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ON SOME BANACH SPACE SEQUENCES

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We introduce the Banach space of vector valued sequences $l^{p,q}\langle E\rangle$, $1 \le p, q \le \infty$, where E is a Banach space. Then we study the relation between $l^{p,q}\langle E\rangle$ and the Schur multipliers of $l^p \hat{\otimes} E$, where E is taken to be some l^r .

0. Introduction

Let E be a Banach space. Cohen [3], used the spaces $l^p(E)$, $l^p\{E\}$ together with the space he introduced $l^p\langle E\rangle$, to study p-summing operators, and their dual ideal (see [11]). Apiola [1], studied the duality relationships between the spaces $l^p(E)$, $l^p\{E\}$ and $l^p\langle E\rangle$.

In this paper we introduce the space $l^{p,q}\langle E\rangle$, and find its dual. Further, we investigate the relationship between such spaces and the Schur multipliers [2], on discrete spaces.

Throughout the paper, if E and F are Banach spaces, then $E \hat{\otimes} F$ and $E \check{\otimes} F$ will denote the completion of the projective tensor product of E with F , and the injective tensor product, respectively [4]. Let $\phi \in E \otimes F$; then $\|\phi\|_\pi$ designates the projective norm and $\|\phi\|_\varepsilon$ that of the injective norm. The dual of E will be denoted by E^* for any Banach space E . The set of natural numbers is denoted by N , and the complex numbers by $\mathbb{C}$. Let l^p be the space of p-summable sequences, $1 \le p \le \infty$.

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1. The space $l^{p,q}\langle E\rangle$ and its dual

Let E be a Banach space. Then $l^p(E)$ will denote the space of all functions $f : N \to E$, such that $\sum_{n=1}^{\infty} |\langle f(n), x^*\rangle|^p < \infty$, $x^* \in E^*$. The space $l^p(E)$ becomes a Banach space when one introduces the norm

$$\|f\|_{\varepsilon(p)} = \sup_{x^*} \left\{\left(\sum |\langle f(i), x^*\rangle|^p\right)^{1/p}, \|x^*\| \le 1, x^* \in E^*\right\},$$

for all $f \in l^p(E)$, [3]. Grothendieck, [5], showed that $l^p(E)$ is isometrically isomorphic to $\left(l^{p'} \hat{\otimes} F\right)^*$, where $F^* = E$, and $1/p + 1/p' = 1$.

Cohen, [3], introduced the space $l^p\langle E\rangle$ to be the space of all functions $f : N \to E$ such that $\sum_{i=1}^{\infty} |\langle f(i), g(i)\rangle| < \infty$, for all $g \in l^{p'}(E^*)$. The norm of f is given by

$$\|f\|_{\sigma(1,p)} = \sup_{g} \left\{\sum_{i=1}^{\infty} |\langle f(i), g(i)\rangle|, g \in l^{p'}(E^*) \text{ and } \|g\|_{\varepsilon(p')} \le 1\right\}.$$

The space $l^p(E)$ was shown to induce the injective norm on $l^p \otimes E$, [3], and Cohen showed that $l^p\langle E\rangle$ induces the projective norm on $l^p \otimes E$. Further, Apiola, [1], showed that $\left(l^p\langle E\rangle\right)^* \equiv l^{p'}(E^*)$ and $\left(l^p(E)\right)^* \equiv l^{p'}\langle E^*\rangle$.

Now we introduce the space $l^{p,q}\langle E\rangle$ to be the space of all functions $f : N \to E$ such that $\sum_{i=1}^{\infty} |\langle f(i), g(i)\rangle|^p < \infty$ for all $g \in l^{q'}(E^*)$. If $f \in l^{p,q}\langle E\rangle$, then we define

$$\|f\|_{\sigma(p,q)} = \sup_{g} \left(\sum_{i=1} |\langle f(i), g(i)\rangle|^p\right)^{1/p},$$

where $g \in l^{q'}(E^*)$ and $\|g\|_{\varepsilon(q')} \le 1$.

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LEMMA 1.1. *The function* $\| \|_{\sigma(p,q)}$ *is a norm on* $l^{p,q}\langle E\rangle$.

Proof. It is enough to show that $\|f\|_{\sigma(p,q)} < \infty$ for all $f \in l^{p,q}\langle E\rangle$. The rest of the properties of the norm are easy to verify.

Let $f \in l^{p,q}\langle E\rangle$. Define the bilinear form

$$\hat{f} : l^{p'} \times l^{q'}(E^*) \to \mathbb{C} ,$$

$$f(a, g) = \sum_{i=1}^{\infty} a(i)\langle f(i), g(i)\rangle .$$

It is not hard to check that $\hat{f}$ is separately continuous on $l^{p'} \times l^{q'}(E^*)$. Hence, [2, p. 172], $\hat{f}$ is jointly continuous, and consequently $\|f\|_{\sigma(p,q)} < \infty$ for all $f \in l^{p,q}\langle E\rangle$.

THEOREM 1.2. *The space* $l^{p,q}\langle E\rangle$ *with the* $\sigma(p, q)$ *norm is a Banach space.*

Proof. Let $f_n \in l^{p,q}\langle E\rangle$ such that $\sum_{n=1}^{\infty} \|f_n\|_{\sigma(p,q)} < \infty$. It is enough to show that $\left\|\sum_{n=1}^{\infty} f_n\right\| < \infty$, [12]. We first prove this for the case $p = 1$. Since E is a Banach space, then every absolutely summable sequence in E is summable. It follows that for each natural number i , the serries $\sum_{n=1}^{\infty} f_n(i)$ is convergent in E . Define $F : N \to E$ by $F(i) = \sum_{n=1}^{\infty} f_n(i)$.

Let $g \in l^{q'}(E^*)$ and $\|g\|_{\varepsilon(q')} \leq 1$. We have to prove that $\sum_{i=1}^{\infty} |\langle F(i), g(i)\rangle| < \infty$. Now

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$$\sum_{i=1}^{\infty} |\langle F(i),\, g(i)\rangle|$$

$$= \sum_{i=1}^{\infty} \left|\left\langle \sum_{n=1}^{\infty} f_n(i),\, g(i)\right\rangle\right|$$

$$= \sum_{i=1}^{\infty} \left|\sum_{n=1}^{\infty} \langle f_n(i),\, g(i)\rangle\right| \quad \left(\text{since } \sum_{n=1}^{\infty} \|f_n(i)\| < \infty\right)$$

$$\le \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i),\, g(i)\rangle| \quad \left[\text{since } \left|\sum_{n=1} \langle f_n(i),\, g(i)\rangle\right| < \infty\right].$$

If ν is the counting measure on the set of natural numbers N, then

$$\sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i),\, g(i)\rangle|$$

can be considered as

$$\int_N \sum_{n=1} |\langle f_n(i),\, g(i)\rangle| d\nu(i) .$$

As a consequence of the monotone convergence theorem we get

$$\int_N \sum_{n=1}^{\infty} |\langle f_n(i),\, g(i)\rangle| d\nu(i) = \sum_{n=1}^{\infty} \int_N |\langle f_n(i),\, g(i)\rangle| d\nu(i) .$$

It follows that

$$\sum_{i=1}^{\infty} |\langle F(i),\, g(i)\rangle|$$

$$\le \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\langle f_n(i),\, g(i)\rangle| < \infty \quad \left(\text{since } \sum_{n=1}^{\infty} \|f_n\|_{\sigma(1,q)} < \infty\right).$$

Hence $\sum_{i=1}^{\infty} |\langle F(i),\, g(i)\rangle| < \infty$ for all $g \in l^{q'}(E^*)$ with $\|g\|_{\varepsilon(q')} < \infty$.

Consequently $F \in l^{1,q}\langle E\rangle$, and so $\sum_{n=1}^{\infty} f_n \in l^{1,q}\langle E\rangle$.

For general p, the result follows from the fact that

$$\|f\|_{\sigma(p,q)} = \sup_{\theta,g} \left|\sum_{i=1}^{\infty} \theta(i)\langle f(i),\, g(i)\rangle\right| ,$$

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where $\theta \in l^{p'}$, $g \in l^{q'}(E^*)$ and $\|\theta\|_{p'} \le 1$, $\|g\|_{\varepsilon(q')} \le 1$. Hence the proof of the theorem is complete.

Let $l^r \cdot l^s(E^*)$ be the set of all elements of the form $a \cdot f$ such that $a \in l^r$, $f \in l^s(E^*)$ and $(a \cdot f)(i) = a(i) \cdot f(i)$.

THEOREM 1.3. *A linear functional F on $l^{p,q}\langle E\rangle$ is bounded if and only if F is of the form $a \cdot f$, for some $a \in l^{p'}$ and $f \in l^{q'}(E^*)$.*

REMARK. The space $l^{1,q}\langle E\rangle$ is just $l^q\langle E\rangle$ in Cohen [3]. Apiola, [1], proved that $(l^{1,q}\langle E\rangle)^*$ is isometrically isomorphic to $l^{q'}(E^*)$ which is in turn isomorphic to $l^\infty \cdot l^{q'}(E^*)$.

Proof of Theorem 1.3. Let $a \in l^{p'}$ and $f \in l^{q'}(E^*)$. Consider the linear functional $F : l^{p,q}\langle E\rangle \to \mathbb{C}$ defined by

$$F(g) = \sum_{i=1}^{\infty} a(i)\langle f(i), g(i)\rangle .$$

Then

$$|F(g)| \le \|a\|_{p'} \cdot \|f\|_{\varepsilon(q')} \cdot \|g\|_{\sigma(p,q)} .$$

Hence F is bounded and $\|F\| \le \|a\|_{p'} \cdot \|f\|_{\varepsilon(q')}$.

Conversely, let $F \in (l^{p,q}\langle E\rangle)^*$. Hence $|F(f)| \le \lambda \cdot \|f\|_{\sigma(p,q)}$ for some constant λ. Let e_i be the natural embedding of E in $l^{p,q}\langle E\rangle$, so

$$e_i(x)(j) = \begin{cases} x , & i = j , \\ 0 , & i \ne j . \end{cases}$$

Put $x_i^* = F \circ e_i$. Clearly $x_i^* \in E^*$, and if $f \in l^{p,q}\langle E\rangle$, then

$$F(f) = \sum_{i=1}^{\infty} \langle f(i), x_i^*\rangle .$$

Assume F to be of norm one; then there is an $a \in l^{p'}$ such that

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$\|a\|_{p'} \le 1$ and $|F(f)| \le \sup_g \sum_{i=1}^{\infty} |a(i)\langle f(i), g(i)\rangle|$, $\|g\|_{\varepsilon(q)} \le 1$. Thus

$$\left|\sum_{i=1}^{\infty} \langle f(i), x_i^*\rangle\right| \le \sup_g \sum_{i=1}^{\infty} |ai\langle f(i), g(i)\rangle| , \quad \|g\|_{\varepsilon(q')} \le 1 .$$

Now let D be the unit disc and πD be the countable product of D with itself. Since D is compact, then πD is compact. Let B_1 be the unit ball of $l^{q'}(E^*)$. As a dual of $l^{1,q}\langle E\rangle$, [1], B_1 is compact with respect to the w^*-topology, and so is the product space $\pi D \times B_1$. Let $C(\pi D \times B_1)$ be the space of continuous functions on $\pi D \times B_1$. Consider the map

$$\psi : l^{p,q}\langle E\rangle \to C(\pi D \times B_1) ,$$

$\psi(f) = G$, where

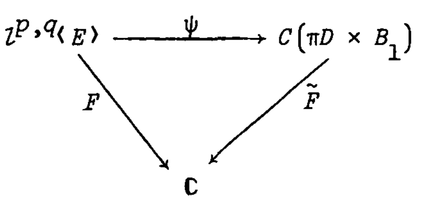
$$G(\theta, u) = \sum_{i=1}^{\infty} a(i)\theta(i)\ f(i),\ u(i) \quad ,$$

for all $f \in l^{p,q}\langle E\rangle$ and $\theta \in \pi D$, and $u \in B_1$. It follows that

$$\|G\|_\infty = \sup_{\theta,u} |G(\theta, u)| = \sup_{\theta,u} \left|\sum_{i=1}^{\infty} a(i)\theta(i)\langle f(i), u(i)\rangle\right|$$

$$= \sup_u \sum_{i=1}^{\infty} |a(i)\langle f(i), u(i)\rangle| .$$

Hence $|F(f)| \le \|\psi(f)\|$. This implies that $\ker \psi \subseteq \ker F$.



This implies that there exists an $\tilde{F} : C(\pi D \times B_1) \to \mathbb{C}$ such that $\tilde{F} \circ \psi = F$. The Riesz representation theorem implies that there exists a regular Borel measure μ on $\pi D \times B_1$ such that

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$$F(f) = \mu(\psi(f)) = \iint_{\pi D\times B_1} \sum_{i=1}^{\infty} a(i)\theta(i)\langle f(i), u(i)\rangle d\mu(\theta, u) .$$

Let f_n denote the function $f_n : N \to E$,

$$f_n(i) = \begin{cases} f(i) , & i = n , \\ 0 , & i \neq n . \end{cases}$$

Then

$$F(f_1) = a(1) \iint_{\pi D\times B_1} \theta(1)\langle f(1), u(1)\rangle d\nu(\theta, u) .$$

But $F(f_1) = \langle f(1), x_1^*\rangle$. It follows that

$$x_1^* = a(1) \cdot \iint_{\pi D\times B_1} \theta(1)\cdot u(1)d\nu(\theta, u) ,$$

where the integral here is the Pettis integral, [4]. Set

$$Z_1^* = \iint_{\pi D\times B_1} \theta(1)u(1)d\nu(\theta, u) .$$

Hence $x_1^* = a(1)\cdot Z_1^*$. Similarly $x_i^* = a(i)\cdot Z_i^*$, $i = 2, 3, \ldots$. It remains to show that the function $g : N \to E^*$, defined by $g(i) = Z_i^*$, is an element of $l^{q'}(E^*)$. To see that, consider

$$\begin{aligned} |\langle g(i), x\rangle| &\le \iint_{\pi D\times B_1} |\theta(i)\langle u(i), x\rangle| d\mu(\theta, u) \quad (x \in E, \|X\| \le 1) \\ &\le \iint_{\pi D\times B_1} |\langle u(i), x\rangle| d\mu(\theta, u) \\ &\le \left(\iint_{\pi D\times B_1} |\langle u(i), x\rangle|^{q'} d\mu(\theta, u)\right)^{1/q'} . \end{aligned}$$

Hence

$$\sum_{i=1}^{\infty} |\langle g(i), x\rangle|^{q'} \le \sum_{i=1}^{\infty} \iint_{\pi D\times B_1} |\langle u(i), x\rangle|^{q'} d\mu(\theta, u) .$$

The monotone convergence theorem implies that

$$\sum_{i=1}^{\infty} |\langle g(i), x\rangle|^{q'} \le \iint_{\pi D\times B_1} \sum_{i=1}^{\infty} |\langle u(i), x\rangle|^{q'} d\mu(\theta, u)$$

$$\le \sup_{u\in B_1} \sum_{i=1}^{\infty} |\langle u(i), x\rangle|^{q'}\cdot|\mu| ,$$

where $|\mu|$ is the total variation of μ. Thus $g \in l^{q'}(E^*)$. So $F = a\cdot g$, $a \in l^{p'}$, $g \in l^{q'}(E^*)$. This completes the proof of the theorem.

2. Schur multipliers

Let $p, q \ge 1$. A bounded function ϕ on $N \times N$ is called a Schur multiplier of $l^p \hat{\otimes} l^q$ if $\phi\cdot\psi \in l^p \hat{\otimes} l^q$ for all $\psi \in l^p \hat{\otimes} l^q$, where $\phi\cdot\psi$ denotes pointwise multiplication.

If X and Y are Banach spaces, then a bounded linear map $A : X \to Y$ is called p-summing operator if

$$\sum_{i=1}^{n} \|Ax_i\|^p \le \zeta \cdot \sup_{x^*} \sum_{i=1}^{n} |\langle x_i, x^*\rangle|^p ,$$

for all $x_1, \ldots, x_n$ in X and some constant ζ independent of n. The supremum is taken over all elements x^* in the unit ball of X^*, [10]. Bennett [2] proved that a bounded function ϕ is a multiplier of $l^p \hat{\otimes} l^q$ if and only if $\phi\cdot u \otimes v : l^{p^*} \to l^\infty$ is q^* summing operator for all $u \otimes v \in l^\infty \hat{\otimes} l^p$. For more about multipliers we refer to [2], [6], [7] and [3].

LEMMA 2.1. *Let $A : l^p \to l^\infty$ be a bounded operator. If A is q-summing, then $A \in l^{q,q'}\langle l^{p'}\rangle$.*

Proof. Let $f : N \to l^{p'}$ be the function defined by $f(i) = A_i$, where $A_i(j) = A(i, j)$ (considering A as an infinite matrix). If $g \in l^q(l^p)$, then

$$\sum_{i=1}^{\infty} |\langle f(i), g(i)\rangle|^q = \sum_{i=1}^{\infty} |\langle A_i, g(i)\rangle|^q$$

$$= \sum_{i=1}^{\infty} \left|\sum_{j=1}^{\infty} A(i, j)g(i)(j)\right|^q$$

$$\leq \sum_{i=1}^{\infty} \sup_k \left|\sum_{j=1}^{\infty} A(k, j)g(i)(j)\right|^q$$

$$= \sum_{i=1}^{\infty} \|A(g(i))\|^q$$

$$\leq \zeta \sup_h \sum_{i=1}^{\infty} |\langle g(i), h\rangle|^q \quad \text{(by assumption)},$$

where h is the unit ball of $l^{p'}$. Hence $f \in l^{q,q'}\langle l^{p'}\rangle$. Let $M(l^p \hat{\otimes} l^q)$ denote the space of all multipliers of $l^p \hat{\otimes} l^q$. Then:

THEOREM 2.2. *Let* ϕ *be a bounded function on* $N \times N$. *Then* $\phi \in M(l^p \hat{\otimes} l^q)$ *if and only if* $\phi \cdot 1 \otimes u \in l^{q',q}\langle l^p\rangle$, *for all* $u \in l^p$.

Proof. Let $\phi \in M(l^p \hat{\otimes} l^q)$. Then by Bennett's result [2], $\phi \cdot 1 \otimes u : l^{p'} \to l^\infty$ is q'-summing for all $u \in l^p$. Lemma 2.1 then implies that $\phi \cdot 1 \otimes u \in l^{q',q}\langle l^p\rangle$.

Conversely, let $\phi \cdot 1 \otimes u \in l^{q',q}\langle l^p\rangle$ for all $u \in l^p$. It is enough to show that $\phi \in M(l^q \hat{\otimes} l^p)$. So let $u \otimes v \in l^q \hat{\otimes} l^p$, and $\psi \in l^{q'} \check{\otimes} l^{p'}$. Then

$$|\langle \phi \cdot u \otimes v, \psi\rangle| = \left|\sum_{i,j=1}^{\infty} \phi(i, j)u(i)v(j)\psi(i, j)\right|$$

$$= \left|\sum_{i=1}^{\infty} u(i)\langle \phi_i \cdot v, \psi_i\rangle\right| ,$$

where $\phi_i(j) = \phi(i, j)$ and $\psi_i(j) = \psi(i, j)$. Since $\psi \in l^{q'} \check{\otimes} l^{p'}$ it follows that $g : Z^+ \to l^{p'}$ defined by $g(i) = \psi_i$ is an element of $l^{q'}(l^{p'})$, [3]. Hence

$$|\langle \phi\cdot u \otimes v,\ \psi\rangle| \le \|u\|_q \left(\sum_{i=1}^{\infty} |\langle \phi_i\cdot v,\ \psi_i\rangle|^{q'}\right)^{1/q'}$$
$$\le \|u\|_q \cdot \|\phi\cdot 1 \otimes v\|_{\sigma(q',q)} \cdot \|\psi\|_{\varepsilon(q')} .$$

This completes the proof of the theorem.

LEMMA 2.3. *If* $A : l^{p'} \to l^{\infty}$ *is* q'*-summing, then* $A \in M(l^p \hat{\otimes} l^q)$.

Proof. It is enough to show that $A\ 1 \otimes v : l^{p'} \to l^{\infty}$ is q'-summing operator for all $v \in l^p$, [2]. But

$$\sum_{i=1}^{\infty} \|(A\cdot 1 \otimes v)f_i\|_{\infty}^{q'} = \sum_{i=1}^{\infty} \|A(v\cdot f_i)\|_{\infty}^{q'}$$
$$\le \zeta \sup_h \sum_{i=1}^{\infty} |\langle v\cdot f_i,\ h\rangle|^{q'}$$
$$\le \zeta \sup_h \sum_{i=1}^{\infty} |\langle f_i,\ v\cdot h\rangle|^{q'}$$
$$\le \zeta \sup_k \sum_{i=1}^{\infty} |\langle f_i,\ k\rangle|^{q'}$$

where h and k are in the unit ball of l^p , and the lemma follows.

It follows from Lemmas 2.3 and 2.1 that the set of all q'-summing maps from $l^{p'}$ into l^{∞} is contained in $M(l^p \hat{\otimes} l^q) \cap l^{q',q}\langle l^p\rangle$, where $\cap$ denotes the intersection of the two sets.

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