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DISTRIBUTIVE p-ALGEBRAS AND DOUBLE p-ALGEBRAS HAVING n-PERMUTABLE CONGRUENCES

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Recent research on aspects of distributive lattices, p-algebras, double p-algebras and de-Morgan algebras (see [2] and the references therein) has led to the consideration of the classes $\mathscr{D}_n$ ($n \geqq 1$) of distributive lattices having no $n+1$-element chain in their poset of prime ideals. In [1] we were obliged to characterize the members of $\mathscr{D}_n$ by a sentence in the first-order theory of distributive lattices. Subsequently (see [2]), it was realised that $\mathscr{D}_n$ coincides with the class of distributive lattices having $n+1$-permutable congruences. This result is hereby employed to describe those distributive p-algebras and double p-algebras having n-permutable congruences. As an application, new characterizations of those distributive p-algebras and double p-algebras having the property that their compact congruences are principal are obtained. In addition, those varieties of distributive p-algebras and double p-algebras having n-permutable congruences are announced.

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1. Preliminaries

Henceforth, n will denote an arbitrary integer $\geqq 2$. If θ and ψ are binary relations on a set L then $\theta \circ \psi$ will denote their relational product and $\theta \circ^n \psi$ will stand for the compound relational product $\theta \circ \psi \circ \theta \ldots$, involving n factors, starting with θ and thereafter alternating between θ and ψ. In the event that $\theta \circ^n \psi = \psi \circ^n \theta$ we will say that θ and ψ *n-permute* (or are *n-permutable*). If θ and ψ 2-permute we will simply say that they *permute* (or are *permutable*). An algebra is said to have *n-permutable congruences* if every pair of congruences on it n-permute.

A (*distributive*) *p-algebra* is an algebra $\langle L; \vee, \wedge, {}^*, 0, 1 \rangle$ in which the deletion of the unary operation* yields a bounded (distributive) lattice and * is the operation of pseudocomplementation; that is, $x \leqq a^*$ iff $a \wedge x = 0$. The binary relation ϕ defined on any p-algebra L by $a \equiv b$ (ϕ) iff $a^* = b^*$ is a congruence, called the *Glivenko congruence* of L, and the set $D^*(L) = \{x \in L : x^{**} = 1\}$ is a filter of L which coincides with the congruence class of ϕ containing 1.

A (*distributive*) *double p-algebra* is an algebra $\langle L; \vee, \wedge, {}^*, {}^+, 0, 1 \rangle$ in which the deletion of $^+$ yields a (distributive) p-algebra and, for every $a \in L$, a^+ is the dual pseudocomplement of a; that is, $x \geqq a^+$ iff $a \vee x = 1$. The binary relation Φ defined on a double p-algebra L by $a \equiv b(\Phi)$ iff $a^* = b^*$ and $a^+ = b^+$ is a congruence, called the *determination congruence* of L, and the members of the quotient L/Φ are called *determination classes* of L. The set $K(L) = D^*(L) \cap D^+(L)$, where $D^+(L)$ is the ideal

$\{x\in L: x^{++}=0\}$, is called the *core* of L and, in the event that $K(L)\neq\emptyset$, is a determination class of L. A filter F of a double p-algebra is said to be *normal* if $f^{+*}\in F$ whenever $f\in F$. If F is a filter of a p-algebra or double p-algebra L then $\Theta(F)$ will denote the smallest congruence of L collapsing F and $\Theta_{\text{lat}L}(F)$ will denote the corresponding congruence of the lattice reduct of L. If θ is a congruence of such an algebra then the *cokernel of* θ is defined to be the class of θ containing 1 and denoted $\text{cok}\theta$.

Henceforth, we will write $\bar{\theta}$ instead of the binary relation $\theta\cap\leqq$ for an arbitrary congruence θ; be it on a distributive lattice, p-algebra or double p-algebra.

For all other unexplained notation and terminology we refer the reader to [3] or [5].

2. The theorems

Perverse as it may seem to the reader, we will state and prove a characterization of distributive double p-algebras having n-permutable congruences before doing the same for distributive p-algebras. However, once the ideas involved in the proof for distributive double p-algebras have been aired, they require only minor modification to establish a proof of an analogous characterization of distributive p-algebras having n-permutable congruences.

We begin by presenting several key results. The first lemma and the equivalence of (i) and (ii) in the second lemma are proved in [2]. A proof of the equivalence of (i) and (iii) in the second lemma and proofs (or, at least, references to proofs) of the third and fourth lemmas may be found in [1].

Lemma 2.1. *Congruences θ and ψ of a lattice n-permute iff the binary relations $\bar{\theta}$ and $\bar{\psi}$ n-permute.*

Lemma 2.2. *For a distributive lattice L, the following are equivalent:*

(i) $L\in\mathscr{D}_{n-1}$,
(ii) *L has n-permutable congruences,*
(iii) *given any $x_i\in L$ with $0\leqq i\leqq n$ and $x_0\leqq x_1\leqq\cdots\leqq x_n$, there exist $x_i'\in L$ with $1\leqq i\leqq n-1$ such that $x_0=x_1\wedge x_1'$, $x_i\vee x_i'=x_{i+1}\wedge x_{i+1}'$ when $1\leqq i<n-1$ and $x_{n-1}\vee x_{n-1}'=x_n$.* □

Lemma 2.3. *Let L be a distributive double p-algebra. Then the lattice reduct of L belongs to $\mathscr{D}_{n+1}$ iff every determination class of L belongs to $\mathscr{D}_{n-1}$.* □

Lemma 2.4. *Let L be a distributive double p-algebra. If θ is an arbitrary congruence on L, ψ is a congruence on L below Φ and F is an arbitrary normal filter of L then*

(i) $\theta=\Theta(\text{cok}\theta)\vee(\theta\wedge\Phi)$,
(ii) $\Theta(F)=\Theta_{\text{lat}L}(F)$,
(iii) $\Theta(F)$ *and* ψ permute. □

Without further ado, we proceed directly to:

Theorem 2.5. *A distributive double p-algebra L has n-permutable congruences iff every determination class of L has the same property.*

Proof. Suppose that L has n-permutable congruences. Let C be a determination class of L and let θ_c, ψ_c be congruences of C. By the congruence extension property of distributive lattices, there exist congruences θ_L, ψ_L of the lattice reduct of L such that $\theta_c=\theta_L\cap C^2$ and $\psi_c=\psi_L\cap C^2$. Let $\theta=\theta_L\cap\Phi$ and $\psi=\psi_L\cap\Phi$. Then it is easy to see that θ and ψ are congruences of L, $\theta_c=\theta\cap C^2$ and $\psi_c=\psi\cap C^2$. To show that θ_c and ψ_c n-permute it is enough, by Lemma 2.1, to show that

$$(\theta_c\cap\leqq_c)\circ^n(\psi_c\cap\leqq_c)=(\psi_c\cap\leqq_c)\circ^n(\theta_c\cap\leqq_c),$$

where $\leqq_c$ is $\leqq\cap C^2$—the partial ordering of C. First note that $\theta_c\cap\leqq c=\bar{\theta}\cap C^2$ and $\psi_c\cap\leqq_c=\bar{\psi}\cap C^2$ and observe that $(\bar{\theta}\cap C^2)\circ^n(\bar{\psi}\cap C^2)=(\bar{\theta}\circ^n\bar{\psi})\cap C^2$. Indeed, if $(x,y)\in(\bar{\theta}\circ^n\bar{\psi})\cap C^2$ then $x,y\in C$ and there exist $x_1,\dots,x_{n-1}\in L$ such that $x\leqq x_1\leqq\cdots\leqq x_{n-1}\leqq y$ and $x\equiv x_1(\theta)$, $x_1\equiv x_2(\psi),\dots,x_{n-1}\equiv y(\alpha)$, where

$$\alpha=\begin{cases}\theta, & n\text{ odd}\\ \psi, & n\text{ even}\end{cases}.$$

But C is a convex sublattice of L, so that $\{x_1,\dots,x_{n-1}\}\subseteq C$, and therefore $(x,y)\in(\bar{\theta}\cap C^2)\circ^n(\bar{\psi}\cap C^2)$. Thus, $(\bar{\theta}\circ^n\bar{\psi})\cap C^2\subseteq(\bar{\theta}\cap C^2)\circ^n(\bar{\psi}\cap C^2)$ and so $(\bar{\theta}\cap C^2)\circ^n(\bar{\psi}\cap C^2)=(\bar{\theta}\circ^n\bar{\psi})\cap C^2$, since the reverse inclusion is obvious. Consequently,

$$(\theta_c\cap\leqq_c)\circ^n(\psi_c\cap\leqq_c)=(\bar{\theta}\circ^n\bar{\psi})\cap C^2 \text{ and, similarly,}$$

$$(\psi_c\cap\leqq_c)\circ^n(\theta_c\cap\leqq_c)=(\bar{\psi}\circ^n\bar{\theta})\cap C^2.$$

Therefore $(\theta_c\cap\leqq_c)\circ^n(\psi_c\cap\leqq_c)=(\psi_c\cap\leqq_c)\circ^n(\theta_c\cap\leqq_c)$, since $\bar{\theta}$ and $\bar{\psi}$ n-permute by Lemma 2.1. In summary, C has n-permutable congruences.

Suppose now that every determination class of L has n-permutable congruences. We start by showing that any pair of congruences θ, ψ of L below Φ n-permute. Suppose that $(x,y)\in\theta\circ^n\psi$. Then there exist $x_1,\dots,x_{n-1}\in L$ such $x\equiv x_1(\theta)$, $x_1\equiv x_2(\psi),\dots,x_{n-1}\equiv y(\alpha)$, where

$$\alpha=\begin{cases}\theta, & n\text{ odd}\\ \psi, & n\text{ even}\end{cases}.$$

Now $\{x,x_1,\dots x_{n-1},y\}$ is a subset of some determination class C (say) of L, since $\theta,\psi\leqq\Phi$, and so $(x,y)\in(\theta\cap C^2)\circ^n(\psi\cap C^2)=(\psi\cap C^2)\circ^n(\theta\cap C^2)$, since C has n-permutable congruences. Therefore $(x,y)\in\psi\circ^n\theta$. Similarly, $\psi\circ^n\theta\subseteq\theta\circ^n\psi$ and so $\theta\circ^n\psi=\psi\circ^n\theta$. Thus, any pair of congruences on L below Φ n-permute. We show that this implies that L has n-permutable congruences. Let θ and ψ be congruences on L. Then $\theta=\Theta(\operatorname{cok}\theta)\vee(\theta\wedge\Phi)$ and $\psi=\Theta(\operatorname{cok}\psi)\vee(\psi\wedge\Phi)$, by Lemma 2.4(i). For brevity, let us write θ_c, ψ_c, θ_d, ψ_d instead of $\Theta(\operatorname{cok}\theta)$, $\Theta(\operatorname{cok}\psi)$, $\theta\wedge\Phi$, $\psi\wedge\Phi$, respectively, so that $\theta\circ^n\psi=(\theta_c\circ\theta_d)\circ^n(\psi_c\circ\psi_d)$. Now, $\theta_c\circ\theta_d=\theta_d\circ\theta_c$, $\theta_c\circ\psi_d=\psi_d\circ\theta_c$, $\psi_c\circ\theta_d=\theta_d\circ\psi_c$ and $\psi_c\circ\psi_d=\psi_d\circ\psi_c$, by Lemma 2.4 (iii). Repeated and judicious applications of these four equations yields $\theta\circ^n\psi=(\theta_c\circ^n\psi_c)\circ(\theta_d\circ^n\psi_d)$. However, θ_c and ψ_c permute, since they are both of the form $\Theta_{\mathrm{lat}\,L}(F)$ for some filter F (by Lemma 2.4(ii)) and it is well-known that such congruences permute on arbitrary distributive lattices. Therefore, $\theta_c\circ^n\psi_c=\theta_c\circ\psi_c$ and so $\theta\circ^n\psi=(\theta_c\circ\psi_c)\circ(\theta_d\circ^n\psi_d)$. Similarly we have $\psi\circ^n\theta=(\psi_c\circ\theta_c)\circ(\psi_d\circ^n\theta_d)=(\theta_c\circ\psi_c)\circ(\psi_d\circ^n\theta_d)$. But $\theta_d\circ^n\psi_d=\psi_d\circ^n\theta_d$, since θ_d and ψ_d are below Φ and therefore n-permute. Thus, $\theta\circ^n\psi=\psi\circ^n\theta$ and the proof is complete. □

Theorem 2.5, in conjunction with Lemma 2.2 and Lemma 2.3, yields:

Corollary 2.6. *For a distributive double p-algebra L, the following are equivalent:*

(i) *L has n-permutable congruences,*
(ii) *the lattice reduct of L belongs to $\mathcal{D}_{n-1}$,*
(iii) *the lattice reduct of L has $n+2$-permutable congruences.* □

In [1], we showed that every compact congruence of a distributive double p-algebra is principal iff its lattice reduct belongs to $\mathcal{D}_4$. Thus, we have

Corollary 2.7. *Every compact congruence of a distributive double p-algebra L is principal iff L has 3-permutable congruences.* □

Of course, Lemma 2.2(iii) permits the formulation of alternative versions of Corollaries 2.6 and 2.7.

In the event that L has a non-empty core $K(L)$, we have the following refinement of Theorem 2.5:

Corollary 2.8. *Let L be a distributive double p-algebra having non-empty core $K(L)$. Then L has n-permutable congruences iff $K(L)$ has the same property.*

Proof. If L has n-permutable congruences then $K(L)$, being a determination class of L, has the same property by Theorem 2.5.

Suppose that $K(L)$ has n-permutable congruences. In order to show that L has the same property it is enough, on examination of the proof of Theorem 2.5, to show that any pair of congruences θ, ψ of L below Φ are n-permutable. Suppose that $(x,y)\in\bar{\theta}\circ^n\bar{\psi}$. We can always choose $k\in K(L)$ satisfying $y\wedge y^+\leqq k$; indeed, $k=(y\wedge y^+)\vee k'$, where $k'\in K(L)$ is arbitrary, will suffice. Now, there exist $x_1,\ldots,x_{n-1}\in L$ such that

$$x\leqq x_1\leqq\cdots\leqq x_{n-1}\leqq y\quad\text{and}\quad x\equiv x_1(\theta),\ x_1\equiv x_2(\psi),\ldots,x_{n-1}\equiv y(\alpha),$$

where

$$\alpha=\begin{cases}\theta, & n \text{ odd}\\ \psi, & n \text{ even}\end{cases}.$$

Note that $x^*=x_1^*=\cdots=x_{n-1}^*=y^*$, since θ, $\psi\leqq\Phi$, and, for $1\leqq i\leqq n-1$, let $k_i=(x_i \vee x_i^*)\wedge k$. Then $k_i\in K(L)$, $(x\vee x^*)\wedge k\leqq k_1\leqq\cdots\leqq k_{n-1}\leqq(y\vee y^*)\wedge k$ and $(x\vee x^*)\wedge k\equiv k_1(\theta)$, $k_1\equiv k_2(\psi),\ldots,k_{n-1}\equiv(y\vee y^*)\wedge k(\alpha)$. Therefore $((x\vee x^*)\wedge k,$ $(y\vee y^*)\wedge k)\in(\bar\theta\cap K(L)^2)\circ^n(\bar\psi\cap K(L)^2)=(\bar\psi\cap K(L)^2\circ^n(\bar\theta\cap K(L)^2)$, by Lemma 2.1, and so there exist $k_i'\in K(L)$, $1\leqq i\leqq n-1$, such that

$$(x\vee x^*)\wedge k\leqq k_1'\leqq\cdots\leqq k_{n-1}'\leqq(y\vee y^*)\wedge k \text{ and}$$

$$(x\vee x^*)\wedge k\equiv k_1'(\psi),\ k_1'\equiv k_2'(\theta),\ldots,k_{n-1}'\equiv(y\vee y^*)\wedge k(\alpha'),$$

where $\alpha'\in\{\theta,\psi\}\backslash\{\alpha\}$. Now, on recalling that the identity $a=a^{**}\wedge(a\vee a^*)$ holds in L and $x^{**}=y^{**}$, we see that

$$x\wedge k\leqq x^{**}\wedge k_1'\leqq\cdots\leqq x^{**}\wedge k_{n-1}'\leqq y\wedge k \quad \text{and}$$

$$x\wedge k\equiv x^{**}\wedge k_1'(\psi), x^{**}\wedge k_1'\equiv x^{**}\wedge k_2'(\theta),\ldots,x^{**}\wedge k_{n-1}'\equiv y\wedge k(c').$$

On writing $x_i'=x\vee(x^{**}\wedge k_i')$, $1\leqq i\leqq n-1$ and recalling that $x\leqq y$, we infer that $x\leqq x_1'\leqq\cdots\leqq x_{n-1}'\leqq y$ and $x\equiv x_1'(\psi)$, $x_1'\equiv x_2'(\theta),\ldots,$ $x_{n-1}'\equiv y\wedge(x\vee k)(\alpha')$.

However, $y\wedge y^+\leqq k$ so that $y=y^{++}\vee(y\wedge y^+)\leqq y^{++}\vee k=x^{++}\vee k\leqq x\vee k$ and therefore $y\wedge(x\vee k)=y$. Thus, $(x,y)\in\bar\psi\circ^n\bar\theta$. We conclude that $\bar\theta\circ^n\bar\psi=\bar\psi\circ^n\bar\theta$ and therefore θ and ψ are n-permutable by Lemma 2.1. □

For distributive p-algebras, we have the following counterpart of Theorem 2.5:

Theorem 2.9 *A distributive p-algebra L has n-permutable congruences iff $D^*(L)$ has the same property.*

Proof. If L has n-permutable congruences then, on replacing C by $D^*(L)$ in the first part of the proof of Theorem 2.5, we see that $D^*(L)$ has the same property.

Suppose that $D^*(L)$ has n-permutable congruences. To show that L has the same property it is enough to show that any pair of congruences θ, ψ below ϕ (the Glivenko congruence) n-permute because we can then virtually copy the last part of the proof of Theorem 2.5 (bearing in mind that Lemma 2.4 is known to have an obvious analogue in the context of distributive p-algebras). So, let $(x,y)\in\bar\theta\circ^n\bar\psi$. Using ideas similar to those used in the proof of Corollary 2.8, we can show that

$$(x\vee x^*,y\vee y^*)\in(\bar\theta\cap D^*(L)^2)\circ^n(\bar\psi\cap D^*(L)^2)=(\bar\psi\cap D^*(L)^2)\circ^n(\bar\theta\cap D^*(L)^2).$$

On taking the meet of the members of any sequence guaranteeing that

$$(x \vee x^*, y \vee y^*) \in (\bar{\psi} \cap D^*(L)^2) \circ^n (\bar{\theta} \cap D^*(L)^2)$$

with $x^{**}(=y^{**})$, we see that $(x,y) \in \bar{\psi} \circ^n \bar{\theta}$. In conclusion, $\bar{\theta} \circ^n \bar{\psi} = \bar{\psi} \circ^n \bar{\theta}$ so that θ and ψ are n-permutable by Lemma 2.1 □

Theorem 2.9, in conjunction with Lemma 2.2 and the analogue of Lemma 2.3 in the context of distributive p-algebras (see [4]), yields the following counterpart of Corollary 2.6:

Corollary 2.10. *For a distributive p-algebra L the following are equivalent:*

(i) *L has n-permutable congruences,*
(ii) *the lattice reduct of* $L \in \mathscr{D}_n$,
(iii) *the lattice reduct of L has* $n+1$*-permutable congruences.* □

In [4], we saw that every compact congruence of a distributive p-algebra is principal iff its lattice reduct belongs to $\mathscr{D}_3$. Thus, we have:

Corollary 2.11. *Every compact congruence of a distributive p-algebra is principal iff it has 3-permutable congruences.* □

Remarks on varieties of (double) *p*-algebras. For an integer $n \geqq 2$, let us say that a variety of distributive p-algebras or double p-algebras has n-permutable congruences if every algebra in it has that property. It is natural to ask which varieties have n-permutable congruences. The situation for varieties of distributive p-algebras is trivial. Indeed, it is known that the lattice of subvarieties of the variety $\mathscr{B}_\omega$ of all distributive p-algebras forms a chain $\mathscr{B}_{-1} \subset \mathscr{B}_0 \subset \mathscr{B}_1 \subset \cdots \subset \mathscr{B}_\omega$ of type $\omega+1$, where $\mathscr{B}_{-1}$ is the trivial variety, $\mathscr{B}_0$ is the variety of Boolean algebras and $\mathscr{B}_1$ is the variety of Stone algebras (see [3]). Bearing Corollary 2.10 in mind and the fact that $\mathscr{B}_1$ contains all chain algebras, it is clear that a variety of distributive p-algebras having n-permutable congruences cannot contain $\mathscr{B}_1$. Consequently, $\mathscr{B}_0$ is the only non-trivial variety of distributive p-algebras having n-permutable congruences.

The situation for varieties of distributive double p-algebras is only marginally more difficult. Recall that an algebra is said to be *regular* if every congruence of it is uniquely determined by any one of its classes and that a variety is said to be regular if every algebra in it is regular. Regular double p-algebras are distributive and have been characterized in numerous ways (see [1] and the references therein). Let V be a variety of distributive double p-algebras. In [1]. it is shown that V is congruence regular iff it is congruence permutable and, furthermore, if k is an integer $\geqq 1$ and V is non-regular then there is an algebra $L^{[k]} \in V$ having the property that the poset of prime ideals of its lattice reduct contains a k-element chain. Bearing Corollary 2.6 in mind, it follows that a variety of distributive double *p-algebras has* n-permutable congruences iff it is congruence regular.

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