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ON THE TRIPLE CHARACTERIZATION FOR STONE ALGEBRAS

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1. Introduction. In [**1**], C. C. Chen and G. Grätzer developed a method for studying Stone algebras by associating with each Stone algebra L, a uniquely determined triple $(C(L), D(L), \phi(L))$, consisting of a Boolean algebra $C(L)$, a distributive lattice $D(L)$, and a connecting map $\phi(L)$. This approach has been successfully exploited by various investigators to determine properties of Stone algebras (e.g. H. Lakser [**9**] characterized the injective hulls of Stone algebras by means of this technique). The present paper is a continuation of this program.

After summarizing the properties of the category of triples, the epimorphisms in this category are determined confirming a conjecture of G. Grätzer. The prime ideals, $\mathscr{P}(L)$, of a Stone algebra L are characterized in terms of its triple. As a first application of this result it is shown that

$$|\mathscr{P}(L)| = |\mathscr{P}(C(L))| + |\mathscr{P}(D(L))|.$$

Another application yields a construction for the Stone algebra having a given triple. In the last section necessary and sufficient conditions are given in order that a Boolean algebra and a distributive lattice with 1 uniquely determine a triple.

2. Preliminaries. Let $\mathbf{B}$ be the class of Boolean algebras, $\mathbf{D}_{01}$ the class of distributive lattices with 0, 1 and $\mathbf{D}_1$ the class of distributive lattices with 1 ($\mathscr{B}$, $\mathscr{D}_{01}$, and $\mathscr{D}_1$ are the corresponding categories respectively). For $L \in \mathbf{D}_{01}$, let $C(L)$ be the Boolean algebra of complemented elements of L. If $L \in \mathbf{D}_1$, $\bar{D}(L)$ is the lattice of filters of L. Recall that $\bar{D}(L) \in \mathbf{D}_{01}$; in fact, for F_1, $F_2 \in \bar{D}(L)$, $F_1 \cdot F_2 = F_1 \cap F_2$, $F_1 + F_2 = \{x + y | x \in F_1, y \in F_2\}$, $0_{\bar{D}(L)} = [1)$ and $1_{\bar{D}(L)} = L$. The poset of prime ideals of a distributive lattice L is $\mathscr{P}(L)$ and we set $\mathscr{P}_0(L) = \mathscr{P}(L) \cup \{\emptyset\}$. Let $\mathbf{n}$ be the n-element chain $0 < 1 < \ldots < n - 1$. For $J \in \mathscr{P}(L)$, $f_J : L \rightarrow \mathbf{2}$ is the $\mathbf{D}_{01}$-homomorphism defined by

$$xf_J = \begin{cases} 1, & \text{if } x \notin J \\ 0, & \text{if } x \in J. \end{cases}$$

We introduce the category $\mathscr{K}$, called the category of triples, as follows. The objects of $\mathscr{K}$ are triples (C, D, ϕ) where $C \in \mathbf{B}$, $D \in \mathbf{D}_1$ and $\phi : C \rightarrow \bar{D}(D)$ is a $\mathbf{D}_{01}$-homomorphism. The morphisms in $[(C, D, \phi), (C_1, D_1, \phi_1)]_{\mathscr{K}}$

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are the pairs (f, g) where $f \in [C, C_1]_{\mathscr{B}}$, $g \in [D, D_1]_{\mathscr{D}_1}$ and $(a\phi)g \subseteq af\phi_1$ for each $a \in C$. The composition of morphisms is defined by $(f_1, g_1)(f_2, g_2) = (f_1f_2, g_1g_2)$ for $(f_i, g_i) \in [(C_i, D_i, \phi_i), (C_{i+1}, D_{i+1}, \phi_{i+1})]_{\mathscr{K}}$ for $i = 1, 2$. We see that $(1_C, 1_D)$ is the identity on (C, D, ϕ) where 1_A is the identity on a set A. Moreover $(f, g) \in [(C, D, \phi), (C_1, D_1, \phi_1)]_{\mathscr{K}}$ is an isomorphism (in $\mathscr{K}$) if and only if f is an isomorphism in $\mathscr{B}$, g is an isomorphism in $\mathscr{D}_{01}$ and $(a\phi)g = af\phi'$ for each $a \in C$.

Recall from [**1**] that for a Stone algebra L, we can associate the triple $(C(L), D(L), \phi(L))$ where $D(L)$ is the member of $\mathbf{D}_1$ consisting of the dense elements of L and $\phi(L) : C(L) \rightarrow \bar{D}(D(L))$ is the $\mathbf{D}_{01}$-homomorphism defined by $a\phi(L) = \{d \in D(L) | d \geqq a^*\}$ for each $a \in C(L)$.

The assignment $L \mapsto (C(L), D(L), \phi(L))$ can be extended to a functor (implicit in [**1**]) which establishes an equivalence from the category of Stone algebras and Stone homomorphisms to the category $\mathscr{K}$. Indeed the functor takes the Stone homomorphism $f : L \rightarrow L_1$ into $(f|C(L), g|D(L))$ – the codomain of $f|C(L)$ and $g|D(L)$ are taken to be $C(L_1)$ and $D(L_1)$ respectively. The following result from [**1**] will be needed.

LEMMA 2.1. *If $(C, C, \phi) \in \mathrm{Ob}\,\mathscr{K}$ then for each $a \in C$ and $d \in D$, there is an element $d_{\rho_a} \in D$ such that $[d_{\rho_a}) = a\phi \cap [d)$. Moreover $(d_{\rho_a})(d_{\rho_{\bar{a}}}) = d$.*

Proof. For $a \in C$ and $d \in D$, we have $d \in a\phi + \bar{a}\phi$ so $d = xy$ for some $x \in a\phi$, $y \in \bar{a}\phi$. It is easy to see that $d + x$ is the required element, d_{ρ_a}.

For $(C, D, \phi) \in \mathrm{Ob}\,\mathscr{K}$ and $J \in \mathscr{P}(D)$, define $I(J) = \{c \in C | \bar{c}\phi \cap J \neq \emptyset\}$.

LEMMA 2.2. *If $(C, D, \phi) \in \mathscr{K}$ then for each $J \in \mathscr{P}(D)$, $I(J) \in \mathscr{P}(C)$ and $(f_{I(J)}, f_J) \in [(C, D, \phi), (\mathbf{2}, \mathbf{2}, \phi_{\bar{2}})]_{\mathscr{K}}$, where $\phi_{\bar{2}} : \mathbf{2} \rightarrow \bar{D}(\mathbf{2})$ is defined by $0\phi_{\bar{2}} = [1)$ and $1\phi_{\bar{2}} = [0)$.*

Proof. It is routine to verify that $I(J)$ is a proper ideal. If $c_1 \in I(J)$ and $c_2 \notin I(J)$ then $\bar{c}_1\phi \cap J = \bar{c}_2\phi \cap J = \emptyset$ so $\bar{c}_1\phi \subseteq D \sim J$ and $\bar{c}_2\phi \subseteq D \sim J$. But $D \sim J \in \bar{D}(D)$ so $\overline{c_1c_2}\phi = \bar{c}_1\phi + \bar{c}_2\phi \subseteq D \sim J$ and hence $c_1c_2 \notin I(J)$. Thus $I(J) \in \mathscr{P}(C)$.

It follows that $f_J \in [D, \mathbf{2}]_{\mathbf{D}_1}$ and $f_{I(J)} \in [C, \mathbf{2}]_{\mathscr{B}}$. To prove that $(a\phi)f_J \subseteq af\phi_{I(J)}\phi_{\bar{2}}$, first suppose $a \notin I(J)$ then $af_{I(J)} = 1$ so $(a\phi)f_J \subseteq [0) = (af_{I(J)})\phi_{\bar{2}}$. Next suppose $a \in I(J)$. So there is an element $x \in \bar{a}\phi \cap J$. Now if $d \in a\phi$ then $df_J = 1$. Indeed if $df_J = 0$ then $d \in J$ so $d + x \in a\phi \cap \bar{a}\phi = 0\phi = \{1\}$ and hence $1 = d + x \in J$, a contradiction. Thus, $(a\phi)f_J = \{df_J | d \in a\phi\} = \{1\} \subseteq af_{I(J)}\phi_{\bar{2}}$.

We close the section with an application of Lemma 2.2.

THEOREM 2.3. *A morphism $(f, g) \in [(C, D, \phi), (C_1, D_1, \phi_1)]_{\mathscr{K}}$ is an epimorphism if and only if f is an epimorphism in $\mathscr{B}$ and g is epimorphism in $\mathscr{D}_1$.*

Proof. The sufficiency of the condition is trivial. Conversely, suppose that (f, g) is epic in $\mathscr{K}$, $f_1, f_1' \in [C_1, C_2]_{\mathscr{B}}$ and $ff_1 = ff_1'$. Let $g_1 \in [D_1, \mathbf{1}]_{\mathscr{D}}$ and $\phi_2 \in$

$[C_2, \bar{D}(\mathbf{1})]_{\mathscr{B}}$ be constant maps. Then $(C_2, \mathbf{1}, \phi_2) \in \text{Ob}\,\mathscr{K}$ and $(f_1, g_1), (f_1', g_1) \in [(C_1, D_1, \phi_1), (C_2, \mathbf{1}, \phi_2)]_{\mathscr{K}}$. But then $(f, g)(f_1, g_1) = (ff_1, gg_1) = (ff_1', gg_1) = (f, g)(f_1', g_1)$ so $(f_1, g_1) = (f_1', g_1)$ and hence $f_1 = f_1'$.

Again suppose that (f, g) is epic in $\mathscr{K}$ but that g is not epic in $\mathscr{D}_1$. Since $\mathbf{2}$ is the only subdirectly irreducible in $\mathscr{D}_1$, there exist distinct prime ideals J_1, J_1' in D_1 such that $J_1 \cap Dg = J_1' \cap Dg$. We first show:

(1) For each $x \in D_1$ there exists $d \in D$ such that $dg \leqq x$.

In order to verify (1), suppose that for some $x \in D$, $dg \nleqq x$ for any $d \in D$. Then $(x] \cap [(D)g) = \emptyset$ and hence there exists $J \in \mathscr{P}(D_1)$ with $x \in J$ and $J \cap (D)g = \emptyset$. Let $g_1 : D_1 \rightarrow \mathbf{2}$ be the constant map with value 1, then $(f_{I(J)}, g_1), (f_{I(J)}, f_J) \in [(C_1, D_1, \phi_1), (\mathbf{2}, \mathbf{2}, \phi_{\bar{2}})]_{\mathscr{K}}$ and $(f, g)(f_{I(J)}, f_J) = (f, g)(f_{I(J)}, g_1)$, contradicting the fact that (f, g) is an epimorphism.
Next we prove:

(2) If $x \in a\phi_1$ then there exist $d \in D$ and $c \in C$ such that
$(d_{\rho_c})g \leqq x$ and $cf = a$.

Indeed, since f is epic in $\mathscr{B}$ (and hence onto) there exists $c \in C$ such that $cf = a$. By (1) we obtain an element $d \in D$ such that $dg \leqq x$. Now $(d_{\rho_{\bar{c}}})g \in (\bar{c}\phi)g \subseteq (\bar{c}f)\phi_1 = (\overline{cf})\phi_1 = \bar{a}\phi_1$, so $x + (d_{\rho_{\bar{c}}})g \in a\phi_1 \cap \bar{a}\phi_1 = \{1\}$ and hence $x + (d_{\rho_{\bar{c}}})g = 1$. Thus,

$$(d_{\rho_c})g = x((d_{\rho_c})g) + ((d_{\rho_{\bar{c}}})g)((d_{\rho_c})g) \leqq x + dg \leqq x.$$

We can now show that $(f_{I(J_1)}, f_{J_2}) \in [(C_1, D_1, \phi_1), (\mathbf{2}, \mathbf{2}, \phi_{\bar{2}})]_{\mathscr{K}}$. It suffices to prove that $(a\phi_1)f_{J_2} \subseteq (af_{I(J_1)})\phi_{\bar{2}}$ for $a \in I(J_1)$. But $a \in I(J_1)$ implies the existence of an element $y \in \bar{a}\phi_1 \cap J_1$. We will prove that $x \in a\phi_1$ implies $x \notin J_2$.

Indeed suppose $x \in a\phi_1 \cap J_2$. But by (2) there exists $d \in D$ and $c \in C$ such that $(d_{\rho_c})g \leqq x$ so $(d_{\rho_c})g \in J_2$. Hence $(d_{\rho_c})g \in J_2 \cap Dg \subseteq J_1$ and therefore $(d_{\rho_c})g + y \in J_1$. Now $(d_{\rho_c})g \in (c\phi)g \subseteq (cf)\phi_1 = a\phi_1$ so $y + (d_{\rho_c})g \in \bar{a}\phi_1 \cap a\phi_1 = \{1\}$ which implies the contradiction $1 = y + (d_{\rho_c})g \in J_1$. Thus $x \in a\phi_1$ implies $x \notin J_2$ so

$$(a\phi_1)f_{J_2} = \{xf_{J_2} | x \in a\phi_1\} = \{1\} \subseteq (af_{J_2})\phi_2.$$

Finally, $J_1 \cap Dg = J_2 \cap Dg$ implies $gf_{J_1} = gf_{J_2}$ so $(f, g)(f_{I(J_1)}, f_{J_2}) = (ff_{I(J_1)}, gf_{J_2}) = (ff_{I(J_1)}, gf_{J_1}) = (f, g)(f_{I(J_1)}, f_{J_1})$. But (g, f) is epic so $f_{J_2} = f_{J_1}$, a contradiction.

This establishes a conjecture of G. Grätzer that a Stone homomorphism $f : L \rightarrow L_1$ is an epimorphism if and only if $(C(L))f = C(L_1)$ and $f\,|D(L)$, with codomain restricted to $D(L_1)$, is an epimorphism in $\mathscr{D}_1$.

3. Prime ideals. We begin by characterizing $\mathscr{P}(L)$ in terms of the triple $(C(L), D(L), \phi(L))$.

THEOREM 3.1. *Let L be a Stone algebra. Then*

$$(1)\quad \mathscr{P}(L) \cong \{(I, J) | I \in \mathscr{P}(C(L)), J \in \mathscr{P}_0(D(L)), a^*\phi(L) \cap J = \emptyset \text{ or } a \in I \text{ for all } a \in C(L)\}.$$

Proof. Let P be the poset on the right side of (1). For $K \in \mathscr{P}(L)$ it is easily verified that $K \cap C(L) \in \mathscr{P}(C(L))$ and $K \cap D(L) \in \mathscr{P}_0(D(L))$. If $d \in a^*\phi(L) \cap K \cap D(L)$ then $d \geqq a^{**} = a$ so $a \in K$.

Thus, the map $h : \mathscr{P}(L) \to P$ given by $Kh = (K \cap C(L), K \cap D(L))$ is well defined and obviously preserves order. Suppose $K, K_1 \in \mathscr{P}(L)$, $K \cap C(L) \subseteq K_1 \cap C(L)$ and $K \cap D(L) \subseteq K_1 \cap D(L)$. For $x \in K, x = x^{**}(x + x^*)$ so $x^{**} \in K$ or $x + x^* \in K$. In the first case, $x^{**} \in K \cap C(L) \subseteq K_1$ so $x \in K_1$. Otherwise, $x + x^* \in K \cap D(L) \subseteq K_1$ so $x \in K_1$.

Suppose that $(I, J) \in P$. Let $K = (I \cup J]_L$. Since $I \neq \emptyset$, K is an ideal. If $K = L$ then $1 = a + d$ for some $a \in I, d \in J \cup \{0\}$. But $d \neq 0$ since I is proper so $d \in J$. Thus $d \geqq a^*$ implies $d \in a\phi(L) \cap J$. Since $(I, J) \in P$, $a^* \in I$ which leads to the contradiction $1 = a + a^* \in I$. To prove that $K \in \mathscr{P}(L)$, suppose $uv \in K$. Then there exists $a \in I, d \in J \cup \{0\}$ such that $uv \leqq a + d$. If $d = 0$ then $u^{**}v^{**} \leqq a^{**} = a$ so $u^{**} \in I$ or $v^{**} \in I$, in which case $u \in K$ or $v \in K$. On the other hand suppose $d \in J$. Then $uva^* \leqq d$ so $(u + d)(v + d)(a^* + d) \leqq d$. But $d \in J$ and $\{u + d, v + d, a^* + d\} \subseteq D(L)$ so one of the three elements is in J. If $a^* + d \in J$ then $a^* + d \in a\phi(L) \cap J$ and hence the contradiction $a^* \in I$. Thus $u + d \in J$ or $v + d \in J$. It follows that $u \in K$ or $v \in K$.

Since $I \subseteq K \cap C(L)$, $J \subseteq K \cap D(L)$ it remains to verify that $K \cap C(L) \subseteq I$ and $K \cap D(L) \subseteq J$. First let $a \in K \cap C(L)$ so $a \leqq b + d$ where $b \in I$, $d \in J \cup \{0\}$. We can assume $d \neq 0$. Then $ab^* \leqq d$ so $d \in (ab^*)^*\phi(L) \cap J$ and hence $ab^* \in I$. But I is prime so $a \in I$. Finally let $d \in K \cap D(L)$, $d \leqq a + d_1$, where $a \in I, d_1 \in J \cup \{0\}$. If $d_1 = 0, 1 = a \in I$ so assume $d_1 \in J$. Then $(a^* + d_1)(d + d_1) \leqq d_1$ so $a^* + d_1 \in J$ or $d + d_1 \in J$. Now $a^* + d_1 \in J$ implies $a^* + d_1 \in a\phi(L) \cap J$ which means $a^* \in I$. So we can assume $d + d_1 \in J$ and hence $d \in J$.

It is well known (and can easily be seen from (1)) that the poset of minimal prime ideals of L is isomorphic with $\mathscr{P}(C(L))$. Moreover, recalling the definition of $I(J)$ preceding Lemma 2.2, we have:

COROLLARY 3.2. *For a Stone algebra L,*

$$\mathscr{P}(L) \cong \{(I, \emptyset) | I \in \mathscr{P}(C(L))\} \cup \{(I(J), J) | J \in \mathscr{P}(D(L))\}.$$

In particular $|\mathscr{P}(L)| = |\mathscr{P}(C(L))| + |\mathscr{P}(D(L))|$.

Proof. Again let P represent the right side of (1). For $I \in \mathscr{P}(C(L))$ it is obvious that $(I, \emptyset) \in P$. Next let $J \in \mathscr{P}(D(L))$. By Lemma 2.2, $I(J) \in \mathscr{P}(C(L))$ and if $a \notin I(J)$ then $a^*\phi(L) \cap J = \emptyset$. Conversely, let $(I, J) \in P$. We can assume $J \neq \emptyset$ so $J \in \mathscr{P}(D(L))$. But then $I(J) \subseteq I$ for if $a \in I(J)$ then $a^*\phi(L) \cap J \neq \emptyset$ so $a \in I$. By Nachbin's theorem, $I(J) = I$.

In showing that the functor in Section 2 is an equivalence, it is necessary to prove that for $(C, D, \phi) \in \mathscr{K}$, there exists a Stone algebra L such that $(C, D, \phi) \cong (C(L), D(L), \phi(L))$. This was accomplished in Section 4 of [**1**]. Recently, in [**7**], T. Katrinák has given a new shorter construction of L (see [**3**, Problem 55]). Theorem 3.1 also leads to a more direct construction of L by replacing each abstract symbol $\langle a, d \rangle$, used in [**1**], by a set. Specifically we obtain, for the objects of $\mathscr{K}$, the analogue of the Stone representation theorem.

Let $(C, D, \phi) \in \mathrm{Ob}\,\mathscr{K}$ and set

$$P = \{(I, J) | I \in \mathscr{P}(C), J \in \mathscr{P}_0(D), \bar{a}\phi \cap J = \emptyset \text{ or } a \in I \text{ for all } a \in C\}.$$

For each $a \in C$ and $d \in a\phi$, let $\langle a, d \rangle = \{(I, J) \in P | a \notin I, d \notin J\}$ and $R = \{\langle a, d \rangle | a \in C, d \in a\phi\}$. It follows immediately from Lemma 2.1 that $d \in a\phi$, $e \in b\phi$ implies $e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}}) \in (a + b)\phi$ and $(d_{\rho b})(e_{\rho a}) \in (ab)\phi$. We will show that R is a ring of sets by establishing:

(2) $\langle a, d \rangle \cup \langle b, e \rangle = \langle a + b, e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}}) \rangle$, and

(3) $\langle a, d \rangle \cap \langle b, e \rangle = \langle ab, (d_{\rho b})(e_{\rho a}) \rangle$.

For (2), suppose $(I, J) \in \langle a, d \rangle$. Then $a \notin I$, $d \notin J$. Now $e_{\rho\bar{a}} \notin J$ since $e_{\rho\bar{a}} \in J \cap \bar{a}\phi$ implies $a \in I$, so $d(e_{\rho\bar{a}}) \notin J$ and hence $(I, J) \in \langle a + b, e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}}) \rangle$. Similarly $\langle b, e \rangle \subseteq \langle a + b, e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}}) \rangle$. Conversely, suppose $a + b \notin I$ and $e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}}) \notin J$. Without loss of generality, assume $a \notin I$. First suppose $b \in I$. Then $d_{\rho b} \notin J$. Indeed, $d_{\rho b} \in b\phi \cap J$ implies $\bar{b} \in I$, a contradiction. But $d_{\rho\bar{b}} \geqq e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}})$ so $d_{\rho\bar{b}} \notin J$. Since $d \geqq (d_{\rho b})(d_{\rho\bar{b}})$ we conclude that $d \notin J$ and hence $(I, J) \in \langle a, d \rangle$. On the other hand suppose $b \notin I$. Then $e + d \geqq e(d_{\rho\bar{b}}) + d(e_{\rho\bar{a}})$ implies $d \notin J$ or $e \notin J$ so $(I, J) \in \langle a, d \rangle$ of $(I, J) \in \langle b, e \rangle$. (3) is verified in a similar manner.

It is obvious that $\langle a, 1 \rangle$ and $\langle 1, d \rangle$ are members of R for all $a \in C$ and $d \in D$ and that $\emptyset = \langle 0, 1 \rangle = 0_R$ and $P = \langle 1, 1 \rangle = 1_R$. Moreover R is pseudocomplemented with

(4) $\langle a, d \rangle^* = \langle \bar{a}, 1 \rangle$.

Indeed it is clear that $\langle a, d \rangle \cap \langle \bar{a}, 1 \rangle = \emptyset$. Conversely suppose $\langle a, d \rangle \cap \langle b, e \rangle = \emptyset$ but $b \nleqq \bar{a}$. Then there exists $(I, \emptyset) \in P$ such that $(I, \emptyset) \in \langle a, d \rangle \cap \langle b, e \rangle$, a contradiction, so $\langle b, e \rangle \subseteq \langle \bar{a}, 1 \rangle$.

Since $\langle a, d \rangle^* \cup \langle a, d \rangle^{**} = 1_R$, R is a Stone algebra with $C(R) = \{\langle \bar{a}, 1 \rangle | a \in C\}$ and $D(R) = \{(1, d) | d \in D\}$.

To show $(C, D, \phi) \cong (C(R), D(R), \phi(R))$, we note that it is easy to verify that the map $f : C \to C(R)$, defined by $af = \langle a, 1 \rangle$ is an isomorphism in $\mathscr{B}$. It is clear that the map $g : D \to D(R)$ defined by $dg = \langle 1, d \rangle$ preserves order and is onto. Suppose $d \nleqq d_1$, $\{d, d_1\} \subseteq D$. Then there exists $J \in \mathscr{P}(D)$ such that $d_1 \in J$, $d \notin J$. But $(I(J), J) \in P$ and $(I(J), J) \in \langle 1, d \rangle \sim \langle 1, d_1 \rangle$ so g is an isomorphism in $\mathscr{D}$. It remains to verify that for $a \in C$, $(a\phi)g = (af)\phi(R)$. First suppose $\langle 1, d \rangle \in (af)\phi(R)$. Then $\langle \bar{a}, 1 \rangle \subseteq \langle 1, d \rangle$ but suppose $d \notin a\phi$.

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Then there exists $J \in P(D)$ such that $a\phi \cap J = \emptyset$ and $d \in J$. Then $(I(J), J) \in P$ and $(I(J), J) \in \langle \bar{a}, 1 \rangle$ for if $\bar{a} \in I(J)$ then $a \notin I(J)$ implies $\bar{a}\phi \cap J = \emptyset$. But $a\phi \cap J = \emptyset$ and hence the contradiction $J = \emptyset$. We conclude that $(I(J), J) \in \langle 1, d \rangle$, contradicting $d \in J$. For the converse, assume $d \in a\phi$ and $(I, J) \in \langle \bar{a}, 1 \rangle$ then $\bar{a} \notin I$ so $a\phi \cap J = \emptyset$. But $d \in a\phi$ so $d \notin J$ and hence $(I, J) \in \langle 1, d \rangle$.

We close by noting that the above construction is a concrete representation of the Chen-Grätzer construction. This follows from the fact that for $\langle a, d \rangle$ and $\langle b, e \rangle \in R$, $\langle a, d \rangle \subseteq \langle b, e \rangle$ if and only if $a \leqq b$ and $d \leqq e_{\rho_a}$ (cf. [**1**, p. 887]).

4. Uniqueness of ϕ. In [**1**], it is shown that for any $C \in \mathbf{B}$, $C \neq 1$ and any $D \in \mathbf{D}_1$ there exists $\phi : C \rightarrow \bar{D}(D)$ such that $(C, D, \phi) \in \mathrm{Ob}\mathscr{K}$; if $C = \mathbf{1}$ then there exists ϕ such that $(C, D, \phi) \in \mathrm{Ob}\mathscr{K}$ if and only if $D = \mathbf{1}$. Thus, for a given C and D the existence of a ϕ for which $(C, D, \phi) \in \mathscr{K}$ is completely settled. In this section we will answer the corresponding uniqueness question. There are three trivial cases to handle first: if $(C, D, \phi) \in \mathscr{K}$ and $C = \mathbf{1}$ or $C = \mathbf{2}$ or $D = \mathbf{1}$ then ϕ is uniquely determined (as well as D in the first case) since ϕ preserve 0, 1. We now proceed to the general case.

THEOREM 4.1. *Let $C \in \mathbf{B}$, $D \in \mathbf{D}_1$ and $C \neq \mathbf{2}$, $C \neq \mathbf{1}$, $D \neq \mathbf{1}$. There exists exactly one member* (up to isomorphism) *of* $\mathrm{Ob}\mathscr{K}$ *of the form* (C, D, ϕ) *if and only if*

(i) $C(\bar{D}(D)) = \{0_{\bar{D}(D)}, 1_{\bar{D}(D)}\}$, *and*

(ii) *If I_1, I_2 are prime ideals in C then there exists a $\mathbf{B}$-automorphism f of L such that $I_1 f = I_2$.*

Proof. ($\Rightarrow$) Suppose $\bar{D}(D)$ contains a complemented element d, other than $0_{\bar{D}(D)}$ and $1_{\bar{D}(D)}$. Since $C \neq \mathbf{1}, \mathbf{2}$, there exist distinct prime ideals I_1, I_2 in C. Then the maps $\phi_i : C \rightarrow \bar{D}(D)$, $i = 1, 2$, defined by

$$c\phi_1 = \begin{cases} 1_{\bar{D}(D)}, & \text{if } c \notin I_1 \cup I_2 \\ \bar{d}, & \text{if } c \in I_2 \sim I_1 \\ d, & \text{if } c \in I_1 \sim I_2 \\ 0_{\bar{D}(D)}, & \text{if } c \in I_1 \cap I_2 \end{cases} \quad \text{and } c\phi_2 = \begin{cases} 1_{\bar{D}(D)}, & \text{if } c \notin I_2 \\ 0_{\bar{D}(D)}, & \text{if } c \in I_2 \end{cases}$$

are $\mathbf{D}_{01}$-homomorphisms. But then (C, D, ϕ_i), $i = 1, 2$ are objects in $\mathscr{K}$ and by hypothesis there is an isomorphism $(f, g) \in [(C, D, \phi_1), (C, D, \phi_2)]_{\mathscr{K}}$. Now choose $b \in I_2 \sim I_1$. Then $b\phi_1 = \bar{d}$ so $(\bar{d})g = (b\phi_1)g = (bf)\phi_2 \in \{0_{\bar{D}(D)}, 1_{\bar{D}(D)}\}$. Since g is an automorphism of D, the map $F \rightarrow (F)g$ is an automorphism of $\bar{D}(D)$ and hence $(\bar{d})g \in \{0_{\bar{D}(D)}, 1_{\bar{D}(D)}\}$, implies $\bar{d} \in \{0_{\bar{D}(D)}, 1_{\bar{D}(D)}\}$.

In order to prove (ii), let I_1, I_2 be prime ideals in C. Define $\phi_i : C \rightarrow \bar{D}(D)$ by

$$c\phi_i' = \begin{cases} 1_{\bar{D}(D)}, & \text{if } c \notin I_i, \\ 0_{\bar{D}(D)}, & \text{if } c \in I_i \end{cases}$$

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for $i = 1, 2$. Then (C, D, ϕ_i'), $i = 1, 2$ are objects in $\mathscr{K}$ so there is an isomorphism $(f', g') \in [(C, D, \phi), (C, D, \phi')]_{\mathscr{K}}$. But then $f' : C \to C$ is the required automorphism.

($\Leftarrow$) Suppose $(C, D, \phi_1), (C, D, \phi_2) \in \text{Ob}\, \mathscr{K}$. Set $I_i = \{c \in C | c\phi_i = 0_{\bar{D}(D)}\}$. Since ϕ_i is a $\mathbf{D}_{01}$-homomorphism, it preserves complemented elements. It follows from (i) that $C\phi_i \subseteq \{0_{\bar{D}(D)}, 1_{\bar{D}(D)}\}$ and, in particular that I_i is a prime ideal for $i = 1, 2$. But then by (ii) there is a $\mathbf{B}$-automorphism $f : C \to C$ such that $(I_1)f = I_2$. It can be verified that $(f, 1_D)$ is an isomorphism in $\mathscr{K}$ from (C, D, ϕ_1) to (C, D, ϕ_2).

For any finite Boolean algebra C, condition (ii) holds: we can extend to a $\mathbf{B}$-automorphism, any map which permutes the coatoms of C. However, in the infinite case, condition (ii) does not hold in general. For example there exist Boolean algebras with no non-trivial automorphisms (e.g., see [**6**]). On the other hand, if C is any free Boolean algebra, the condition is satisfied. Indeed if S freely $\mathbf{B}$-generates C and $\{I_1, I_2\} \subseteq \mathscr{P}(C)$ define $f : S \to C$ by

$$f(s) = \begin{cases} s, & \text{if } s \in (I_1 \cap I_2) \cup (\tilde{I}_1 \cap \tilde{I}_2) \\ \bar{s}, & \text{otherwise.} \end{cases}$$

Then f extends to a homomorphism g such that $g^2 = 1$, $I_1 g = I_2$.

It is easy to verify that for a distributive lattice D, with 0, 1, condition (i) is equivalent to: $C(D) = \{0, 1\}$. We have:

COROLLARY. *Let* $C \in \mathbf{B}$, $D \in \mathbf{D}_1$, $2 < |C| < \infty$, $1 < |D| < \infty$. *Then* (C, D, ϕ) *is uniquely determined by* C *and* D *if and only if* $C(D) = \{0, 1\}$. *Thus, the finite Stone algebras which are uniquely determined by their center* $\mathbf{2}^n$, $2 \leqq n < \infty$ *and set of dense elements* D, *are the algebras of the form* $\mathbf{2}^{n-1} \times (1 \oplus D)$, *where the symbol* $\oplus$ *denotes ordinal sum and* D *is a finite distributive lattice with* $C(D) = \{0, 1\}$.

The "smallest" non-isomorphic Stone algebras with isomorphic centers and dense elements are $\mathbf{3} \times \mathbf{3}$ and $(\mathbf{1} \oplus \mathbf{2}^2) \times \mathbf{2}$.

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