$ at infinity can be used to formulate a "theory of the constant term" for the space of distributions spanned by the Fourier transforms of orbital integrals. It can also be used to show that the Fourier transforms of orbital integrals are "linearly independent at infinity."

1 Introduction

Let F be a p-adic field of characteristic zero. Let G be the set of F-rational points of a connected reductive group defined over F, and let $\mathfrak{g}$ be its Lie algebra. For $X \in \mathfrak{g}$, let $\mathcal{O} = \mathcal{O}_X$ denote the G-orbit of X, and let $\mu_{\mathcal{O}}$ denote the orbital integral corresponding to $\mathcal{O}$, so that

$$(1.1) \qquad \mu_{\mathcal{O}}(f) = \int_{G/G_X} f(x \cdot X)\, dx^*, \quad f \in C_c^\infty(\mathfrak{g}).$$

Here G_X denotes the centralizer of X in G and dx^* is an invariant measure on G/G_X. Let B denote a symmetric, nondegenerate, G-invariant bilinear form on $\mathfrak{g}$, and fix an additive character ψ of F. Then we have the Fourier transform

$$(1.2) \qquad \hat{f}(X) = \int_{\mathfrak{g}} f(Y)\psi\big(B(X,Y)\big)\, dY, \quad f \in C_c^\infty(\mathfrak{g}).$$

The distribution $\hat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\hat{f})$, $f \in C_c^\infty(\mathfrak{g})$, is the Fourier transform of the orbital integral. Harish-Chandra [2] proved that it is a locally constant function on $\mathfrak{g}'$, the set of regular semisimple elements of $\mathfrak{g}$.

For $X \in \mathfrak{g}$, let $\eta_{\mathfrak{g}}(X)$ denote the coefficient of t^l in the polynomial $\det(t - \operatorname{ad} X)$, where t is an indeterminate and l is the rank of $\mathfrak{g}$. Then $\mathfrak{g}' = \{X \in \mathfrak{g} : \eta_{\mathfrak{g}}(X) \neq 0\}$. For any G-orbit $\mathcal{O}$ in $\mathfrak{g}$, we normalize $\hat{\mu}_{\mathcal{O}}$ by defining

$$(1.3) \qquad \Phi(\mathfrak{g}, \mathcal{O}, X) = |\eta_{\mathfrak{g}}(X)|^{1/2}\hat{\mu}_{\mathcal{O}}(X), \quad X \in \mathfrak{g}'.$$

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Harish-Chandra [2] proved that the normalized Fourier transform $\Phi(\mathfrak{g}, \mathcal{O})$ is locally bounded on $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since there is a unique semisimple orbit $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O})$, $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O}) = \mathfrak{h} \cap \mathcal{O}_0$ is a finite set, possibly empty. For $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$, let $\mathfrak{g}_Y$ denote the centralizer of Y in $\mathfrak{g}$ and let G_Y denote the centralizer of Y in G. Then $\mathfrak{g}_Y$ is reductive, and there is a unique nilpotent G_Y-orbit of $\mathfrak{g}_Y$, which we denote by $\xi_Y(\mathcal{O})$, such that $Y+\xi_Y(\mathcal{O}) \subset \mathcal{O}$. Note that $Y + \xi_Y(\mathcal{O})$ is also a G_Y-orbit in $\mathfrak{g}_Y$. Let $\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O})\big)$ and $\Phi\big(\mathfrak{g}_Y, Y + \xi_Y(\mathcal{O})\big)$ denote the normalized Fourier transforms of the orbital integrals on $\mathfrak{g}_Y$ corresponding to the G_Y-orbits $\xi_Y(\mathcal{O})$ and $Y + \xi_Y(\mathcal{O})$ respectively. They are functions on $\mathfrak{g}_Y'$, and hence on $\mathfrak{h}' \subset \mathfrak{g}_Y'$. Further, since Y is central in $\mathfrak{g}_Y$, it is easy to see that for all $X \in \mathfrak{g}_Y'$,

$$\Phi\big(\mathfrak{g}_Y, Y + \xi_Y(\mathcal{O}), X\big) = \psi\big(B(Y, X)\big)\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), X\big). \tag{1.4}$$

The main result of this paper is the following theorem.

Theorem 1.1 *Let $\mathcal{O}$ be any G-orbit in $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then given any compact subset ω of $\mathfrak{h}'$, there is $C > 0$ so that for all $H \in \omega$ and $t \in F^\times$ such that $|t| \geq C$,*

$$\Phi(\mathfrak{g}, \mathcal{O}, tH) = \sum_{Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})} \Phi\big(\mathfrak{g}_Y, Y + \xi_Y(\mathcal{O}), tH\big) c_Y(tH).$$

Here for each $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$, $c_Y : \mathfrak{h}' \to \mathbf{C}$ is a locally constant function (independent of ω) satisfying

(i) $|c_Y(H)|$ is non-zero and independent of $H \in \mathfrak{h}'$;
(ii) $c_Y(t^2H) = c_Y(H)$ for all $t \in F^\times$, $H \in \mathfrak{h}'$.

Remark 1.1 The functions $\Phi(\mathfrak{g}, \mathcal{O})$ and $\Phi\big(\mathfrak{g}_Y, Y + \xi_Y(\mathcal{O})\big)$ are only determined up to constants which depend on the choices of invariant measures on the orbits $\mathcal{O}$ and $\xi_Y(\mathcal{O})$, $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$. In Section 2 we will normalize these measures consistently. When we do this, the functions c_Y in Theorem 1.1 are independent of $\mathcal{O}$. That is, given $Y \in \mathfrak{h}$, we use the same function c_Y in the expansion of $\Phi(\mathfrak{g}, \mathcal{O})$ for any orbit $\mathcal{O}$ such that $Y \in \mathrm{cl}(\mathcal{O})$.

Suppose that $\mathcal{O}$ is a regular semisimple orbit. Then it is closed, so that $\mathrm{cl}(\mathcal{O}) = \mathcal{O}$. Further, for all $Y \in \mathfrak{h} \cap \mathcal{O}$, $\mathfrak{g}_Y = \mathfrak{h}$ is abelian, so that $\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O})\big) \equiv 1$. Thus in this case, the equation in Theorem 1.1 reduces to

$$\Phi(\mathfrak{g}, \mathcal{O}, tH) = \sum_{Y \in \mathfrak{h} \cap \mathcal{O}} \psi\big(B(Y, tH)\big) c_Y(tH), \quad H \in \omega,\ |t| \geq C. \tag{1.5}$$

This result was proven by Waldspurger in [10].

Suppose on the other hand that $\mathcal{O}$ is a nilpotent orbit. Then for any Cartan subalgebra $\mathfrak{h}$, $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O}) = \{0\}$, and for $Y = 0$, $\mathfrak{g}_Y = \mathfrak{g}$ and $\xi_Y(\mathcal{O}) = \mathcal{O}$. Thus in this case, the equation in Theorem 1.1 reduces to

$$\Phi(\mathfrak{g}, \mathcal{O}, tH) = \Phi(\mathfrak{g}, \mathcal{O}, tH) c_0(tH), \quad H \in \omega,\ |t| \geq C. \tag{1.6}$$

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Of course $c_0(H) \equiv 1$ and the theorem gives no information. However, it follows from the homogeneity property of nilpotent orbital integrals (see Section 3.1 of [2]), that for all $X \in \mathfrak{g}'$ and $t \in F^\times$,

$$\Phi(\mathfrak{g}, \mathcal{O}, t^2X) = |t|^{d_0(\mathcal{O})}\Phi(\mathfrak{g}, \mathcal{O}, X) \tag{1.7}$$

where

$$d_0(\mathcal{O}) = \dim \mathfrak{g} - \dim \mathcal{O} - \operatorname{rank} \mathfrak{g} = \dim \mathfrak{g}_{X_0} - \operatorname{rank} \mathfrak{g} \tag{1.8}$$

Here $\mathfrak{g}_{X_0}$ denotes the centralizer in $\mathfrak{g}$ of a representative $X_0 \in \mathcal{O}$. Thus the dependence of $\Phi(\mathfrak{g}, \mathcal{O}, t^2H)$ on t is already known in this case.

In the general case, let $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$. Then it follows from (1.7) applied to $\mathfrak{g}_Y$ and $\xi_Y(\mathcal{O})$ that

$$\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), t^2X\big) = |t|^{d_0(\mathcal{O})}\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), X\big), \quad X \in \mathfrak{g}_Y',\ t \in F^\times. \tag{1.9}$$

Here $d_0(\mathcal{O})$ is again defined using (1.8) since if $Z \in \xi_Y(\mathcal{O})$, then $X_0 = Y + Z \in \mathcal{O}$ and $\mathfrak{g}_{X_0} = (\mathfrak{g}_Y)_Z$, rank $\mathfrak{g} = $ rank $\mathfrak{g}_Y$. Thus in general, for $H \in \omega$ and $|t^2| \geq C$, we can write

$$\Phi(\mathfrak{g}, \mathcal{O}, t^2H) = |t|^{d_0(\mathcal{O})} \sum_{Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})} \psi\big(B(Y, t^2H)\big)\Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big)c_Y(H). \tag{1.10}$$

This shows the dependence on t of $\Phi(\mathfrak{g}, \mathcal{O}, t^2H)$ precisely for large t.

Let K be any compact open subgroup of G and let $X \in \mathfrak{g}$. Define

$$T_K(X, H) = \int_{G/G_X} \int_K \psi\big(B(k \cdot H, x \cdot X)\big)\, dk\, dx^*, \quad H \in \mathfrak{g}', \tag{1.11}$$

where dx^* is an invariant measure on G/G_X and dk is normalized Haar measure on K. Then it follows from Lemma 7.1 of [2] in the regular semisimple case and Theorem 3 of [3] for the general case, that the above integral converges. Further, if $\mathcal{O} = \mathcal{O}_X$, and dx^* is the invariant measure on G/G_X used to define $\mu_{\mathcal{O}}$ in (1.1), then

$$\hat{\mu}_{\mathcal{O}}(H) = T_K(X, H), \quad H \in \mathfrak{g}'. \tag{1.12}$$

To prove Theorem 1.1 we show that if $X \in \mathfrak{g}$ and ω is a compact subset of $\mathfrak{h}'$, then we can evaluate $T_K(X, t^2H)$, $H \in \omega$, $t \in F^\times$, if K is small enough and t is large enough.

The expansion at infinity of $\Phi(\mathfrak{g}, \mathcal{O})$ given in Theorem 1.1 can be used to develop a "theory of the constant term" as follows. Since in the Lie algebra we can go to infinity in any direction, we have "constant terms" corresponding to each Cartan subalgebra instead of constant terms corresponding to split components of Cartan subgroups as in the group case.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For any integer $d \geq 0$, we let $\mathcal{C}(\mathfrak{h}, d)$ denote the set of all measureable functions $f : \mathfrak{h} \to \mathbf{C}$ which are locally constant on $\mathfrak{h}'$ and satisfy

$$f(t^2H) = |t|^d f(H), \quad t \in F^\times,\ H \in \mathfrak{h}'. \tag{1.13}$$

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Let $\mathcal{A}(\mathfrak{h})$ denote the set of all measureable functions $f\colon \mathfrak{h} \to \mathbf{C}$ of the form

$$f(H) = \sum_{Y,d} \psi\big(B(Y,H)\big) f_{Y,d}(H), \quad H \in \mathfrak{h}, \tag{1.14}$$

where Y runs over a finite set of elements in $\mathfrak{h}$, d runs over a finite set of non-negative integers, and $f_{Y,d} \in \mathcal{C}(\mathfrak{h}, d)$ for all Y, d.

Let $f_i\colon \mathfrak{h} \to \mathbf{C}$, $i = 1, 2$, be measureable functions which are locally constant on $\mathfrak{h}'$. Then we say $f_1 \sim_{\mathfrak{h}} f_2$ if given any compact subset ω of $\mathfrak{h}'$ there is $C > 0$ so that $f_1(tH) = f_2(tH)$ for all $H \in \omega$ and $t \in F^{\times}$ such that $|t| \geq C$. We prove in Proposition 6.3 that functions in $\mathcal{A}(\mathfrak{h})$ are uniquely determined by their behavior at infinity. That is, if $f \in \mathcal{A}(\mathfrak{h})$ with $f \sim_{\mathfrak{h}} 0$, then $f = 0$.

Let $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$, and for $H \in \mathfrak{h}'$, define

$$\Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}, Y, H) = \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big) c_Y(H); \tag{1.15}$$

$$\Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}, H) = \sum_{Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})} \psi\big(B(Y,H)\big) \Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}, Y, H). \tag{1.16}$$

It follows from property (ii) of the functions $c_Y(H)$ in Theorem 1.1 and (1.9) that for all $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$, $\Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}, Y) \in \mathcal{C}\big(\mathfrak{h}, d_0(\mathcal{O})\big)$. Thus $\Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}) \in \mathcal{A}(\mathfrak{h})$.

Using this notation we can restate Theorem 1.1 as follows.

Theorem 1.2 *Let $\mathcal{O}$ be an orbit in $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then*

$$\Phi(\mathfrak{g}, \mathcal{O}) \sim_{\mathfrak{h}} \Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}).$$

Further, $\Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O})$ is the unique element of $\mathcal{A}(\mathfrak{h})$ with this property.

Let $\mathcal{T}(\mathfrak{g})$ denote the set of all G-invariant distributions T on $\mathfrak{g}$ which are finite linear combinations of normalized Fourier transforms of orbital integrals. Thus every $T \in \mathcal{T}(\mathfrak{g})$ can be written as

$$T = \sum_{\mathcal{O}} c_T(\mathcal{O}) \Phi(\mathfrak{g}, \mathcal{O}) = |\eta_{\mathfrak{g}}|^{1/2} \sum_{\mathcal{O}} c_T(\mathcal{O}) \hat{\mu}_{\mathcal{O}}, \tag{1.17}$$

where $\mathcal{O}$ runs over the set of G-orbits in $\mathfrak{g}$, $c_T(\mathcal{O}) \in \mathbf{C}$ for all orbits $\mathcal{O}$, and $c_T(\mathcal{O}) = 0$ for all but finitely many orbits $\mathcal{O}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we define

$$\Phi(T, \mathfrak{h}) = \sum_{\mathcal{O}} c_T(\mathcal{O}) \Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}). \tag{1.18}$$

As an immediate consequence of Theorem 1.2 we have the following.

Theorem 1.3 *Let $T \in \mathcal{T}(\mathfrak{g})$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\Phi(T, \mathfrak{h}) \in \mathcal{A}(\mathfrak{h})$, and is the unique element of $\mathcal{A}(\mathfrak{h})$. such that*

$$T \sim_{\mathfrak{h}} \Phi(T, \mathfrak{h}).$$

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Let $f \in \mathcal{A}(\mathfrak{h})$, and as in (1.14) write

$$f(H) = \sum_Y \sum_d \psi\big(B(Y,H)\big) f_{Y,d}(H), \quad H \in \mathfrak{h}.$$

We prove in Proposition 6.3 that this expansion is unique. Thus we can define

$$X(f) = \{Y \in \mathfrak{h} : f_{Y,d} \neq 0 \text{ for some } d \geq 0\}. \tag{1.19}$$

For $T \in \mathcal{T}(\mathfrak{g})$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, we write

$$X(T,\mathfrak{h}) = X\big(\Phi(T,\mathfrak{h})\big). \tag{1.20}$$

We think of the set $X(T,\mathfrak{h})$ as the "exponents" of T along $\mathfrak{h}$.

Let $S(T)$ denote the set of all semisimple elements $Y \in \mathfrak{g}$ such that $Y \in \mathrm{cl}(\mathcal{O})$ for some orbit $\mathcal{O}$ such that $c_T(\mathcal{O}) \neq 0$.

Theorem 1.4 *Let $T \in \mathcal{T}(\mathfrak{g})$. Then*

$$S(T) = \bigcup_{\mathfrak{h}} X(T,\mathfrak{h}).$$

Remark 1.2 Let $T \in \mathcal{T}(\mathfrak{g})$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then it follows from Theorem 1.4 that $X(T,\mathfrak{h}) \subset S(T) \cap \mathfrak{h}$. However, it is not necessarily true that $S(T) \cap \mathfrak{h} \subset X(T,\mathfrak{h})$. For example, let $\mathfrak{g} = \mathrm{sl}(2,F)$, where F is a p-adic field such that -1 is not a square. Let Z be a non-zero nilpotent element of $\mathfrak{g}$, and let $\mathcal{O}^{\pm}$ denote the G-orbit of $\pm Z$. Then $\mathcal{O}^+ \neq \mathcal{O}^-$, and it is easy to see from (1.11) and (1.12) that we can normalize measures so that for all $X \in \mathfrak{g}'$,

$$\Phi(\mathfrak{g},\mathcal{O}^+,-X) = \Phi(\mathfrak{g},\mathcal{O}^-,X). \tag{1.21}$$

Let $T = \Phi(\mathfrak{g},\mathcal{O}^+) - \Phi(\mathfrak{g},\mathcal{O}^-)$. Then it follows from (1.21) that $T(-X) = -T(X)$ for all $X \in \mathfrak{g}'$. Let $\mathfrak{h}$ be a split Cartan subalgebra of $\mathfrak{g}$, and let $H \in \mathfrak{h}'$. Then $T(-H) = T(H)$ since T is a class function on $\mathfrak{g}'$, and $-H$ is G-conjugate to H. Thus $\Phi(T,\mathfrak{h},H) = T(H) = 0$ for all $H \in \mathfrak{h}'$, and so $X(T,\mathfrak{h}) = \varnothing$. But $S(T) \cap \mathfrak{h} = \{0\}$.

Corollary 1.5 *Let $T \in \mathcal{T}(\mathfrak{g})$ such that $T \sim_{\mathfrak{h}} 0$ for every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then $T = 0$.*

Corollary 1.5 can also be stated as follows. Let $T = \sum_{\mathcal{O}} c_{\mathcal{O}}(T)\Phi(\mathfrak{g},\mathcal{O}) \in \mathcal{T}(\mathfrak{g})$ as in (1.17), and suppose $T \sim_{\mathfrak{h}} 0$ for all Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$. Then $c_{\mathcal{O}}(T) = 0$ for all orbits $\mathcal{O}$ of $\mathfrak{g}$. That is the Fourier transforms of orbital integrals over arbitrary orbits of $\mathfrak{g}$ are "linearly independent at infinity." This contrasts to the situation in a neighborhood of the identity. It follows from Theorem 5.11 of [2] that for every $T \in \mathcal{T}(\mathfrak{g})$ there is a neighborhood V of the identity in $\mathfrak{g}$ such that the restriction of T to V is a linear combination of the $\Phi(\mathfrak{g},\mathcal{O})$ where $\mathcal{O}$ runs over the finite set of nilpotent orbits of $\mathfrak{g}$.

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The Fourier transforms of orbital integrals play an important role in p-adic representation theory. As mentioned above, Theorem 1.1 generalizes a result of Waldspurger [10] for regular semisimple orbits. Waldspurger needed to control the behavior at infinity of the functions $\hat{\mu}_{\mathcal{O}}$ for his work on local trace formulas for p-adic Lie algebras.

In addition, the Fourier transforms of orbital integrals are related to character formulas for representations of G. Roughly, what frequently happens is that we have an orbit $\mathcal{O}$ of $\mathfrak{g}$ and a representation π of G, with character Θ_π, such that

$$\Theta_\pi(\exp X) = c_\pi \hat{\mu}_{\mathcal{O}}(X)$$

for regular X in some neighborhood of 0 in $\mathfrak{g}$ and some non-zero constant c_π. Murnaghan [6], [7], [8], [9] has found many cases of formulas of this type when π is supercuspidal and $\mathcal{O}$ is a regular elliptic orbit. Although this formula is for small $X \in \mathfrak{g}'$, by twisting with characters it is possible to relate character formulas on G to values of $\hat{\mu}_{\mathcal{O}}(X)$ for large elements $X \in \mathfrak{g}'$.

For small values of $X \in \mathfrak{g}'$, the local expansion in terms of nilpotent orbits holds for $\hat{\mu}_{\mathcal{O}}(X)$. The region of validity for the local expansion has been studied by Waldspurger [11] and is related to work of Moy and Prasad [4], [5] in the group case. DeBacker [1] has shown that for regular elliptic orbits in the Lie algebra of $\mathrm{GL}_l(F)$, l prime and F sufficiently tame, the local expansion and the expansion at infinity give the entire formula for $\hat{\mu}_{\mathcal{O}}$ up to a single shell.

In Section 2 of this paper we normalize invariant measures in a consistent way and state Theorems 2.1 and 2.2. These are stronger versions of Theorem 1.1 which have some uniformity as the orbit $\mathcal{O}$ varies. In Section 3 we prove Theorem 2.1. The proof of Theorem 2.2 is given in Section 4 and Section 5. The proofs of Theorems 1.2, 1.3, 1.4 are given in Section 6. I'd like to thank Allen Moy for asking questions that got me started on this problem, and also thank Tom Hales and Julee Kim for useful discussions.

2 Expanded Versions of Theorem 1.1

In this section we state stronger versions of Theorem 1.1 which have some uniformity as the orbit $\mathcal{O}$ varies.

We first look at the special case of orbits $\mathcal{O}$ and Cartan subalgebras $\mathfrak{h}$ such that $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O}) = \varnothing$. If ω is a subset of $\mathfrak{g}$, we let $\omega^G = \{x \cdot X : x \in G, X \in \omega\}$. Let $J(\omega)$ denote the space of all G-invariant distributions T on $\mathfrak{g}$ such that the support of T is contained in $\mathrm{cl}(\omega^G)$. When ω is compact, Harish-Chandra proved in [2] that $\hat{T}$ is a locally constant function on $\mathfrak{g}'$ for all $T \in J(\omega)$.

Theorem 2.1 *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and suppose that ω is a compact subset of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$. Then given any compact subset ω_1 of $\mathfrak{h}'$ there is $C > 0$ so that $\hat{T}(tH) = 0$ for all $H \in \omega_1$, $t \in F^\times$ such that $|t| \geq C$, and every $T \in J(\omega)$.*

Suppose that $\mathcal{O}$ is an orbit of $\mathfrak{g}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O}) = \varnothing$. Let $X \in \mathcal{O}$. Then for $\omega = \{X\}$, $\mathrm{cl}(\omega^G) = \mathrm{cl}(\mathcal{O})$, and $\mu_{\mathcal{O}} \in J(\omega)$. Thus Theorem 1.1 in this case follows from Theorem 2.1. The proof of Theorem 2.1 is given in Section 3.

We now look at the general case. If Y is any semisimple element of $\mathfrak{g}$, then $\mathfrak{m} = \mathfrak{g}_Y$ is a reductive subalgebra of $\mathfrak{g}$ with rank $\mathfrak{m} = $ rank $\mathfrak{g}$.

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Fix a reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ with rank $\mathfrak{m}$ = rank $\mathfrak{g}$, and define

$$\mathfrak{g}(\mathfrak{m}) = \{Y \in \mathfrak{g} : Y \text{ is semisimple and } \mathfrak{g}_Y = \mathfrak{m}\}. \tag{2.1}$$

Note that $\mathfrak{g}(\mathfrak{m})$ is contained in the center of $\mathfrak{m}$, and hence in every Cartan subalgebra of $\mathfrak{m}$. Define $\eta_{\mathfrak{g}}$ as in (1.3), and for $Y \in \mathfrak{g}(\mathfrak{m})$, define

$$\eta_{\mathfrak{g}/\mathfrak{m}}(Y) = \det \operatorname{ad} Y|_{\mathfrak{g}/\mathfrak{m}}. \tag{2.2}$$

Then $\eta_{\mathfrak{g}/\mathfrak{m}}(Y) \neq 0$ for $Y \in \mathfrak{g}(\mathfrak{m})$.

Let $\mathcal{N}_M$ denote the set of nilpotent elements in $\mathfrak{m}$ and let Ξ_M denote the set of nilpotent M-orbits in $\mathfrak{m}$. Fix a G-invariant measure dy^* on G/M and an M-invariant measure ν_ξ on ξ for each $\xi \in \Xi_M$. For each $Z \in \xi$, we can define an M-invariant measure dm^* on M/M_Z so that dm^* corresponds to ν_ξ via $m \mapsto m \cdot Z$. Let K be a compact open subgroup of G and let dk be normalized Haar measure on K. Then for $Y \in \mathfrak{g}(\mathfrak{m})$, $Z \in \mathcal{N}_M$, $H \in \mathfrak{g}'$, we define

$$T_K(Y,Z,H) = \int_{G/M} \int_{M/M_Z} \int_K \psi\Big(B\big(k \cdot H, ym \cdot (Y+Z)\big)\Big)\, dk\, dm^*\, dy^*, \tag{2.3}$$

$$\Phi_K(Y,Z,H) = |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{\frac{1}{2}} |\eta_{\mathfrak{g}}(H)|^{\frac{1}{2}} T_K(Y,Z,H). \tag{2.4}$$

Fix $Y \in \mathfrak{g}(\mathfrak{m})$ and $Z \in \mathcal{N}_M$, and let $X = Y + Z$. This is the Jordan decomposition of X, so that $G_X = M_Z$ and $G/G_X = G/M \cdot M/M_Z$. Now $dx^* = dm^* dy^*$ is an invariant measure on G/G_X, so that $T_K(Y,Z,H) = T_K(X,H)$, where $T_K(X,H)$ is defined as in (1.11). Now by (1.3) and (1.12), we can normalize the invariant measure on $\mathcal{O} = \mathcal{O}_X$ so that $\Phi_K(Y,Z,H) = \Phi(\mathfrak{g}, \mathcal{O}, H)$, $H \in \mathfrak{g}'$. Since $\Phi_K(Y,Z,H)$ is independent of K, and depends only on the M-orbit ξ of Z, we also write $\Phi(Y,\xi,H) = \Phi_K(Y,Z,H)$, $Z \in \xi$, $Y \in \mathfrak{g}(\mathfrak{m})$, $H \in \mathfrak{g}'$. When $\mathfrak{m} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{g}(\mathfrak{m}) = \mathfrak{h}'$ and the function Φ above was studied by Harish-Chandra in Section 7 of [2].

Recall that for each $\xi \in \Xi_M$ we have fixed an invariant measure ν_ξ. Use this normalization of the invariant measure to define the orbital integral μ^M_ξ corresponding to ξ. Let $\eta_{\mathfrak{m}}$ be the function on $\mathfrak{m}$ defined as in (1.3). Now for $H \in \mathfrak{m}'$ we define

$$\Phi(\mathfrak{m}, Y+\xi, H) = |\eta_{\mathfrak{m}}(H)|^{\frac{1}{2}} \psi\big(B(Y,H)\big) \hat{\mu}^M_\xi(H) = |\eta_{\mathfrak{m}}(H)|^{\frac{1}{2}} \hat{\mu}^M_{Y+\xi}(H). \tag{2.5}$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $N(\mathfrak{h},\mathfrak{m}) = \{y \in G : y^{-1}\mathfrak{h} \subset \mathfrak{m}\}$. Then for all $y \in N(\mathfrak{h},\mathfrak{m})$, $m \in M$, we have $ym \in N(\mathfrak{h},\mathfrak{m})$. Define $W(\mathfrak{h},\mathfrak{m}) = N(\mathfrak{h},\mathfrak{m})/M$. If $f \colon \mathfrak{m} \to \mathbf{C}$ is an M-invariant function on $\mathfrak{m}$, we write $f(w^{-1}H) = f(y^{-1} \cdot H)$, $H \in \mathfrak{m}$, for $w = yM \in W(\mathfrak{h},\mathfrak{m})$. Let $\mathfrak{m}'' = \mathfrak{m} \cap \mathfrak{g}'$.

Theorem 2.2 *Let ω_1 be a compact subset of $\mathfrak{h}'$ and let ω_2 be a compact subset of $\mathfrak{g}(\mathfrak{m})$. Then there is $C > 0$ such that for all $H \in \omega_1$, $Y \in \omega_2$, $\xi \in \Xi_M$, and $|t| \geq C$,*

$$\Phi(Y,\xi,tH) = \sum_{w \in W(\mathfrak{h},\mathfrak{m})} \Phi(\mathfrak{m}, Y+\xi, tw^{-1}H) c(\mathfrak{m}, Y, tw^{-1}H).$$

Here $c(\mathfrak{m}) \colon \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}'' \to \mathbf{C}$ is a locally constant function on $\mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$ satisfying

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(i) $|c(\mathfrak{m}, Y, H)|$ *is non-zero and independent of* $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$;
(ii) $c(\mathfrak{m}, Y, t^2 H) = c(\mathfrak{m}, Y, H)$ *for all* $t \in F^\times$, $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$;
(iii) $c(\mathfrak{m}, tY, H) = c(\mathfrak{m}, Y, tH)$ *for all* $t \in F^\times$, $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$;
(iv) $c(\mathfrak{m}, Y, m \cdot H) = c(\mathfrak{m}, Y, H)$ *for all* $m \in M$, $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$.

Let $\mathcal{O}$ be an orbit and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Pick a semisimple element $Y \in \mathrm{cl}(\mathcal{O})$, and let $\mathfrak{m} = \mathfrak{g}_Y$. Then we can normalize the invariant measure on $\mathcal{O}$ so that in the notation above,

$$\Phi(\mathfrak{g}, \mathcal{O}, H) = \Phi\big(Y, \xi_Y(\mathcal{O}), H\big), \quad H \in \mathfrak{g}'.$$

For $w = yM \in W(\mathfrak{h}, \mathfrak{m})$ we write $wY = y \cdot Y$. By Lemma 5.1, $w \mapsto wY$ is a bijection between $W(\mathfrak{h}, \mathfrak{m})$ and $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$. For $w \in W(\mathfrak{h}, \mathfrak{m})$, we can normalize the invariant measure on the orbit $wY + \xi_{wY}(\mathcal{O})$ of $\mathfrak{g}_{wY}$ so that

$$\Phi(\mathfrak{m}, Y + \xi, w^{-1} H) = \Phi\big(\mathfrak{g}_{wY}, wY + \xi_{wY}(\mathcal{O}), H\big), \quad H \in \mathfrak{h}'.$$

Finally, we can define

$$c_{wY}(H) = c(\mathfrak{m}, Y, w^{-1} H), \quad H \in \mathfrak{h}'.$$

Thus Theorem 2.2 gives Theorem 1.1 as a special case when $\omega_2 = \{Y\}$.

3 Proof of Theorem 2.1

Let $\mathcal{R}$ denote the ring of integers of F, $\mathcal{P}$ the maximal ideal in $\mathcal{R}$, and ϖ a uniformizing parameter so that $\mathcal{P} = \varpi\mathcal{R}$. Let $|\cdot|$ denote the absolute value on F such that $|\varpi| = q^{-1}$ where $q = [\mathcal{R}/\mathcal{P}]$. We assume that the character ψ of F used to define the Fourier transform in (1.2) has conductor $\mathcal{R}$.

There is $n \geq 1$ so that $\mathfrak{g}$ and G are subsets of $M_n(F)$. We have the usual norm $\|\cdot\|$ on $\mathfrak{g} \subset M_n(F)$ given by

(3.1) $$\|X\| = \max_{i,j} |X_{ij}|, \quad X = [X_{ij}] \in M_n(F).$$

Let B denote the symmetric, nondegenerate, bilinear form on $\mathfrak{g}$ given by

(3.2) $$B(X, Y) = \mathrm{tr}\, XY, \quad X, Y \in \mathfrak{g} \subset M_n(F).$$

In this section we prove Theorem 2.1. If K is a compact open subgroup of G and dk is normalized Haar measure on K, we define

(3.3) $$\phi_K(X, Y) = \int_K \psi\big(B(k \cdot X, Y)\big)\, dk, \quad X, Y \in \mathfrak{g}.$$

Lemma 3.1 *Suppose that* $\mathfrak{g}$ *is semisimple. Let* K *be a compact open subgroup of* G *and let* ω_i, $i = 1, 2$, *be compact subsets of* $\mathfrak{g}$ *such that* $[k \cdot X_1, X_2] \neq 0$ *for all* $X_i \in \omega_i$, $k \in K$. *Then there is* $C > 0$ *so that* $\phi_K(tX_1, X_2) = 0$ *for all* $X_i \in \omega_i$, $t \in F^\times$, $|t| \geq C$.

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Proof By Corollary 7.3 of [2] the map $k \to B(k \cdot X_1, X_2)$ is submersive at k for all $k \in K$, $X_i \in \omega_i$, $i = 1, 2$. Thus there are compact open neighborhoods ω_i' of ω_i in $\mathfrak{g}$, $i = 1, 2$, such that the map $(X_1, X_2, k) \to \big(X_1, X_2, B(k \cdot X_1, X_2)\big)$ of $\omega_1' \times \omega_2' \times K$ into $\omega_1' \times \omega_2' \times F$ is everywhere submersive. Now as in the proof of Lemma 7.1 of [2] there is $C > 0$ so that $\phi_K(tX_1, X_2) = 0$ for all $X_i \in \omega_i'$, $i = 1, 2$, and $t \in F^\times$ such that $|t| \geq C$. ■

Lemma 3.2 *Let ω be a compact subset of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$, let ω_1 be a compact subset of $\mathfrak{h}'$, and let K be a compact open subgroup of G. Then there is $C > 0$ so that $\phi_K(tH, X) = 0$ for all $H \in \omega_1$, $|t| \geq C$, $X \in \mathrm{cl}(\omega^G)$.*

Proof Assume first that $\mathfrak{g}$ is semisimple. Let ω be a compact subset of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$. Then $0 \notin \mathrm{cl}(\omega^G)$. Thus there is $\epsilon > 0$ so that $\|X\| \geq \epsilon$ for all $X \in \mathrm{cl}(\omega^G)$. For $X \neq 0 \in \mathfrak{g}$, define the integer $\nu(X)$ by $\|X\| = q^{-\nu(X)}$. Then $\|X\| = |\varpi^{\nu(X)}|$. Let $S = \{X \in \mathfrak{g} : \|X\| = 1\}$. Then for all $X \neq 0 \in \mathfrak{g}$, $\varpi^{-\nu(X)} X \in S$. Let S_0 denote the closure in S of

$$\{\varpi^{-\nu(X)} X : X \in \mathrm{cl}(\omega^G)\}.$$

It is a compact set. Now $\mathrm{cl}(\omega^G)$ is a closed, G-invariant set. Further, since ω is compact, the eigenvalues of $\mathrm{ad}\, X$, $X \in \mathrm{cl}(\omega^G)$ are bounded. Thus as in Lemma 7.4 of [2], every element of S_0 is either nilpotent or is of the form $\varpi^{-\nu(X)} X$ for some $X \in \mathrm{cl}(\omega^G)$.

Let $X' \in S_0$, $H \in \omega_1$, and suppose that $[k \cdot H, X'] = 0$ for some $k \in K$. Then $k^{-1} X' \in \mathfrak{h}$, so that X' is semisimple, and hence of the form $X' = \varpi^{-\nu(X)} X$ for some $X \in \mathrm{cl}(\omega^G)$. But then $k^{-1} X \in \mathfrak{h} \cap \mathrm{cl}(\omega^G)$. This contradicts the assumption that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$. Thus $[k \cdot H, X'] \neq 0$ for all $k \in K$, $H \in \omega_1$, $X' \in S_0$, and so by Lemma 3.1 there is $C' > 0$ so that $\phi_K(tH, X') = 0$ for all $H \in \omega_1$, $X' \in S_0$, $t \in F$ such that $|t| \geq C'$.

Let $t \in F$, $|t| \geq C = \epsilon^{-1} C'$, and $H \in \omega_1$. Then for all $X \in \mathrm{cl}(\omega^G)$, $X' = \varpi^{-\nu(X)} X \in S_0$ and $|t \varpi^{\nu(X)}| \geq C\epsilon = C'$, so that $\phi_K(tH, X) = \phi_K(t\varpi^{\nu(X)} H, X') = 0$.

Now we drop the assumption that $\mathfrak{g}$ is semisimple. Let $\mathfrak{g}$ be reductive and write $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}_s$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}_s$ is the derived subalgebra. Given any $X \in \mathfrak{g}$, we write $X = X_0 + X_s$ where $X_0 \in \mathfrak{z}$ and $X_s \in \mathfrak{g}_s$. Let $p\colon \mathfrak{g} \to \mathfrak{g}_s$ denote the projection $p(X) = X_s$, $X \in \mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{z} + \mathfrak{h}_s$ where $\mathfrak{h}_s = p(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}_s$, and $\mathfrak{h}' = \mathfrak{z} + \mathfrak{h}_s'$. Let Z denote the connected subgroup of G corresponding to $\mathfrak{z}$ and let $G_s = G/Z$.

Let ω_1 be a compact subset of $\mathfrak{h}'$ and let ω be a compact subset of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$. Then $p(\omega_1)$ is a compact subset of $\mathfrak{h}_s'$ and $\omega_s = p(\omega)$ is a compact subset of $\mathfrak{g}_s$. It is easy to check that $p\big(\mathrm{cl}(\omega^G)\big) \subset \mathrm{cl}(\omega_s^{G_s})$. Suppose that $Y_s \in \mathfrak{h}_s \cap \mathrm{cl}(\omega_s^{G_s})$. Then there are $X_{n,s} \in \omega_s$, $x_n \in G$, such that $x_n \cdot X_{n,s} \to Y_s$. Let $n \geq 1$. Since $X_{n,s} \in \omega_s = p(\omega)$, there is $X_{n,0} \in \mathfrak{z}$ such that $X_n = X_{n,0} + X_{n,s} \in \omega$. Further, since $\{X_n\}$ is a sequence in the compact set ω, there is a convergent subsequence. Thus by passing to a subsequence we can assume that $X_n \to X \in \omega$. Thus $X_{n,0} \to X_0 \in \mathfrak{z}$. Now $x_n \cdot X_n = X_{n,0} + x_n \cdot X_{n,s} \to X_0 + Y_s \in \mathfrak{z} + \mathfrak{h}_s = \mathfrak{h}$. Thus $X_0 + Y_s \in \mathfrak{h} \cap \mathrm{cl}(\omega^G)$. This contradiction shows that $\mathfrak{h}_s \cap \mathrm{cl}(\omega_s^{G_s}) = \varnothing$.

Suppose that K is a compact open subgroup of G. Then $K_s = K/(Z \cap K)$ is a compact open subgroup of G_s. Since $\mathfrak{g}_s \perp \mathfrak{z}$ with respect to B, for $X, Y \in \mathfrak{g}$,

$$\phi_K(X, Y) = \psi\big(B(X_0, Y_0)\big) \int_{K_s} \psi\big(B(k \cdot X_s, Y_s)\big)\, dk = \psi\big(B(X_0, Y_0)\big) \phi_{K_s}(X_s, Y_s).$$

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Now by the semisimple case above, there is $C > 0$ so that $\phi_{K_s}(tH_s, X_s) = 0$ for all $H_s \in p(\omega_1)$, $X_s \in \mathrm{cl}(\omega_s^{G_s})$, $|t| \geq C$. Thus for $H \in \omega_1$, $X \in \mathrm{cl}(\omega^G)$, $|t| \geq C$,

$$\phi_K(tH, X) = \psi\big(B(tH_0, X_0)\big)\phi_{K_s}(tH_s, X_s) = 0. \qquad \blacksquare$$

Proof of Theorem 2.1 Let ω be a compact subset of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathrm{cl}(\omega^G) = \varnothing$, let ω_1 be a compact subset of $\mathfrak{h}'$, and let K be a compact open subgroup of G. Then by Lemma 3.2 there is $C > 0$ so that $\phi_K(tH, X) = 0$ for all $H \in \omega_1$, $|t| \geq C$, $X \in \mathrm{cl}(\omega^G)$. Fix $H \in \omega_1$, $|t| \geq C$. As in [3], there is $\phi \in C_c^\infty(\mathfrak{g})$ such that $\phi(X) = \phi_K(tH, X)$ for all $X \in \mathrm{cl}(\omega^G)$. Let $T \in J(\omega)$. Then by Theorem 3 of [3], $\hat{T}(tH) = T(\phi) = 0$ since $\phi(X) = \phi_K(tH, X) = 0$ for all X in the support of T. $\blacksquare$

4 Evaluation of an Integral

In this section we begin the proof of Theorem 2.2. Fix a reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ with rank $\mathfrak{m}$ = rank $\mathfrak{g}$. In this section we evaluate integrals of the form

$$\phi_K(tH, Y) = \int_K \psi\big(tB(k \cdot H, Y)\big)\, dk, \quad Y \in \mathfrak{g}(\mathfrak{m}),\ H \in \mathfrak{m}'',\ t \in F^\times,$$

for K sufficiently small and $|t|$ sufficiently large. This calculation is similar to those in Section VIII of [10], but in our case Y may not be regular. The main result is Proposition 4.6. In the next section we will use this result to prove Theorem 2.2.

We first need to define new norms on $\mathfrak{g}$ which depend on $\mathfrak{m}$. Since $\mathfrak{m}$ is reductive, the restriction of B to $\mathfrak{m}$ is non-degenerate, and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp$ where $\mathfrak{m}^\perp = \{X \in \mathfrak{g} : B(X, Y) = 0\ \forall Y \in \mathfrak{m}\}$. For $X \in \mathfrak{g}$, write $X = X_0 + X_1$ where $X_0 \in \mathfrak{m}$, $X_1 \in \mathfrak{m}^\perp$. Then we define

$$\|X\|' = \max\{\|X_0\|, \|X_1\|\}, \tag{4.1}$$

where $\|\cdot\|$ is the norm on $\mathfrak{g}$ defined in (3.1). Then $\|X\| \leq \|X\|'$, $X \in \mathfrak{g}$.

For $X \in \mathfrak{g}$, we also define

$$\|X\|'' = \sup_{Z \in \mathfrak{g}, \|Z\|' \leq 1} |B(Z, X)|. \tag{4.2}$$

This is also a norm on $\mathfrak{g}$. Let $X, Z \in \mathfrak{g}$ with $\|Z\|' \leq 1$. Then

$$|B(X, Z)| \leq \|X\|\,\|Z\| \leq \|X\|\,\|Z\|' \leq \|X\|.$$

Thus

$$\|X\|'' \leq \|X\| \leq \|X\|', \quad X \in \mathfrak{g}. \tag{4.3}$$

Since $\|\cdot\|'$ and $\|\cdot\|''$ are equivalent norms on $\mathfrak{g}$ there is a constant $0 < C_0 \leq 1$ so that

$$C_0\|X\|' \leq \|X\|'' \leq \|X\|', \quad X \in \mathfrak{g}. \tag{4.4}$$

For any integer $c \geq 0$, define

$$\mathfrak{k}_c = \{X \in \mathfrak{g} : \|X\|' \leq q^{-c}\}.$$

It is a lattice in $\mathfrak{g}$. The following lemma is elementary.

Lemma 4.1 *There is an integer $c_0 > 0$ so that $q^{-c_0} \le |2|$, and for any $k \ge 2$,*

$$\left|\frac{1}{k!}\right| q^{-c_0(k-2)} \le |2|^{-1}.$$

Further, for any $c \ge c_0$, $\exp\colon \mathfrak{k}_c \to G$ is well-defined, and for $Z \in \mathfrak{k}_c$, $X \in \mathfrak{g}$, $\exp Z \cdot X = X + [Z, X] + W$ where $W = \sum_{k\ge 2}(1/k!)(\operatorname{ad} Z)^k X$ satisfies $\|W\| \le q^{-2c}|2|^{-1}\|X\|$.

Remark 4.1 We can choose c_0 so that

$$\left|\frac{1}{k!}\right| q^{-c_0(k-2)} \le 1$$

for $k \ge 3$. When $k = 2$, for any c_0 we have $|1/k!| = |2|^{-1}$.

For any $c \ge c_0$, let $K_c = \exp(\mathfrak{k}_c)$. It is a compact open subgroup of G contained in $\mathrm{GL}(n, \mathcal{R})$. For $c \ge c_0$, write

$$\phi_c(X, Y) = \phi_{K_c}(X, Y) = \int_{K_c} \psi\big(B(k \cdot X, Y)\big)\, dk, \quad X, Y \in \mathfrak{g}.$$

Proposition 4.2 *Suppose that $c \ge c_0$. Let $X, Y \in \mathfrak{g}$ such that $\|X\|\,\|Y\| \le |2|$, and $t \in F^\times$ such that $|t| \ge q^c$. Then*

$$\phi_c(t^2X, Y) = \int_{K_c(X,Y,t)} \psi\big(t^2 B(k \cdot X, Y)\big)\, dk,$$

where $K_c(X, Y, t) = \{k \in K_c : \|[X, k^{-1} \cdot Y]\|'' \le |t|^{-1}\}$.

Proof Write $|t| = q^r$, and let $k \in K_c$, $Z \in \mathfrak{k}_r$. Then using Lemma 4.1 we can write $B(k \exp Z \cdot X, Y) = B(\exp Z \cdot X, k^{-1} \cdot Y) =$

$$B(X, k^{-1} \cdot Y) + B([Z, X], k^{-1} \cdot Y) + B(W, k^{-1} \cdot Y)$$

where $\|W\| \le |2|^{-1} q^{-2r}\|X\|$. But

$$|B\big(W, k^{-1} \cdot Y\big)| \le \|W\|\, \|k^{-1}Y\| \le |2|^{-1} q^{-2r}\|X\|\, \|Y\| \le q^{-2r},$$

since by assumption $\|X\|\,\|Y\| \le |2|$. Thus $\psi\big(t^2 B(W, k^{-1} \cdot Y)\big) = 1$ since $|t| = q^r$ and we have assumed that ψ has conductor $\mathcal{R}$. Now since $B(X, k^{-1} \cdot Y) = B(k \cdot X, Y)$ and $B([Z, X], k^{-1} \cdot Y) = B(Z, [X, k^{-1} \cdot Y])$, we have

$$\psi\big(t^2 B(k \exp Z \cdot X, Y)\big) = \psi\big(t^2 B(k \cdot X, Y)\big)\psi\big(t^2 B(Z, [X, k^{-1} \cdot Y])\big).$$

Since $r \ge c$, we have $K_r \subset K_c$, so that we can write K_c as a finite union $K_c = \bigcup_i k_i K_r$. Thus there is a normalization of Haar measure dZ on $\mathfrak{k}_r$ so that

$$\begin{aligned}\phi_c(t^2X, Y) &= \int_{K_c} \psi\big(t^2 B(k \cdot X, Y)\big)\, dk = \sum_i \int_{\mathfrak{k}_r} \psi\big(t^2 B(k_i \exp Z \cdot X, Y)\big)\, dZ \\ &= \sum_i \psi\big(t^2 B(k_i \cdot X, Y)\big) \int_{\mathfrak{k}_r} \psi\big(t^2 B(Z, [X, k_i^{-1}\dot{Y}])\big)\, dZ.\end{aligned}$$

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But for each i, $Z \mapsto \psi\big(t^2 B(Z, [X, k_i^{-1}\dot{Y}])\big)$ is a unitary character of the additive group $\mathfrak{k}_r$, and is trivial if and only if $\|[X, k_i^{-1}\dot{Y}]\|'' \leq q^{-r}$, so that $k_i \in K_c(X, Y, t)$.

We have shown that for each coset $k_i K_r$ of K_c,

$$\int_{k_i K_r} \psi\big(t^2 B(k \cdot X, Y)\big)\, dk = 0 \Leftrightarrow k_i \in K_c(X, Y, t).$$

Since the left hand side of the equation is independent of the coset representative k_i for $k_i K_r$, we must have $k_i k \in K_c(X, Y, t)$ if and only if $k_i \in K_c(X, Y, t)$ for all $k \in K_r$, and so the union of the cosets $k_i K_r$ with $k_i \in K_c(X, Y, t)$ is equal to $K_c(X, Y, t)$. ■

For any linear transformation $T\colon \mathfrak{m}^\perp \to \mathfrak{m}^\perp$, we write

$$\|T\| = \sup_{Z \in \mathfrak{m}^\perp, Z \neq 0} \|T(Z)\| / \|Z\|.$$

Let $X \in \mathfrak{m}$. Since $[\mathfrak{m}, \mathfrak{m}^\perp] \subset \mathfrak{m}^\perp$, the restriction of $\operatorname{ad} X$ to $\mathfrak{m}^\perp$ is a linear transformation $T_X\colon \mathfrak{m}^\perp \to \mathfrak{m}^\perp$. Let $\mathfrak{m}^{\mathrm{reg}}$ denote the set of all $H \in \mathfrak{m}$ such that H is semisimple and T_H is invertible. Note that $\mathfrak{g}(\mathfrak{m}) \subset \mathfrak{m}^{\mathrm{reg}}$ and $\mathfrak{m}'' = \mathfrak{m} \cap \mathfrak{g}' \subset \mathfrak{m}^{\mathrm{reg}}$.

For any integer $s > 0$, we let

$$\mathfrak{m}_s^{\mathrm{reg}} = \{H \in \mathfrak{m}^{\mathrm{reg}} : \|H\| \leq |2|^{1/2}, \|T_H^{-1}\| \leq q^s\};$$
$$\mathfrak{g}(\mathfrak{m})_s = \mathfrak{g}(\mathfrak{m}) \cap \mathfrak{m}_s^{\mathrm{reg}}.$$

Then for all $H \in \mathfrak{m}_s^{\mathrm{reg}}$, $Z_1 \in \mathfrak{m}^\perp$,

(4.5) $$q^{-s}\|Z_1\| \leq \|\operatorname{ad} H Z_1\| \leq |2|^{1/2}\|Z_1\|.$$

Define C_0 as in (4.4).

Lemma 4.3 *Let $H \in \mathfrak{m}_s^{\mathrm{reg}}$, $Y \in \mathfrak{g}(\mathfrak{m})_s$.*

(i) For all $Z = Z_0 + Z_1$, $Z_0 \in \mathfrak{m}$, $Z_1 \in \mathfrak{m}^\perp$,

$$\|Z_1\| \leq q^{2s} C_0^{-1} \|\operatorname{ad} H \operatorname{ad} Y Z\|''.$$

(ii) Suppose c is large enough that $q^{-c} < q^{-2s} C_0$. Then for all $Z \in \mathfrak{k}_c$,

$$\|[H, \exp(-Z) \cdot Y]\|'' = \|\operatorname{ad} H \operatorname{ad} Y Z\|''.$$

Proof Let $Z = Z_0 + Z_1$, $Z_0 \in \mathfrak{m}$, $Z_1 \in \mathfrak{m}^\perp$. Then since $Y \in \mathfrak{g}(\mathfrak{m})$, $\operatorname{ad} Y Z = \operatorname{ad} Y Z_1 \in \mathfrak{m}^\perp$. Further, since $H, Y \in \mathfrak{m}_s^{\mathrm{reg}}$, using (4.5)

$$\|\operatorname{ad} H \operatorname{ad} Y Z\|' = \|\operatorname{ad} H \operatorname{ad} Y Z_1\| \geq q^{-2s}\|Z_1\|.$$

Thus by (4.4)

$$\|Z_1\| \leq q^{2s}\|\operatorname{ad} H \operatorname{ad} Y Z\|' \leq q^{2s} C_0^{-1} \|\operatorname{ad} H \operatorname{ad} Y Z\|''.$$

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Now let $k = \exp Z$, $Z = Z_0 + Z_1 \in \mathfrak{k}_c$. Then since $[H, Y] = 0$ and $\big[H, [-Z, Y]\big] = \operatorname{ad} H \operatorname{ad} YZ$, we have $[H, k^{-1} \cdot Y] = \operatorname{ad} H \operatorname{ad} YZ + V$ where

$$V = \sum_{k \geq 2} \frac{1}{k!} [H, (-\operatorname{ad} Z)^k Y] = \sum_{k \geq 2} \frac{1}{k!} \big[H, (-\operatorname{ad} Z)^{k-1} [Y, Z]\big].$$

Let $X \in \mathfrak{g}$, $\|X\|' \leq 1$. Using Lemma 4.1, for each $k \geq 2$,

$$\Big| B\Big(X, \frac{1}{k!} \big[H, (-\operatorname{ad} Z)^{k-1} [Y, Z]\big]\Big) \Big| \leq \Big| \frac{1}{k!} \Big| \|X\| \, \|H\| \, \|Z\|^{k-2} \|Z\| \cdot \|[Y, Z]\| \leq q^{-c} \|Z_1\|$$

since

$$\|H\| \, \|[Y, Z]\| = \|H\| \, \|[Y, Z_1]\| \leq \|H\| \, \|Y\| \, \|Z_1\| \leq |2|^{-1} \|Z_1\|.$$

But by the above,

$$q^{-c} \|Z_1\| \leq q^{-c} q^{2s} C_0^{-1} \| \operatorname{ad} H \operatorname{ad} YZ\|'' < \| \operatorname{ad} H \operatorname{ad} YZ\|''$$

when $q^{-c} < q^{-2s} C_0$. Thus for such c we have $|B(X, V)| < \| \operatorname{ad} H \operatorname{ad} YZ\|''$. Thus

$$\begin{aligned} \|[H, k^{-1} \cdot Y]\|'' &= \sup_{X \in \mathfrak{g}, \|X\|' \leq 1} |B(X, [H, k^{-1} \cdot Y])| \\ &= \sup_{X \in \mathfrak{g}, \|X\|' \leq 1} |B(X, \operatorname{ad} H \operatorname{ad} YZ) + B(X, V)| = \| \operatorname{ad} H \operatorname{ad} YZ\|''. \end{aligned}$$ ■

Let $d(\mathfrak{m}^\perp)$ denote the dimension of $\mathfrak{m}^\perp$. Normalize Haar measure dZ_1 on $\mathfrak{m}^\perp$ so that $\{Z_1 \in \mathfrak{m}^\perp : \|Z_1\| \leq 1\}$ has volume one. Let $V(K_c M/M)$ denote the volume of $K_c M/M$ with respect to the invariant measure dy^* on G/M normalized as in (2.3).

Lemma 4.4 *There is $V_M > 0$ so that if $q^{-c} < q^{-4s-c_0} C_0^2$ and $|t| \geq q^{2s+c} C_0^{-1}$, then for all $H \in \mathfrak{m}_s^{\text{reg}}$, $Y \in \mathfrak{g}(\mathfrak{m})_s$, $V(K_c M/M) \phi_c(t^2 H, Y) =$*

$$V_M |t|^{-d(\mathfrak{m}^\perp)} \psi\big(t^2 B(H, Y)\big) \int_{\mathfrak{m}^\perp(H,Y)} \psi\big(1/2 B(Z_1, \operatorname{ad} H \operatorname{ad} Y Z_1)\big) \, dZ_1.$$

Here

$$\mathfrak{m}^\perp(H, Y) = \{Z_1 \in \mathfrak{m}^\perp : \| \operatorname{ad} H \operatorname{ad} Y Z_1\|'' \leq 1\}.$$

Proof Fix $c > 0$ such that $q^{-c} < q^{-4s-c_0} C_0^2$. For $H \in \mathfrak{m}_s^{\text{reg}}$, $Y \in \mathfrak{g}(\mathfrak{m})_s$, define $\mathfrak{k}_c(H, Y, t) = \{Z \in \mathfrak{k}_c : \| \operatorname{ad} H \operatorname{ad} YZ\|'' \leq |t|^{-1}\}$. Using Lemmas 4.2, 4.3, since $c \geq c_0$ and $q^{-c} < q^{-2s} C_0$, for all $t \in F^\times$ such that $|t| \geq q^c$, we have

$$\phi_c(t^2 H, Y) = \int_{\mathfrak{k}_c(H,Y,t)} \psi\big(t^2 B(\exp Z \cdot H, Y)\big) \, dZ,$$

where dZ is the Haar measure on $\mathfrak{g}$ for which $\mathfrak{k}_c$ has volume one.

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Let $Z \in \mathfrak{k}_c(H, Y, t)$. Then

$$B(\exp Z \cdot H, Y) = B(H, Y) + B([Z, H], Y) + 1/2B\big((\operatorname{ad} Z)^2 H, Y\big) + b$$

where

$$b = \sum_{k \geq 3} \frac{1}{k!} B\big((\operatorname{ad} Z)^k H, Y\big).$$

But $B([Z, H], Y) = B(Z, [H, Y]) = 0$ and $B\big((\operatorname{ad} Z)^2 H, Y\big) = B(Z, \operatorname{ad} H \operatorname{ad} Y Z)$. Fix $k \geq 3$. Then

$$\left| \frac{1}{k!} B\big((\operatorname{ad} Z)^k H, Y\big) \right| = \left| \frac{1}{k!} B\big((\operatorname{ad} Z)^{k-1} H, [Y, Z]\big) \right|.$$

Write $Z = Z_0 + Z_1$ where $Z_0 \in \mathfrak{m}$, $Z_1 \in \mathfrak{m}^\perp$. Then

$$\|Z\|' = \max\{\|Z_0\|, \|Z_1\|\},$$

so that $\|Z_0\| \leq q^{-c}$ and $\|Z_1\| \leq q^{-c}$. Now $[Y, Z] = [Y, Z_1] \in \mathfrak{m}^\perp$, and

$$(\operatorname{ad} Z)^{k-1} H = \sum \operatorname{ad} Z_{\epsilon_1} \operatorname{ad} Z_{\epsilon_2} \cdots \operatorname{ad} Z_{\epsilon_{k-1}} H$$

where the sum is over multi-indices $\epsilon = \{\epsilon_i\}_{i=1}^{k-1}$, $\epsilon_i \in \{0, 1\}$, $1 \leq i \leq k - 1$. If $\epsilon_i = 0$, $1 \leq i \leq k - 1$, then this term is $(\operatorname{ad} Z_0)^{k-1} H \in \mathfrak{m}$, and

$$B\big((\operatorname{ad} Z_0)^{k-1} H, [Y, Z_1]\big) = 0$$

since $[Y, Z_1] \in \mathfrak{m}^\perp$. Thus

$$\left| \frac{1}{k!} B\big((\operatorname{ad} Z)^{k-1} H, [Y, Z]\big) \right| \leq \max_\epsilon \left| \frac{1}{k!} \right| |B(\operatorname{ad} Z_{\epsilon_1} \cdots \operatorname{ad} Z_{\epsilon_{k-1}} H, [Y, Z_1])|,$$

where the sum is over multi-indices $\epsilon = \{\epsilon_i\}_{i=1}^{k-1}$ for which at least one $\epsilon_i = 1$. For each such ϵ, using Lemma 4.1 we have

$$\left| \frac{1}{k!} \right| |B(\operatorname{ad} Z_{\epsilon_1} \cdots \operatorname{ad} Z_{\epsilon_{k-1}} H, [Y, Z_1])| \leq \left| \frac{1}{k!} \right| q^{-c(k-2)} \|H\| \, \|Y\| \, \|Z_1\|^2 \leq q^{-(c-c_0)(k-2)} \|Z_1\|^2.$$

But by Lemma 4.3, for $k \geq 3$,

$$q^{-(c-c_0)(k-2)} \|Z_1\|^2 \leq q^{-(c-c_0)} q^{4s} C_0^{-2} (\| \operatorname{ad} H \operatorname{ad} Y Z\|'')^2 \leq |t|^{-2}$$

for $Z \in \mathfrak{k}_c(H, Y, t)$ since $q^{-c} \leq q^{-4s-c_0} C_0^2$. Thus $|b| \leq |t|^{-2}$ so that $\psi(t^2 b) = 1$ and

$$\phi_c(t^2 H, Y) = \psi\big(t^2 B(H, Y)\big) \int_{\mathfrak{k}_c(H,Y,t)} \psi\big(t^2 B(Z, \operatorname{ad} H \operatorname{ad} Y Z)\big) \, dZ.$$

Write $Z = Z_0 + Z_1$, $Z_0 \in \mathfrak{m}$, $Z_1 \in \mathfrak{m}^\perp$, and let $\mathfrak{m}_c = \mathfrak{m} \cap \mathfrak{k}_c$, $\mathfrak{m}_c^\perp = \mathfrak{m}^\perp \cap \mathfrak{k}_c$. Let dZ_0 denote the Haar measure on $\mathfrak{m}$ for which $\mathfrak{m}_c$ has volume one. Now

$$\|Z\|' = \max\{\|Z_0\|, \|Z_1\|\}, \quad \| \operatorname{ad} H \operatorname{ad} Y Z\|'' = \| \operatorname{ad} H \operatorname{ad} Y Z_1\|'',$$

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since $\operatorname{ad} YZ = \operatorname{ad} YZ_1$. Thus $\mathfrak{k}_c = \mathfrak{m}_c \oplus \mathfrak{m}_c^\perp$, $dZ = q^{cd(\mathfrak{m}^\perp)}\, dZ_0\, dZ_1$, and $\mathfrak{k}_c(H,Y,t) = \mathfrak{m}_c \oplus \mathfrak{k}_c^1(H,Y,t)$ where $\mathfrak{k}_c^1(H,Y,t) = \mathfrak{k}_c(H,Y,t) \cap \mathfrak{m}^\perp$. Further,

$$B\big(Z_0 + Z_1, \operatorname{ad} H \operatorname{ad} Y(Z_0 + Z_1)\big) = B(Z_0 + Z_1, \operatorname{ad} H \operatorname{ad} YZ_1) = B(Z_1, \operatorname{ad} H \operatorname{ad} YZ_1).$$

Thus using the change of variables $W_1 = tZ_1$, we have

$$\begin{aligned}\int_{\mathfrak{k}_c(H,Y,t)} &\psi\big(t^2 1/2B(Z, \operatorname{ad} H \operatorname{ad} YZ)\big)\, dZ \\ &= q^{cd(\mathfrak{m}^\perp)} \int_{\mathfrak{m}_c} dZ_0 \int_{\mathfrak{k}_c^1(H,Y,t)} \psi\big(t^2 1/2B(Z_1, \operatorname{ad} H \operatorname{ad} YZ_1)\big)\, dZ_1 \\ &= |t|^{-d(\mathfrak{m}^\perp)} q^{cd(\mathfrak{m}^\perp)} \int \psi\big(1/2B(W_1, \operatorname{ad} H \operatorname{ad} YW_1)\big)\, dW_1,\end{aligned}$$

where the last integral is over $\{W_1 \in \mathfrak{m}^\perp(H,Y) : \|W_1\| \le q^{-c}|t|\}$. But recall from Lemma 4.3, that

$$\|W_1\| \le q^{2s} C_0^{-1} \|\operatorname{ad} H \operatorname{ad} YW_1\|''.$$

Thus for $|t| \ge q^{2s+c} C_0^{-1}$ and $W_1 \in \mathfrak{m}^\perp(H,Y)$, we automatically have

$$\|W_1\| \le q^{2s} C_0^{-1} \le q^{-c}|t|.$$

Finally, $V_M = V(K_c M/M) q^{cd(\mathfrak{m}^\perp)}$ is independent of c. ■

For $H \in \mathfrak{m}^{\mathrm{reg}}$, define $\eta_{\mathfrak{g}/\mathfrak{m}}(H) = \det \operatorname{ad} H|_{\mathfrak{m}^\perp} = \det T_H$. Note that for $Y \in \mathfrak{g}(\mathfrak{m})$, this agrees with our definition in (2.2). Let

(4.6) $$\Lambda(\mathfrak{m}) = \{(Y,H) \in \mathfrak{m}^{\mathrm{reg}} \times \mathfrak{m}^{\mathrm{reg}} : \|Y\|\,\|H\| \le |2|\}.$$

Note that for $Y, H \in \mathfrak{m}^{\mathrm{reg}}$, $t \in F^\times$, $(tY,H) \in \Lambda(\mathfrak{m})$ if and only if $(Y,tH) \in \Lambda(\mathfrak{m})$. Further, for all $s > 0$, $\mathfrak{m}_s^{\mathrm{reg}} \times \mathfrak{m}_s^{\mathrm{reg}} \subset \Lambda(\mathfrak{m})$. For $(Y,H) \in \Lambda(\mathfrak{m})$, define

(4.7) $$c(\mathfrak{m},Y,H) = V_M |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2} \int_{\mathfrak{m}^\perp(H,Y)} \psi\big(1/2B(Z, \operatorname{ad} H \operatorname{ad} YZ)\big)\, dZ.$$

Lemma 4.5 *(i) $c(\mathfrak{m},Y,H)$ is locally constant for $(Y,H) \in \Lambda(\mathfrak{m})$;*

(ii) $c(\mathfrak{m},tY,H) = c(\mathfrak{m},Y,tH)$ for all $Y,H \in \mathfrak{h}$, $t \in F^\times$, such that $(tY,H) \in \Lambda(\mathfrak{m})$;
(iii) $|c(\mathfrak{m},Y,H)|$ is non-zero and independent of $(Y,H) \in \Lambda(\mathfrak{m})$;
(iv) $c(\mathfrak{m},Y,t^2H) = c(\mathfrak{m},Y,H)$ for all $Y,H \in \mathfrak{m}^{\mathrm{reg}}$, $t \in F^\times$, such that $(Y,H) \in \Lambda(\mathfrak{m})$ and $(Y,t^2H) \in \Lambda(\mathfrak{m})$.

Proof Parts (i) and (ii) are obvious from the definition. Let $Z \in \mathfrak{m}^\perp$. Then

$$\|Z\|'' = \sup_{X \in \mathfrak{g}, \|X\|' \le 1} |B(Z,X)| = \sup_{W \in \mathfrak{m}^\perp, \|W\| \le 1} |B(Z,W)|$$

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since if $X = W + W'$, $W \in \mathfrak{m}^\perp$, $W' \in \mathfrak{m}$, then $\|X\|' = \max\{\|W\|, \|W'\|\}$, and $B(Z,X) = B(Z,W)$.

Fix $(Y,H) \in \Lambda(\mathfrak{m})$, and for $Z, W \in \mathfrak{m}^\perp$, define $q(Z,W) = B(Z, \operatorname{ad} H \operatorname{ad} YW) = B(Z, T_H T_Y W)$. Since $T_H T_Y$ is a nonsingular, self-adjoint linear transformation on $\mathfrak{m}^\perp$, q is a non-degenerate symmetric bilinear form on $\mathfrak{m}^\perp$. Let $L = \mathfrak{m}^\perp(H,Y)$. It is a lattice in $\mathfrak{m}^\perp$. Define $\tilde{L} = \{W \in \mathfrak{m}^\perp : |q(W,Z)| \le 1 \; \forall Z \in L\}$. Then

$$\begin{aligned}\tilde{L} &= \{W \in \mathfrak{m}^\perp : |B(W, \operatorname{ad} H \operatorname{ad} YZ)| \le 1 \; \forall Z \in \mathfrak{m}^\perp \text{ with } \|\operatorname{ad} H \operatorname{ad} YZ\|'' \le 1\} \\ &= \{W \in \mathfrak{m}^\perp : |B(W,Z)| \le 1 \; \forall Z \in \mathfrak{m}^\perp \text{ with } \|Z\|'' \le 1\}\end{aligned}$$

is independent of $(Y,H) \in \Lambda(\mathfrak{m})$.

Suppose that $W \in \tilde{L}$. Then $\|\operatorname{ad} H \operatorname{ad} YW\|'' =$

$$\begin{aligned}\sup_{X \in \mathfrak{m}^\perp, \|X\| \le 1} |B(X, \operatorname{ad} H \operatorname{ad} YW)| &= \sup_{X \in \mathfrak{m}^\perp, \|X\| \le 1} |B(\operatorname{ad} H \operatorname{ad} YX, W)| \\ &\le \sup_{X \in \mathfrak{m}^\perp, \|X\| \le |2|} |B(X,W)| \\ &= |2| \sup_{X \in \mathfrak{m}^\perp, \|X\| \le 1} |B(X,W)|\end{aligned}$$

since for all $X \in \mathfrak{m}^\perp$,

$$\|\operatorname{ad} H \operatorname{ad} YX\| \le |2| \|X\|.$$

But for all $X \in \mathfrak{m}^\perp$ such that $\|X\| \le 1$, $\|X\|'' \le \|X\| \le 1$ so that $|B(X,W)| \le 1$. Thus $\|\operatorname{ad} H \operatorname{ad} YW\|'' \le |2|$ so that $W \in 2L$. Thus $\tilde{L} \subset 2L$.

Define

$$I(L) = \int_L \psi\big(1/2q(Z,Z)\big)\, dZ = \int_{\mathfrak{m}^\perp(H,Y)} \psi\big(1/2B(Z, \operatorname{ad} H \operatorname{ad} YZ)\big)\, dZ.$$

Then as in [W], $|I(L)| = \big(m(L)m(\tilde{L})\big)^{1/2}$ where $m(L)$ and $m(\tilde{L})$ denote the measures of L and $\tilde{L}$ with respect to the Haar measure dZ on $\mathfrak{m}^\perp$. But $m(\tilde{L})$ is a positive constant independent of $(Y,H) \in \Lambda(\mathfrak{m})$. Further, using the change of variables $Z_1 = \operatorname{ad} H \operatorname{ad} YZ$, we see that

$$m(L) = \int_{\|\operatorname{ad} H \operatorname{ad} YZ\|'' \le 1} dZ = |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{-1} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1} \int_{\|Z_1\|'' \le 1} dZ_1.$$

Thus

$$|I(L)| = |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1/2} C$$

where $C > 0$ is independent of (Y,H). This proves (iii) since

$$|c(\mathfrak{m}, Y, H)| = V_M |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{1/2} |\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2} |I(L)| = V_M C.$$

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Finally, let $Y, H \in \mathfrak{m}^{\text{reg}}$, $t \in F^\times$, such that (Y,H), $(Y, t^2H) \in \Lambda(\mathfrak{m})$. We may as well assume that $|t| \le 1$, since if $|t| > 1$, we can replace H by $H' = t^2H$ and t by $t' = t^{-1}$. Now $c(\mathfrak{m}, Y, t^2H) =$

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(t^2H)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2}\int_{\|\,\mathrm{ad}\, t^2H\,\mathrm{ad}\,YZ\|''\le 1}\psi\big(1/2B(Z,\mathrm{ad}\,t^2H\,\mathrm{ad}\,YZ)\big)\,dZ$$
$$= |\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2}\int_{\|\,\mathrm{ad}\,H\,\mathrm{ad}\,YZ\|''\le |t|^{-1}}\psi\big(1/2B(Z,\mathrm{ad}\,H\,\mathrm{ad}\,YZ)\big)\,dZ$$

using the change of variables $Z' = tZ$, since

$$|\eta_{\mathfrak{g}/\mathfrak{m}}(t^2H)|^{1/2} = |t|^{d(\mathfrak{m}^\perp)}|\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{1/2}.$$

Thus for (iii) it suffices to prove that

$$I(L_t) = \int_{L_t}\psi\big(1/2q(Z,Z)\big)\,dZ = I(L) = \int_L \psi\big(1/2q(Z,Z)\big)\,dZ$$

where $L_t = \{Z \in \mathfrak{m}^\perp : \|\,\mathrm{ad}\,H\,\mathrm{ad}\,YZ\|'' \le |t|^{-1}\} = t^{-1}L$. But $\tilde{L}_t = t\tilde{L} \subset t2L \subset t^{-1}2L = 2L_t$ since $|t| \le 1$. Thus as above, we have

$$|I(L_t)| = \big(m(L_t)m(\tilde{L}_t)\big)^{1/2} = \big(m(t^{-1}L)m(t\tilde{L})\big)^{1/2} = \big(m(L)m(\tilde{L})\big)^{1/2} = |I(L)|.$$

Further, as in [W]

$$I(L_t)/|I(L_t)| = I(L)/|I(L)|. \qquad \blacksquare$$

Remark 4.2 It is possible to analyse the terms $c(\mathfrak{m}, Y, H)$ further as in Section VIII.5 of [10].

Using Lemma 4.5 we can extend the definition of $c(\mathfrak{m})$ to $\mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$ as follows. Let $(Y,H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$. Then $Y, H \in \mathfrak{m}^{\text{reg}}$, and there is $t \in F^\times$ such that $\|Y\|\,\|t^2H\| \le |2|$, so that $(Y, t^2H) \in \Lambda(\mathfrak{m})$. Now we define

$$c(\mathfrak{m}, Y, H) = c(\mathfrak{m}, Y, t^2H). \tag{4.8}$$

By Lemma 4.5 this is independent of the choice of t.

Proposition 4.6 *Let ω_1 be a compact subset of $\mathfrak{m}''$ and let ω_2 be a compact subset of $\mathfrak{g}(\mathfrak{m})$. Then there is $c_1 > 0$ with the following property. For each $c \ge c_1$ there is $C(c) > 0$ so that for all $H \in \omega_1$, $Y \in \omega_2$, and $t \in F^\times$ such that $|t| \ge C(c)$,*

$$V(K_cM/M)\phi_c(tH, Y) = |\eta_{\mathfrak{g}/\mathfrak{m}}(tH)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1/2}\psi\big(B(tH,Y)\big)c(\mathfrak{m}, Y, tH).$$

Here $c(\mathfrak{m})$ is a locally constant function on $\mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$ satisfying

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(i) $|c(\mathfrak{m}, Y, H)|$ *is non-zero and independent of* $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$*;*
(ii) $c(\mathfrak{m}, Y, t^2H) = c(\mathfrak{m}, Y, H)$ *for all* $t \in F^\times$, $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$*;*
(iii) $c(\mathfrak{m}, tY, H) = c(\mathfrak{m}, Y, tH)$ *for all* $t \in F^\times$, $(Y, H) \in \mathfrak{g}(\mathfrak{m}) \times \mathfrak{m}''$.

Proof The properties of the function $c(\mathfrak{m})$ follow immediately from Lemma 4.5 and the definition (4.8).

Let $\omega \subset \mathfrak{m}''$ be compact. We will prove that there is $c_1 > 0$ so that for $c \geq c_1$ there is $C'(c) > 0$ such that for all $|t| \geq C'(c)$, $H \in \omega$, $Y \in \omega_2$,

$$V(K_cM/M)\phi_c(t^2H, Y) = |\eta_{\mathfrak{g}/\mathfrak{m}}(t^2H)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1/2}\psi\big(B(t^2H, Y)\big)c(\mathfrak{m}, Y, t^2H).$$

This is sufficient to prove the proposition, since given $\omega_1 \subset \mathfrak{m}''$ compact,

$$\omega = \{t_0H : H \in \omega_1, q^{-1} \leq |t_0| \leq 1\}$$

is also a compact subset of $\mathfrak{m}''$. Now suppose $|t| \geq C(c) = C'(c)^2$ and $H \in \omega_1$. Now we can write $tH = t_1^2H_0$ where $H_0 \in \omega$ and $|t_1| \geq C'(c)$.

Let ω_1 be a compact subset of $\mathfrak{m}''$ and ω_2 be a compact subset of $\mathfrak{g}(\mathfrak{m})$. Since ω_1 and ω_2 are compact, there is r_0 such that for all $H \in \omega_1$, $Y \in \omega_2$,

$$\|Y\| \leq |2|^{1/2}q^{r_0}, \quad \|H\| \leq |2|^{1/2}q^{r_0}.$$

Write $t_0 = \varpi^{r_0}$ and let $\omega_1' = \{t_0H : H \in \omega_1\}$, $\omega_2' = \{t_0Y : Y \in \omega_2\}$. Then for all $H' \in \omega_1'$, $Y' \in \omega_2'$,

$$\|Y'\| \leq |2|^{1/2}, \quad \|H'\| \leq |2|^{1/2}.$$

Since $\omega_1' \subset \mathfrak{m}''$ and $\omega_2' \subset \mathfrak{g}(\mathfrak{m})$ are compact, there is $s_0 > 0$ so that for all $H' \in \omega_1'$, $Y' \in \omega_2'$,

$$\|T_{H'}^{-1}\| \leq q^{s_0}, \quad \|T_{Y'}^{-1}\| \leq q^{s_0}.$$

Thus $\omega_1' \subset \mathfrak{m}_{s_0}^{\mathrm{reg}}$ and $\omega_2' \subset \mathfrak{g}(\mathfrak{m})_{s_0}$.

Pick $c_1 > 0$ big enough that $q^{-c_1} \leq q^{-4s_0-c_0}C_0^{-2}$, and let $c \geq c_1$. Let $H \in \omega_1$, $Y \in \omega_2$, $t \in F$ such that $|t| \geq C(c) = q^{c+2s_0+r_0}C_0^{-1}$. Then $t_0H = H' \in \mathfrak{m}_{s_0}^{\mathrm{reg}}$, $t_0Y = Y' \in \mathfrak{g}(\mathfrak{m})_{s_0}$, and $|tt_0^{-1}| = q^{-r_0}|t| \geq q^{c+2s_0}C_0^{-1}$.

Now combining Lemma 4.4, the definition of $c(\mathfrak{m}, Y, H)$, Lemma 4.5 (ii) and (iv), and the fact that $|\eta_{\mathfrak{g}/\mathfrak{m}}(t^2H)|^{-1/2} = |t|^{-d(\mathfrak{m}^\perp)}|\eta_{\mathfrak{g}/\mathfrak{m}}(H)|^{-1/2}$, we have

$$\begin{aligned}
V(K_cM/M)\phi_c(t^2H, Y) &= V(K_cM/M)\phi_c\big((tt_0^{-1})^2H', Y'\big) \\
&= |\eta_{\mathfrak{g}/\mathfrak{m}}\big((tt_0^{-1})^2H'\big)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y')|^{-1/2}\psi\big((tt_0^{-1})^2B(H', Y')\big) \\
&\qquad c(\mathfrak{m}, Y', H') \\
&= |\eta_{\mathfrak{g}/\mathfrak{m}}(t_0^{-1}t^2H)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(t_0Y)|^{-1/2}\psi\big(t^2B(H, Y)\big)c(\mathfrak{m}, t_0Y, t_0H) \\
&= |\eta_{\mathfrak{g}/\mathfrak{m}}(t^2H)|^{-1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{-1/2}\psi\big(t^2B(H, Y)\big)c(\mathfrak{m}, Y, H). \qquad \blacksquare
\end{aligned}$$

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5 Proof of Theorem 2.2

In this section we use Proposition 4.6 to prove Theorem 2.2.

Fix a reductive subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and define $N(\mathfrak{h},\mathfrak{m}) = \{y \in G : y^{-1}\mathfrak{h} \subset \mathfrak{m}\}$, $W(\mathfrak{h},\mathfrak{m}) = N(\mathfrak{h},\mathfrak{m})/M$, as in Section 2. For $Y \in \mathfrak{g}(\mathfrak{m})$, define $wY = y \cdot Y$ if $w = yM \in W(\mathfrak{h},\mathfrak{m})$.

Lemma 5.1 *Let $\mathcal{O}$ be an orbit in $\mathfrak{g}$ and $Y \in \mathfrak{g}(\mathfrak{m}) \cap \mathrm{cl}(\mathcal{O})$. Then $w \mapsto wY$ gives a bijection between $W(\mathfrak{h},\mathfrak{m})$ and $\mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$.*

Proof Let $Y' \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$. Then there is $y \in G$ such that $Y' = y \cdot Y$. Now $Y \in y^{-1}\mathfrak{h}$ so that $y^{-1}\mathfrak{h} \subset \mathfrak{g}_Y = \mathfrak{m}$. Thus $y \in N(\mathfrak{h},\mathfrak{m})$, and $Y' = wY$ where $w = yM$. Conversely, if $w = yM \in W(\mathfrak{h},\mathfrak{m})$, then $y^{-1}\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{m}$. Since Y is a central semsimple element in $\mathfrak{m}$, $Y \in y^{-1}\mathfrak{h}$ so that $wY = y \cdot Y \in \mathfrak{h}$. Finally, the mapping is one-to one since $M = G_Y$. ■

The following lemma is elementary.

Lemma 5.2 *Let $\mathfrak{h}_1, \dots, \mathfrak{h}_k$ denote a complete set of representatives for the M-conjugacy classes of Cartan subalgebras of $\mathfrak{m}$ which are G-conjugate to $\mathfrak{h}$. For each $1 \leq i \leq k$, let $x_i \in G$ such that $x_i\mathfrak{h}_i = \mathfrak{h}$. Then $N(\mathfrak{h},\mathfrak{m}) = \{x_i n_i m : m \in M, n_i \in N_G(\mathfrak{h}_i), 1 \leq i \leq k\}$, and $W(\mathfrak{h},\mathfrak{m}) = \bigcup_i x_i N_G(\mathfrak{h}_i)/N_M(\mathfrak{h}_i)$. In particular, $W(\mathfrak{h},\mathfrak{m})$ is a finite set.*

For each $w \in W(\mathfrak{h},\mathfrak{m})$, fix $y_w \in N(\mathfrak{h},\mathfrak{m})$ with $w = y_w M$. We can assume that y_w is chosen so that $y_w^{-1}\mathfrak{h} = \mathfrak{h}_i$ for some $1 \leq i \leq k$. Let $\gamma \in \mathfrak{g}(\mathfrak{m})$ and let ω_γ be a compact neighborhood of γ in $\mathfrak{m}$. We can assume that ω_γ is small enough that $y_{w_1} \cdot \omega_\gamma \cap y_{w_2} \cdot \omega_\gamma = \varnothing$ for all $w_1, w_2 \in W(\mathfrak{h},\mathfrak{m})$ with $w_1 \neq w_2$, since $y_{w_1}\gamma \neq y_{w_2}\gamma$ in this case. Let U_γ be an M-domain in $\mathfrak{m}$ such that $\gamma \subset U_\gamma \subset \omega_\gamma^M$ which satisfies the conditions of Corollary 2.3 of [2]. In particular, we can assume that $U_\gamma \cap \mathfrak{h}_i \subset \omega_\gamma$, $1 \leq i \leq k$, $C_\mathfrak{g}(X) \subset \mathfrak{m}$ for all $X \in U_\gamma$, and for every compact subset Q of $\mathfrak{g}$ there is a compact subset Ω of G such that $x \cdot U_\gamma \cap Q \neq \varnothing$ implies that $x \in \Omega M$. Let $\omega_2(\gamma) = U_\gamma \cap \mathfrak{g}(\mathfrak{m})$. Then $\omega_2(\gamma)$ is a compact neighborhood of γ in $\mathfrak{g}(\mathfrak{m})$. Since for any compact $\omega_2 \subset \mathfrak{g}(\mathfrak{m})$, there are finitely many γ_i, $1 \leq i \leq r$, in ω_2 such that $\omega_2 \subset \bigcup_{1 \leq i \leq r} \omega_2(\gamma_i)$, it suffices to prove Theorem 2.2 when ω_2 is of the form $\omega_2 = U_\gamma \cap \mathfrak{g}(\mathfrak{m})$ for some $\gamma \in \mathfrak{g}(\mathfrak{m})$.

Fix $\gamma \in \mathfrak{g}(\mathfrak{m})$ and let $\omega = \omega_\gamma$, $U = U_\gamma$, $\omega_2 = U \cap \mathfrak{g}(\mathfrak{m})$ as above. Define $V = U^G$. By Corollary 2.4 of [2], $V \subset \omega^G$ and is a G-domain in $\mathfrak{g}$, that is an open and closed G-invariant set. For any compact open subgroup K of G, define $V(K) = \{ky \cdot X : k \in K, y \in N(\mathfrak{h},\mathfrak{m}), X \in U\}$. Thus $V(K) = \varnothing$ if $N(\mathfrak{h},\mathfrak{m}) = \varnothing$. For $C > 0$, we write $V_C = \{X \in V : \|X\| \leq C\}$, $V_C(K) = V(K) \cap V_C$.

Lemma 5.3 *Let $y \in G$ such that $y \cdot U \cap \mathfrak{h} \neq \varnothing$. Then $y \in N(\mathfrak{h},\mathfrak{m})$. Let K be a compact open subgroup of G, and let $X = Y + Z$ where $Y \in \omega_2$ and $Z \in \mathcal{N}_M$. Then for $x \in G$, $x \cdot X \notin V(K)$ unless $x \in KN(\mathfrak{h},\mathfrak{m})$.*

Proof Let $y \in G$ and $Y \in U$ such that $y \cdot Y \in \mathfrak{h}$. Then $y^{-1}\mathfrak{h} \subset C_\mathfrak{g}(Y) \subset \mathfrak{m}$, so that $y \in N(\mathfrak{h},\mathfrak{m})$.

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Let $X = Y + Z$ where $Y \in \omega_2$ and $Z \in \mathcal{N}_M$. Then Y is the semisimple part in the Jordan decomposition of X. Let $x \in G$ such that $x \cdot X \in V(K)$. Then there are $k \in K$, $y \in N(\mathfrak{h}, \mathfrak{m})$ so that $y^{-1}k^{-1}x \cdot X \in U$. Write $g = x^{-1}ky$. Now since U is an M-domain and $g^{-1} \cdot Y$ is the semisimple part in the Jordan decomposition for $g^{-1} \cdot X \in U$, we have $g^{-1} \cdot Y \in U$.

Since the statement is trivial if $N(\mathfrak{h}, \mathfrak{m}) = \varnothing$, we may as well assume that $N(\mathfrak{h}, \mathfrak{m}) \neq \varnothing$. Fix $w_0 \in W(\mathfrak{h}, \mathfrak{m})$, and let $y_0 = y_{w_0} \in N(\mathfrak{h}, \mathfrak{m})$ be our fixed representative for w_0. Since $Y \in \omega_2 \subset \mathfrak{g}(\mathfrak{m})$, we have $y_0 \cdot Y \in \mathfrak{h}$ by Lemma 5.1. Now $Y \in gU$ so that $y_0 \cdot Y \in \mathfrak{h} \cap y_0 gU$. By the first part of the lemma, this implies that there are $w \in W(\mathfrak{h}, \mathfrak{m})$, $m \in M$, such that $y_0 g = y_w m$. By assumption there is $1 \leq i \leq k$ so that $y_w^{-1}\mathfrak{h} = \mathfrak{h}_i$. Now $y_w^{-1} y_0 Y \in y_w^{-1}\mathfrak{h} = \mathfrak{h}_i$. But since U is M-stable and $m^{-1}y_w^{-1}y_0 \cdot Y \in U$, then $y_w^{-1}y_0 \cdot Y \in U \cap \mathfrak{h}_i \subset \omega$. Thus $y_0 \cdot Y \in y_0 \cdot \omega \cap y_w \cdot \omega$. By our assumption on ω, this implies that $y_0 = y_w$ so that $g = y_0^{-1} y_w m \in M$. Thus $x = kyg^{-1} \in KN(\mathfrak{h}, \mathfrak{m})M = KN(\mathfrak{h}, \mathfrak{m})$. ■

Lemma 5.4 *Let $K \subset \mathrm{GL}(n, \mathcal{R})$ be a compact open subgroup of G and let ω_1 be a compact subset of $\mathfrak{h}'$. Then there are $C > 0$, $C' > 0$ such that for all $H \in \omega_1$, $X \in V$,*

(i) $\phi_K(tH, X) = 0$ *for all* $|t| \geq 1$ *unless* $X \in V_{C'}$;
(ii) $\phi_K(tH, X) = 0$ *for all* $|t| \geq C$ *unless* $X \in V_{C'}(K)$.

Proof Suppose first that $\mathfrak{g}$ is semisimple. Since $V \subset \omega^G$ where ω is compact, the eigenvalues of ad X, $X \in V \cap \mathfrak{h}$ are bounded. Thus there is $C_1 > 0$ so that $\|X\| \leq C_1$ for all $X \in \mathfrak{h} \cap V$. Define S and $\nu(X)$, $X \neq 0 \in \mathfrak{g}$, as in the proof of Theorem 2.1. Let S_1 denote the closure in S of

$$\{\varpi^{-\nu(X)}X : X \in V, \|X\| > C_1\}.$$

It is a compact set, and as in Lemma 7.4 of [2], every element of S_1 is either nilpotent or is of the form $\varpi^{-\nu(X)}X$ for some $X \in V$, $\|X\| > C_1$.

Let $X' \in S_1$, $H \in \omega_1$, and suppose that $[k \cdot H, X'] = 0$ for some $k \in K$. Then $k^{-1}X' \in \mathfrak{h}$, so that X' is semisimple, and hence of the form $X' = \varpi^{-\nu(X)}X$ for some $X \in V$, $\|X\| > C_1$. But then $k^{-1}X \in \mathfrak{h} \cap V$. But this can't be because $\|k^{-1}X\| = \|X\| > C_1$. Thus $[k \cdot H, X'] \neq 0$ for all $k \in K$, $H \in \omega_1$, $X' \in S_1$, and so by Lemma 3.1 there is $C_2 > 0$ so that $\phi_K(tH, X') = 0$ for all $H \in \omega_1$, $X' \in S_1$, $t \in F$ such that $|t| \geq C_2$. Let $C' = \max\{C_1, C_2\}$, and let $X \in V$, $\|X\| \geq C'$, $t \in F$, $|t| \geq 1$, and $H \in \omega_1$. Then $X' = \varpi^{-\nu(X)}X \in S_1$ and $|t\varpi^{\nu(X)}| \geq C_2$, so that for all $H \in \omega_1$, $\phi(tH, X) = \phi(t\varpi^{\nu(X)}H, X') = 0$. Thus

$$\phi(tH, X) = 0, \quad H \in \omega_1,\ |t| \geq 1,\ X \in V,\ \|X\| > C'.$$

Since $V_{C'}$ is compact, it follows from Corollary 2.3 of [2] that there is a compact subset Ω of G so that if $x \in G$ such that $x \cdot Y \in V_{C'}$ for some $Y \in U$, then $x \in \Omega M$. Now there are finitely many $y_i \in G$, $1 \leq i \leq r$, so that $\Omega M \subset \bigcup_{1 \leq i \leq r} K y_i M$. Thus $V_{C'} \subset \bigcup_{1 \leq i \leq r} V_{C'}(i)$ where $V(i) = \{ky_i \cdot Y : k \in K, Y \in U\}$, $V_{C'}(i) = V(i) \cap V_{C'}$.

Let $1 \leq i \leq r$, and suppose that $V(i) \cap \mathfrak{h} \neq \varnothing$. Then there is $y \in Ky_i$ such that $y \cdot U \cap \mathfrak{h} \neq \varnothing$. Now by Lemma 5.3, $y \in N(\mathfrak{h}, \mathfrak{m})$, so that $Ky_iM = KyM \subset KN(\mathfrak{h}, \mathfrak{m})$. Thus in this case, $V_{C'}(i) \subset V_{C'}(K)$ and we write $C_i = 1$.

Suppose that $V(i) \cap \mathfrak{h} = \varnothing$. Then $[k \cdot H, X] \neq 0$ for any $k \in K$, $H \in \omega_1$, $X \in V_{C'}(i)$, so that by Lemma 3.1 there is $C_i \geq 1$ so that $\phi(tH, X) = 0$ for all $H \in \omega_1$, $X \in V_{C'}(i)$, $|t| \geq C_i$.

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Now let $C = \max\{C_i, 1 \le i \le r\}$. Let $H \in \omega_1$, $|t| \ge C$, and $X \in V$ such that $X \notin V_{C'}(K)$. Then either $\|X\| > C'$, or $X \in V_{C'}(i)$ for some $1 \le i \le r$ such that $V(i) \cap \mathfrak{h} = \varnothing$. In either case, we have $\phi_K(tH, X) = 0$. This completes the proof in the case that $\mathfrak{g}$ is semisimple.

Let $\mathfrak{g}$ be reductive, write $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}_s$, $\mathfrak{m}_s = \mathfrak{m} \cap \mathfrak{g}_s$, and use the notation from the proof of Lemma 3.2. Then $\mathfrak{m} = \mathfrak{z} + \mathfrak{m}_s$, $M_s = M/Z$, and $\mathfrak{g}(\mathfrak{m}) = \mathfrak{z} + \mathfrak{g}_s(\mathfrak{m}_s)$. Let $\gamma \in \mathfrak{g}(\mathfrak{m})$. We can assume that $\omega_\gamma = \omega_0 + \omega_s$ where ω_0 is a compact neighborhood of γ_0 in $\mathfrak{z}$ and $\omega_s = \omega_{\gamma_s}$ is a compact neighborhood of γ_s in $\mathfrak{m}_s$. Then we could take $U_\gamma = \omega_0 + U_{\gamma_s}$. Thus $V = \omega_0 + V_s$ where $V_s = U_{\gamma_s}^G$. Let $C_0 > 0$ so that $\|X_0\| \le C_0$ for all $X_0 \in \omega_0$. Then for $C' \ge C_0$, $V_{C'} = \omega_0 + (V_s)_{C'}$. Let K be a compact open subgroup of G. Then $K_s = K/(Z \cap K)$ is a compact open subgroup of G_s, and $V(K) = \omega_0 + V_s(K_s)$.

Let $\omega_1 \subset \mathfrak{h}'$ be compact, and let $C, C' > 0$ satisfy the conditions of the lemma for $p(\omega_1)$, V_s, and K_s. By making C' larger if necessary, we can assume that $C' \ge C_0$. Let $H \in \omega_1$, $t \in F^\times$, $X \in V$. Then $\phi_K(tH, X) = \psi\big(B(tH_0, X_0)\big)\phi_{K_s}(tH_s, X_s)$. Thus $\phi_K(tH, X) = 0$ for all $|t| \ge 1$ unless $X \in V_{C'}$, and $\phi_K(tH, X) = 0$ for all $|t| \ge C$ unless $X \in V_{C'}(K)$. ■

Let $p\colon \mathfrak{g} \to \mathfrak{m}$ denote the projection corresponding to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}^\perp$. For any $\xi \in \Xi_M$, let $\hat\mu_\xi^M$ denote the Fourier transform of the orbital integral on $\mathfrak{m}$ corresponding to ξ as in (2.5). Recall that for each $w \in W(\mathfrak{h}, \mathfrak{m})$, we have fixed a representative $y_w \in N(\mathfrak{h}, \mathfrak{m})$.

Lemma 5.5 *There is a compact open subgroup K_0 of G so that for all $H \in \omega_1$, $k \in K_0$, $w \in W(\mathfrak{h}, \mathfrak{m})$, $p(ky_w^{-1} \cdot H) \in \mathfrak{m}'$ and $\hat\mu_\xi^M\big(tp(ky_w^{-1} \cdot H)\big) = \hat\mu_\xi^M(ty_w^{-1} \cdot H)$ for all $\xi \in \Xi_M$ and $t \in F^\times$.*

Proof The set Ξ_M of nilpotent orbits of $\mathfrak{m}$ is finite, $W(\mathfrak{h}, \mathfrak{m})$ is finite, and for each $\xi \in \Xi_M$, $\hat\mu_\xi^M$ is locally constant on $\mathfrak{m}'$. Now since ω_1 is compact, p is continuous, and for each $H \in \omega_1$, $w \in W(\mathfrak{h}, \mathfrak{m})$, $p(y_w^{-1} \cdot H) = y_w^{-1} \cdot H \in \mathfrak{m}'$, it is easy to see that we can chose K_0 small enough that for all $H \in \omega_1$, $w \in W(\mathfrak{h}, \mathfrak{m})$, $k \in K_0$, $p(ky_w^{-1} \cdot H) \in \mathfrak{m}'$ and $\hat\mu_\xi^M\big(p(ky_w^{-1} \cdot H)\big) = \hat\mu_\xi^M(y_w^{-1} \cdot H)$ for all $\xi \in \Xi_M$.

Let $\xi \in \Xi_M$ and let $t \in F^\times$. It follows from the discussion in 5.1 of [2] that there is $c_\xi(t) > 0$ so that

$$\hat\mu_\xi^M(tX) = c_\xi(t)\hat\mu_{t\xi}^M(X)$$

for all $X \in \mathfrak{m}'$, where $t\xi = \{tZ : Z \in \xi\} \in \Xi_M$ also. Now for $H \in \omega_1$, $w \in W(\mathfrak{h}, \mathfrak{m})$, and $k \in K_0$,

$$\hat\mu_\xi^M\big(tp(ky_w^{-1} \cdot H)\big) = c_\xi(t)\hat\mu_{t\xi}^M\big(p(ky_w^{-1} \cdot H)\big) = c_\xi(t)\hat\mu_{t\xi}^M(y_w^{-1} \cdot H) = \hat\mu_\xi^M(ty_w^{-1} \cdot H).$$ ■

Let $w = yM \in W(\mathfrak{h}, \mathfrak{m})$. Then for $\xi \in \Xi_M$, $H \in \mathfrak{h}'$, $\hat\mu_\xi^M(w^{-1}H) = \hat\mu_\xi^M(y^{-1} \cdot H)$ is independent of the representative chosen since $\hat\mu_\xi^M$ is an M-invariant function on $\mathfrak{m}'$. For any compact open subgroup K of G, the coset $KwM = KyM$ is also independent of the coset representative. For $Y \in \omega_2$, $Z \in \mathcal{N}_M$, $H \in \mathfrak{h}$, define $T_K(Y, Z, H)$ are in (2.3). It depends only on the orbit ξ of Z, so we also write $T_K(Y, \xi, H) = T_K(Y, Z, H)$, $Z \in \xi$.

Proposition 5.6 *There is a compact open subgroup K_1 of G with the following property. If $K \subset K_1$ is a compact open subgroup of G, there is $C > 0$ so that for all $Y \in \omega_2$, $\xi \in \Xi_M$, $H \in \omega_1$, $|t| \geq C$,*

$$T_K(Y, \xi, tH) = \sum_{w \in W(\mathfrak{h}, \mathfrak{m})} \hat{\mu}_\xi^M(tw^{-1}H) V(KwM/M) \phi_K(tH, wY).$$

Here $V(KwM/M)$ denotes the measure of KwM/M with respect to the measure dy^* on G/M.

Proof Write $W = W(\mathfrak{h}, \mathfrak{m})$, and define K_0 as in Lemma 5.5. By assumption, the sets $y_w \cdot \omega_2$ are disjoint for $w \in W$. Thus we can choose a compact open subgroup $K_1 \subset \mathrm{GL}(n, \mathcal{R})$ which is small enough that the sets $K_1 y_w \cdot \omega_2$ are disjoint for $w \in W$, and $y_w^{-1} K_1 y_w \subset K_0$ for all $w \in W$. Let $w_1, w_2 \in W$, and suppose $x \in K_1 w_1 M \cap K_1 w_2 M$. Let $Y \in \omega_2$. Then $x \cdot Y \in K_1 y_{w_1} \omega_2 \cap K_1 y_{w_2} \omega_2$, so that $w_1 = w_2$. Thus the cosets $K_1 wM$ are disjoint for $w \in W$.

Suppose $K \subset K_1$. By Lemma 5.4, there is $C > 0$ so that for $H \in \omega_1$, $|t| \geq C$, $X \in V$, $\phi_K(tH, X) = 0$ unless $X \in V(K)$. Fix $Y \in \omega_2$, $\xi \in \Xi_M$, $H \in \omega_1$, and $t \in F^\times$, $|t| \geq C$. Pick $Z \in \xi$. Then by Lemma 5.3, for $y \in G$, $m \in M$, $ym \cdot (Y + Z) \in V(K)$ just in case $y \in \bigcup_{w \in W} K y_w M$. Let dk_1 denote normalized Haar measure on K.

Then since the sets $K y_w M$, $w \in W$, are disjoint,

$$T_K(Y, \xi, tH) = \sum_{w \in W} V(KwM/M) T_w(Y, \xi, tH),$$

where for each $w \in W$,

$$T_w(Y, \xi, tH) = \int_K \int_{M/M_Z} \int_K \psi\Big(tB\big(k \cdot H, k_1 y_w m \cdot (Y + Z)\big)\Big)\, dk\, dm^*\, dk_1.$$

Fix $w \in W$. Using the change of variables $k \to k_1 k$ and the invariance of B, we can eliminate the outer integral over K from $T_w(Y, \xi, tH)$. Write $K_w = y_w^{-1} K y_w$, and let dk_w denote normalized Haar measure on K_w. Then using the change of variables $k \to y_w^{-1} k y_w$, $k \in K$, and the invariance of B, we have

$$T_w(Y, \xi, tH) = \int_{M/M_Z} \int_{K_w} \psi\big(tB(k_w y_w^{-1} \cdot H, Y + m \cdot Z)\big)\, dk_w\, dm^*.$$

Write $K_M = K_w \cap M$, and let dk_1 denote normalized Haar measure on K_M. Then $T_w(Y, \xi, tH) =$

$$\int_{M/M_Z} \int_{K_M} \int_{K_w} \psi\big(tB(k_1 k_w y_w^{-1} \cdot H, Y + m \cdot Z)\big)\, dk_w\, dk_1 dm^*.$$

Let $k_1 \in K_M$, $k_w \in K$, $m \in M$. Then

$$\psi\big(tB(k_1 k_w y_w^{-1} \cdot H, Y + m \cdot Z)\big) = \psi\big(tB(k_w y_w^{-1} \cdot H, Y)\big) \psi\Big(tB\big(k_1 p(k_w y_w^{-1} \cdot H), m \cdot Z\big)\Big).$$

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Since $K_w \subset K_0$, by Lemma 5.5, $p(k_w y_w^{-1} \cdot H) \in \mathfrak{m}'$. Now by Lemma 5.4 applied to M in place of G, we can change the order of integration and write $T_w(Y, \xi, tH) =$

$$= \int_{K_w} \psi\big(tB(k_w y_w^{-1} \cdot H, Y)\big) \int_{M/M_Z} \int_{K_M} \psi\Big(tB\big(k_1 p(k_w y_w^{-1} \cdot H), m \cdot Z\big)\Big)\, dk_1\, dm^*\, dk_w.$$

But by Theorem 3 of [3] and Lemma 5.5, for all $k_w \in K_w$,

$$\int_{M/M_Z} \int_{K_M} \psi\Big(tB\big(k_1 p(k_w y_w^{-1} \cdot H), m \cdot Z\big)\Big)\, dk_1\, dm^* = \hat{\mu}_\xi^M\big(tp(k_w y_w^{-1} \cdot H)\big)$$

$= \hat{\mu}_\xi^M(ty_w^{-1} \cdot H)$. Thus

$$\begin{aligned} T_w(Y, \xi, tH) &= \hat{\mu}_\xi^M(ty_w^{-1}H) \int_{K_w} \psi\big(tB(k_w y_w^{-1} \cdot H, Y)\big)\, dk_w \\ &= \hat{\mu}_\xi^M(tw^{-1}H) \int_K \psi\big(tB(k \cdot H, wY)\big)\, dk. \end{aligned}$$ ■

We are now ready to combine the results of this section with Proposition 4.6.

Proof of Theorem 2.2 Write $W = W(\mathfrak{h}, \mathfrak{m})$. By Proposition 5.6, we know that there is a compact open subgroup K_1 of G with the following property. If $K \subset K_1$ is a compact open subgroup of G, there is $C(K) > 0$ so that for all $Y \in \omega_2$, $\xi \in \Xi_M$, $H \in \omega_1$, $|t| \geq C(K)$,

$$T_K(Y, \xi, tH) = \sum_{w \in W} \hat{\mu}_\xi^M(tw^{-1}H) V(KwM/M) \phi_K(tH, wY).$$

Fix $w \in W$, and write $\mathfrak{m}_w = y_w \cdot \mathfrak{m}$ and $M_w = y_w M y_w^{-1}$. Then $\mathfrak{g}(\mathfrak{m}_w) = \{wY : Y \in \mathfrak{g}(\mathfrak{m})\}$. We normalize invariant measure dv^* on G/M_w so that

$$\int_{G/M} f(y \cdot Y)\, dy^* = \int_{G/M_w} f(v \cdot wY)\, dv^*, \quad Y \in \mathfrak{g}(\mathfrak{m}),\ f \in C_c(\mathfrak{g}).$$

Then for any compact open subgroup K, the volume of KwM/M with respect to dy^* is equal to the volume of KM_w/M_w with respect to dv^*, so that $V(KwM/M) = V(KM_w/M_w)$, the volume of KM_w/M_w with respect to dv^*. Further, $\mathfrak{h} \subset \mathfrak{m}_w$, so that $\omega_1 \subset \mathfrak{m}_w \cap \mathfrak{g}' = \mathfrak{m}_w''$. Now applying Proposition 4.6 to $\mathfrak{m}_w$ and $y_w \cdot \omega_2 \subset \mathfrak{g}(\mathfrak{m}_w)$ in place of $\mathfrak{m}$ and ω_2, there is $c_w > 0$ with the following property. For each $c \geq c_w$ there is $C_w(c) > 0$ so that for all $H \in \omega_1$, $Y \in \omega_2$, and $t \in F^\times$ such that $|t| \geq C_w(c)$, $V(K_c wM/M)\phi_c(tH, wY) =$

$$|\eta_{\mathfrak{g}/\mathfrak{m}_w}(tH)|^{-1/2} |\eta_{\mathfrak{g}/\mathfrak{m}_w}(wY)|^{-1/2} \psi\big(B(tH, wY)\big) c(\mathfrak{m}_w, wY, tH).$$

But

$$\psi\big(B(tH, wY)\big) = \psi\big(B(ty_w^{-1} \cdot H, Y)\big), \quad \eta_{\mathfrak{g}/\mathfrak{m}_w}(wY) = \eta_{\mathfrak{g}/\mathfrak{m}}(Y)$$

$$\eta_{\mathfrak{g}/\mathfrak{m}_w}(tH) = \eta_{\mathfrak{g}/\mathfrak{m}}(ty_w^{-1}H) = \eta_{\mathfrak{g}}(ty_w^{-1}H)\eta_{\mathfrak{m}}(ty_w^{-1}H)^{-1} = \eta_{\mathfrak{g}}(tH)\eta_{\mathfrak{m}}(ty_w^{-1}H)^{-1}.$$

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Now pick

$$c \geq \max\{c_w, w \in W\}, \quad C \geq \max\{C(K_c), C_w(c), w \in W\}.$$

Then for all $H \in \omega_1, Y \in \omega_2, \xi \in \Xi_M, |t| \geq C$,

$$\begin{aligned}\Phi(Y,\xi,tH) &= |\eta_{\mathfrak{g}}(tH)|^{1/2}|\eta_{\mathfrak{g}/\mathfrak{m}}(Y)|^{1/2}T_{K_c}(Y,\xi,tH)\\ &= \sum_{w\in W}|\eta_{\mathfrak{m}}(ty_w^{-1}\cdot H)|^{1/2}\psi\big(B(ty_w^{-1}\cdot H,Y)\big)\hat{\mu}_\xi^M(ty_w^{-1}\cdot H)c(\mathfrak{m}_w,wY,tH)\\ &= \sum_{w\in W}\Phi(\mathfrak{m},Y+\xi,tw^{-1}H)c(\mathfrak{m}_w,wY,tH).\end{aligned}$$

In order to complete the proof of the theorem we must show that for all $w \in W$, $c(\mathfrak{m},Y,ty_w^{-1}\cdot H) = c(\mathfrak{m}_w,wY,tH)$. To do this we will use the notation from Section 1 and results from Section 6.

Fix $Y \in \omega_2, \xi_0 = \{0\} \in \Xi_M$, and for $H \in \mathfrak{h}'$, define

$$f(H) = \sum_{w\in W(\mathfrak{h},\mathfrak{m})}\psi\big(B(H,wY)\big)\Phi(\mathfrak{m},\xi_0,w^{-1}H)c(\mathfrak{m}_w,wY,H).$$

Then $f \in \mathcal{A}(\mathfrak{h})$, and by the above for $H \in \omega_1$, $|t| \geq C$, $f(tH) = \Phi(Y,\xi_0,tH)$. Let $y_0 \in N(\mathfrak{h},\mathfrak{m})$ and let $\mathfrak{h}_0 = y_0^{-1}\mathfrak{h}$. Since $\Phi(Y,\xi_0)$ is G-invariant, $\Phi(Y,\xi_0,H) = \Phi(Y,\xi_0,y_0^{-1}\cdot H)$ for all $H \in \mathfrak{h}'$. Applying the above to $\mathfrak{h}_0$, there is $C' > 0$ so that for all $H \in \omega_1$, $|t| \geq C'$, $\Phi(Y,\xi_0,y_0^{-1}\cdot tH) =$

$$\begin{aligned}&\sum_{v\in W(\mathfrak{h}_0,\mathfrak{m})}\psi\big(B(y_0^{-1}tH,vY)\big)\Phi(\mathfrak{m},\xi_0,v^{-1}y_0^{-1}\cdot tH)c(\mathfrak{m}_v,vY,y_0^{-1}\cdot tH)\\ &\qquad = \sum_{w\in W(\mathfrak{h},\mathfrak{m})}\psi\big(B(tH,wY)\big)\Phi(\mathfrak{m},\xi_0,w^{-1}tH)c(y_0^{-1}\mathfrak{m}_w,y_0^{-1}\cdot wY,y_0^{-1}\cdot tH),\end{aligned}$$

since $N(\mathfrak{h},\mathfrak{m}) = \{y_0 y : y \in N(\mathfrak{h}_0,\mathfrak{m})\}$. Define

$$f_0(H) = \sum_{w\in W(\mathfrak{h},\mathfrak{m})}\psi\big(B(H,wY)\big)\Phi(\mathfrak{m},\xi_0,w^{-1}H)c(y_0^{-1}\mathfrak{m}_w,y_0^{-1}\cdot wY,y_0^{-1}\cdot H).$$

Then $f, f_0 \in \mathcal{A}(\mathfrak{h})$, and by the above, for $H \in \omega_1$, $|t| \geq \max\{C,C'\}$, we have

$$f_0(tH) = \Phi(Y,\xi_0,y_0^{-1}tH) = \Phi(Y,\xi_0,tH) = f(tH).$$

That is $f \sim_{\mathfrak{h}} f_0$.

Now by Proposition 6.3, for all $w \in W(\mathfrak{h},\mathfrak{m}), H \in \mathfrak{h}'$,

$$\Phi(\mathfrak{m},\xi_0,w^{-1}H)c(\mathfrak{m}_w,wY,H) = \Phi(\mathfrak{m},\xi_0,w^{-1}H)c(y_0^{-1}\mathfrak{m}_w,y_0^{-1}\cdot wY,y_0^{-1}\cdot H).$$

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But since $\xi_0 = \{0\}$, for all $H \in \mathfrak{h}'$, $w \in W(\mathfrak{h}, \mathfrak{m})$,

$$\Phi(\mathfrak{m}, \xi_0, w^{-1}H) = |\eta_{\mathfrak{m}}(w^{-1}H)|^{1/2} \neq 0,$$

so that

$$c(\mathfrak{m}_w, wY, H) = c(y_0^{-1}\mathfrak{m}_w, y_0^{-1} \cdot wY, y_0^{-1} \cdot H).$$

In particular, if $w = y_0 M$, then

$$c(\mathfrak{m}_w, wY, H) = c(\mathfrak{m}, Y, y_0^{-1} \cdot H).$$

Thus $c(\mathfrak{m}, Y, y_0^{-1} \cdot H)$ depends only on the coset $w = y_0 M$ of y_0 in $W(\mathfrak{h}, \mathfrak{m})$, and we have $c(\mathfrak{m}_w, wY, H) = c(\mathfrak{m}, Y, w^{-1}H)$. This shows that the functions $c(\mathfrak{m})$ satisfy condition (iv) of Theorem 2.2 and finishes the proof that for all $H \in \omega_1$, $Y \in \omega_2$, $\xi \in \Xi_M$, $|t| \geq C$,

$$\Phi(Y, \xi, tH) = \sum_{w \in W(\mathfrak{h}, \mathfrak{m})} \Phi(\mathfrak{m}, Y + \xi, tw^{-1}H) c(\mathfrak{m}, Y, tw^{-1}H). \qquad \blacksquare$$

6 Consequences of the Expansion at Infinity

In this section we will use Theorem 1.1 to prove Theorems 1.2, 1.3, 1.4 and Corollary 1.5. The existence of the constant term in Theorems 1.2, 1.3 follows directly from Theorem 1.1 and (1.9). The first two lemmas in this section will allow us to establish the uniqueness of the constant term in Theorems 1.2, 1.3.

Lemma 6.1 *Assume that $\lambda_1, \ldots, \lambda_k \in F$ are distinct, and let*

$$f(t) = \sum_{i=1}^{k} c_i(t)\psi(t\lambda_i), \qquad t \in F,$$

where the $c_i \colon F \to \mathbf{C}$ are measureable functions satisfying $c_i(t_1^2 t) = c_i(t)$, $t, t_1 \in F^\times$. Suppose that there are $\delta > 0$, $C \geq 0$ so that $|t|^\delta |f(t)| \leq C$ for all $t \in F$. Then $c_i(t) = 0$ for all $t \in F^\times$, $1 \leq i \leq k$.

Proof Fix $1 \leq j \leq k$ and $t_0 \in F^\times$. We must show that $c_j(t_0) = 0$. Let

$$f'(t) = \psi(-\lambda_j t t_0) f(t t_0) = \sum_{i=1}^{k} c_i'(t)\psi(t\lambda_i')$$

where $c_i'(t) = c_i(tt_0)$, $\lambda_i' = t_0(\lambda_i - \lambda_j)$, $1 \leq i \leq k$. Now f' satisfies the same conditions as our original function f, $\lambda_j' = 0$, and $c_j'(1) = c_j(t_0)$. Thus we may as well assume that $j = 1$, $\lambda_1 = 0$, and take $t_0 = 1$.

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Since $\lambda_i \neq 0$, $2 \leq i \leq k$, there is $r_0 > 0$ so that $|\lambda_i| > q^{-r_0}$, $2 \leq i \leq k$. Let $r \geq r_0 + 1$, and let g_r denote the characteristic function of the set $U_r = \varpi^{-2r}(1+\mathcal{P}) = \varpi^{-2r} + \mathcal{P}^{-2r+1}$. Then $U_r \subset (F^\times)^2$, so that for all $t \in U_r$, $c_i(t) = c_i(1)$, $1 \leq i \leq k$. Thus

$$\int_F f(t)g_r(t)\,dt = \sum_{1\leq i\leq k} c_i(1)\int_F \psi(t\lambda_i)g_r(t)\,dt = \sum_{1\leq i\leq k} c_i(1)\hat{g}_r(\lambda_i).$$

But

$$\hat{g}_r(\lambda) = q^{2r-1}\psi(\varpi^{-2r}\lambda)\Phi_{2r-1}(\lambda), \quad \lambda \in F,$$

where Φ_{2r-1} denotes the characteristic function of $\mathcal{P}^{2r-1}$. Thus $\hat{g}_r(\lambda_1) = \hat{g}_r(0) = q^{2r-1}$, and for $2 \leq i \leq k$, $\hat{g}_r(\lambda_i) = 0$ since $|\lambda_i| > q^{-r_0} \geq q^{-2r+1}$. Thus we have

$$\int_F f(t)g_r(t)\,dt = q^{2r-1}c_1(1).$$

But for all $t \in U_r$, $|t| = q^{2r}$, so that $|f(t)| \leq Cq^{-2r\delta}$. Further, U_r has measure q^{2r-1}. Thus for all $r \geq r_0 + 1$,

$$q^{2r-1}|c_1(1)| = \Big|\int_F f(t)g_r(t)\,dt\Big| \leq q^{2r-1}Cq^{-2r\delta}.$$

Since $\delta > 0$, this implies that $c_1(1) = 0$. ∎

Lemma 6.2 *Assume that $\lambda_1, \dots, \lambda_k \in F$ are distinct and $d_1 > d_2 > \cdots > d_r$ are real numbers. Let*

$$f(t) = \sum_{i=1}^{r} |t|^{d_i} \sum_{j=1}^{k} c_{ij}(t)\psi(t\lambda_j), \quad t \in F,$$

where the $c_{ij}\colon F \to \mathbf{C}$ are measureable functions satisfying $c_{ij}(t_1^2 t) = c_i(t)$, $t, t_1 \in F^\times$. Suppose there is $C \geq 0$ so that $f(t) = 0$ for all $t \in F$, $|t| \geq C$. Then $c_{ij}(t) = 0$ for all $t \in F^\times$, $1 \leq i \leq r$, $1 \leq j \leq k$.

Proof Suppose not. Then we may as well assume that there is $1 \leq j \leq k$ so that c_{1j} is not identically zero. Let $C \geq 1$ so that $f(t) = 0$ for all $|t| \geq C$. Let $\delta = d_1 - d_2 > 0$. Write $|t|^{-d_1+\delta} f(t) = |t|^\delta f_1(t) + f_2(t)$ where

$$f_1(t) = \sum_{j=1}^{k} c_{1j}(t)\psi(t\lambda_j), \quad f_2(t) = \sum_{i=2}^{r} |t|^{d_i - d_1 + \delta} \sum_{j=1}^{k} c_{ij}(t)\psi(t\lambda_j).$$

Since each c_{ij} takes on only finitely many values, there is $C_1 > 0$ so that $|c_{ij}(t)| \leq C_1$ for all $1 \leq i \leq r$, $1 \leq j \leq k$. For $|t| \leq C$,

$$|t|^\delta |f_1(t)| \leq C^\delta |f_1(t)| \leq C^\delta \sum_{j=1}^{k} |c_{1j}(t)| \leq C^\delta k C_1.$$

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For $|t| \geq C$, $|t|^{\delta} f_1(t) = -f_2(t)$ since $|t|^{-d_1+\delta} f(t) = 0$. Further, $d_i - d_1 + \delta = d_i - d_2 \leq 0$ for $2 \leq i \leq r$, so that $|t|^{d_i - d_1 + \delta} \leq 1$ for $|t| \geq C \geq 1$. Thus

$$|t|^{\delta} |f_1(t)| = |f_2(t)| \leq \sum_{i=2}^{r} \sum_{j=1}^{k} |c_{ij}(t)| \leq rkC_1.$$

Thus $|t|^{\delta}|f_1(t)|$ is bounded, so that by Lemma 6.1, $c_{1j}(t) = 0$ for all $t \in F^{\times}$, $1 \leq j \leq k$. ■

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and define $\mathcal{C}(\mathfrak{h}, d)$, $d \geq 0$, and $\mathcal{A}(\mathfrak{h})$ as in (1.13), (1.14).

Proposition 6.3 *Let $f \in \mathcal{A}(\mathfrak{h})$, and expand f as in (1.14) as*

$$f(H) = \sum_{Y,d} \psi\big(B(Y, H)\big) f_{Y,d}(H).$$

Then if $f = 0$, we have $f_{Y,d} = 0$ for all Y, d. Further, suppose that $f \sim_{\mathfrak{h}} 0$. Then $f = 0$.

Proof Write

$$f(H) = \sum_{i=1}^{k} \sum_{d=0}^{r} \psi\big(B(Y_i, H)\big) f_{Y_i,d}(H)$$

where the Y_i, $1 \leq i \leq k$, are distinct and $f_{Y_i,d} \in \mathcal{C}(\mathfrak{h}, d)$, $1 \leq i \leq k$, $0 \leq d \leq r$. Let $\mathfrak{h}''$ denote the set of all $H \in \mathfrak{h}'$ such that $B(Y_i, H) \neq B(Y_j, H)$ for $1 \leq i \neq j \leq k$. It is a dense open subset of $\mathfrak{h}'$.

Fix $H \in \mathfrak{h}''$, and for $t \in F$, write $f_H(t) = f(tH)$,

$$\lambda_i = B(Y_i, H), \; c_{id}(t) = |t|^{-d/2} f_{Y_i,d}(tH), \quad 1 \leq i \leq k, \; 0 \leq d \leq r.$$

Then $c_{id}(t^2 t_0) = c_{id}(t_0)$, $t, t_0 \in F^{\times}$, and

$$f_H(t) = \sum_{i=1}^{k} \sum_{d=0}^{r} |t|^{d/2} \psi(\lambda_i t) c_{id}(t), \quad t \in F.$$

Suppose first that $f = 0$. Then $f_H = 0$, so that by Lemma 6.2, $c_{id}(t) = 0$ for all $t \in F^{\times}$, i, d. In particular, $f_{Y_i,d}(H) = c_{id}(1) = 0$ for all i, d. But since $\mathfrak{h}''$ is dense in $\mathfrak{h}'$, this implies that $f_{Y_i,d}(H) = 0$ for all i, d, and $H \in \mathfrak{h}'$.

Now suppose that $f \sim_{\mathfrak{h}} 0$, and fix $H \in \mathfrak{h}''$ as above. Then there is $C > 0$ so that $f(tH) = 0$ for all $|t| \geq C$. Thus $f_H(t) = f(tH) = 0$ for all $|t| \geq C$. Now by Lemma 6.2, $c_{id}(t) = 0$ for all $i, d, t \in F^{\times}$. Thus $f_{Y_i,d}(H) = 0$ for all i, d, so that $f(H) = 0$. But as above this implies that $f(H) = 0$ for all $H \in \mathfrak{h}'$. ■

The following immediate corollary of Proposition 6.3 establishes the uniqueness result in Theorems 1.2, 1.3.

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Corollary 6.4 *Let $F\colon \mathfrak{h} \to \mathbf{C}$ be a measureable function which is locally constant on $\mathfrak{h}'$, and suppose that there are $f_1, f_2 \in \mathcal{A}(\mathfrak{h})$ such that $F \sim_{\mathfrak{h}} f_i$, $i = 1, 2$. Then $f_1 = f_2$.*

Proof of Theorem 1.4 Let $T \in \mathcal{T}(\mathfrak{g})$, and write T as in (1.17) as

$$T = \sum_{\mathcal{O} \in I} c_T(\mathcal{O}) \Phi(\mathfrak{g}, \mathcal{O}),$$

where I is the finite set of orbits such that $c_T(\mathcal{O}) \neq 0$. For any semisimple element Y of $\mathfrak{g}$, let $I(Y) = \{\mathcal{O} \in I : Y \in \mathrm{cl}(\mathcal{O})\}$. Then by definition, $S(T)$ is the set of all semisimple elements Y such that $I(Y) \neq \varnothing$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We can assume that invariant measures on the orbits are consistently normalized as in Section 2 so that the functions c_Y, $Y \in \mathfrak{h}$, are independent of the orbit $\mathcal{O} \in I(Y)$. Now if $Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})$ for some $\mathcal{O} \in I$, then $Y \in S(T) \cap \mathfrak{h}$. Thus

$$\begin{aligned} \Phi(T, \mathfrak{h}) &= \sum_{\mathcal{O} \in I} c_T(\mathcal{O}) \Phi(\mathfrak{g}, \mathfrak{h}, \mathcal{O}) \\ &= \sum_{\mathcal{O} \in I} c_T(\mathcal{O}) \sum_{Y \in \mathfrak{h} \cap \mathrm{cl}(\mathcal{O})} \psi\big(B(Y, H)\big) c_Y(H) \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big) \\ &= \sum_{Y \in S(T) \cap \mathfrak{h}} \psi\big(B(Y, H)\big) c_Y(H) \sum_{\mathcal{O} \in I(Y)} c_T(\mathcal{O}) \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big). \end{aligned}$$

In particular, this shows that $X(T, \mathfrak{h}) \subset S(T) \cap \mathfrak{h}$, so that $\bigcup_{\mathfrak{h}} X(T, \mathfrak{h}) \subset S(T)$.

Suppose that $Y \in S(T)$, but $Y \notin \bigcup_{\mathfrak{h}} X(T, \mathfrak{h})$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_Y$. Then since $Y \in S(T) \cap \mathfrak{h}$, but $Y \notin X(T, \mathfrak{h})$, by the above we have

$$c_Y(H) \sum_{\mathcal{O} \in I(Y)} c_T(\mathcal{O}) \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big) = 0$$

for all $H \in \mathfrak{h}'$. But by Theorem 1.1 $c_Y(H) \neq 0$ for all $H \in \mathfrak{h}'$. Thus

$$\sum_{\mathcal{O} \in I(Y)} c_T(\mathcal{O}) \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), H\big) = 0, \quad H \in \mathfrak{h}'.$$

But since this is true for every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_Y$, and $\mathfrak{g}_Y \cap \mathfrak{g}'$ is dense in $\mathfrak{g}_Y'$, this implies that

$$\sum_{\mathcal{O} \in I(Y)} c_T(\mathcal{O}) \Phi\big(\mathfrak{g}_Y, \xi_Y(\mathcal{O}), X\big) = 0$$

for all $X \in \mathfrak{g}_Y'$. But by [2], $\mathcal{O} \to \xi_Y(\mathcal{O})$ is an injective map from $I(Y)$ into the set of nilpotent orbits in $\mathfrak{g}_Y$. Since the nilpotent orbital integrals are linearly independent by [2], this implies that $c_T(\mathcal{O}) = 0$ for all $\mathcal{O} \in I(Y)$. This contradiction shows that $S(T) \subset \bigcup_{\mathfrak{h}} X(T, \mathfrak{h})$. ■

Proof of Corollary 1.5 Suppose that $T \neq 0$. Then there is an orbit $\mathcal{O}$ of $\mathfrak{g}$ such that $c_T(\mathcal{O}) \neq 0$. But there is a semisimple element Y of $\mathfrak{g}$ such that $Y \in \mathrm{cl}(\mathcal{O})$. Now $Y \in S(T)$, so that by Theorem 1.4 there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $Y \in X(T, \mathfrak{h})$. Thus $\Phi(T, \mathfrak{h}) \neq 0$ and so $T \not\sim_{\mathfrak{h}} 0$. ■

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