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PARAMETRIZING FUCHSIAN SUBGROUPS OF THE BIANCHI GROUPS

Dedicated to the memory of Norbert Wielenberg

C. MACLACHLAN AND A. W. REID

1. **Introduction.** Let d be a positive square-free integer and let O_d denote the ring of integers in $\mathbb{Q}(\sqrt{-d})$. The groups $\mathrm{PSL}_2(O_d)$ are collectively known as the *Bianchi groups* and have been widely studied from the viewpoints of group theory, number theory and low-dimensional topology. The interest of the present article is in geometric Fuchsian subgroups of the groups $\mathrm{PSL}_2(O_d)$. Clearly $\mathrm{PSL}_2(\mathbb{Z})$ is such a subgroup; however results of [18], [19] show that the Bianchi groups are rich in Fuchsian subgroups. Like the Bianchi groups, their Fuchsian subgroups have been studied for a variety of arithmetic, group theoretic and topological reasons (eg. [8],[9],[10],[13],[18],[21],[23],[31],[32]). In this paper, we extend the work in [18] and [21] on Fuchsian subgroups of the Bianchi group and obtain a complete classification in the case of the Picard group $\mathrm{PSL}_2(O_1)$

All non-elementary Fuchsian subgroups of Bianchi groups are subgroups of arithmetic Fuchsian subgroups and each Bianchi group contains infinitely many commensurability classes of arithmetic Fuchsian subgroups [18]. These groups can be either cocompact or non-cocompact. The conjugacy classes of maximal arithmetic Fuchsian subgroups and hence the wide commensurability classes of finite covolume Fuchsian subgroups, can be parametrized by their discriminant (see 3.1) which is a positive integer related to the circle or straight line stabilized by the Fuchsian group. We prove (3.3).

THEOREM 1. *There are finitely many* $\mathrm{PSL}_2(O_d)$*-conjugacy classes of maximal arithmetic Fuchsian subgroups of* $\mathrm{PSL}_2(O_d)$ *of a fixed discriminant.*

The groups $\mathrm{PSL}_2(O_d)$ are closely related to the integer points of Special Orthogonal groups of certain integral quaternary quadratic forms (eg. [6],[20] and § 4) and the number of conjugacy classes in Theorem 1 can be related to the number of equivalence classes of representations of integers by such forms. This is made explicit for the Picard group and by counting the number of classes of representations using the analytic methods of Siegel [25] we prove: (§ 6)

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THEOREM 5. *Let n_D denote the number of* $\mathrm{PSL}_2(O_1)$*-conjugacy classes of maximal arithmetic Fuchsian subgroups of* $\mathrm{PSL}_2(O_1)$ *of discriminant D. Then*

$$n_D = \begin{cases} 1 & \text{if } D \equiv 0, 3 \pmod 4 \\ 3 & \text{if } D \equiv 1 \pmod 4 \\ 2 & \text{if } D \equiv 2 \pmod 4 \end{cases}$$

As an elementary consequence of this theorem we extend and clarify results of Fine [9] which deal with intersection properties of $\mathrm{PSL}_2(\mathbb{Z})$with Fuchsian subgroups of $\mathrm{PSL}_2(O_1)$.

In addition the computations involved in the proof of Theorem 5, together with arithmetic methods for computing the number of conjugacy classes of maximal finite cyclic subgroups of arithmetic Fuchsian groups (eg. [30]) allow us to determine the signature of these Fuchsian groups in a very specific way (see Theorems 7 and 8 of § 7 for the precise statement).

Finally, arithmetic Fuchsian subgroups of torsion-free subgroups Γ of finite index in $\mathrm{PSL}_2(O_d)$ give rise via the action on hyperbolic space to totally geodesic surfaces immersed in the hyperbolic 3-manifold determined by Γ. In particular when the group Γ is the group of the Borromean rings B, our prior results allow us to determine that the minimal genus of a closed totally geodesic surface immersed in $S^3 \setminus B$ is three (see Theorem 9 § 8).

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2. **Preliminaries.**

2.1. Let H^3 denote the upper halfspace model of hyperbolic 3-space so that

$$H^3 = \{(z,t) : z \in \mathbb{C}, \quad t \in \mathbb{R}^+\}$$

endowed with the hyperbolic metric $\frac{|dz|^2+dt^2}{t^2}$.

The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ via linear fractional transformation and the action extends in a natural way to H^3 so that $\mathrm{PSL}_2(\mathbb{C})$ is the full group of orientation-preserving isometries of H^3 (eg. [1] Chap. 4).

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and a *Fuchsian group* is a Kleinian group which stabilizes a circle or straight line $\mathcal{C}$ in $\mathbb{C}$ and preserves the components of $\mathbb{C}/\mathcal{C}$.

If F is a Fuchsian group stabilizing $\mathcal{C}$ then F is *non-elementary* if its limit set on $\mathcal{C}$ consists of more than two points ([1] Chap. 5). In the extended action on H^3, F preserves the hyperbolic plane $H(\mathcal{C})$ represented by the hemisphere or plane on $\mathcal{C}$ in H^3 orthogonal to $\mathbb{C}$ with the restriction of the hyperbolic metric.

The group F is said to have *finite covolume* (resp. F is *cocompact*) if $H(C)/F$ is of finite volume (resp. compact) and we denote the covolume of F by vol(F). Furthermore every Fuchsian group of finite covolume has a presentation of the following form ([1] Chap. 10)

$$\text{Generators: } a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r, p_1, \ldots, p_t$$

$$\text{Relations: } \prod_{j=1}^{g}[a_j, b_j] \prod_{i=1}^{r} x_i \prod_{k=1}^{t} p_k = 1 \; x_i^{m_i} = 1 \; (i = 1, 2, \ldots, r) \tag{1}$$

where the $a_j, b_j (j = 1, 2, \ldots, g)$ are hyperbolic, the $x_i (i = 1, 2, \ldots, r)$ are elliptic and the $p_k (k = 1, 2, \ldots, t)$ parabolic. The integer r (resp. t) is the number of conjugacy classes of maximal finite cyclic (resp. parabolic) subgroups of F, and the quotient $H(C)/F$ is a compact surface of genus g with t punctures and r distinguished points. The integers $m_1, m_2, \ldots, m_r$ are the *periods* of F. The group F is cocompact if and only if $t = 0$ and torsion-free if and only if $r = 0$.

The $(r+2)$-tuple of integers $(g; m_1, m_2, \ldots, m_r : t)$ is the *signature* of F and such an F has covolume given by

$$\text{vol}(F) = 2\pi\left[2(g-1) + t + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right] \tag{2}$$

2.2. Arithmetic Fuchsian groups are obtained as follows ([2],[30])(We use [30] extensively as a reference for the relevant material on quaternion algebras). Let k be a totally real number field and A a quaternion algebra over k ramified at all archimedean places except one. Let ρ be an isomorphism of A into $M_2(\mathbb{C})$, O an order of A and O^1 the elements of norm one in O. Then $P\rho(O^1)$ (where $P\colon GL_2(\mathbb{C}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$) is a Fuchsian group and if C is the invariant circle of $P\rho(O^1)$, then $P\rho(O^1)$ has finite covolume in $H(C)$. The class of *arithmetic Fuchsian groups* is that given by the union of the commensurability classes of all such $P_\rho(O^1)$. ([30] Chap. 4).

An arithmetic Fuchsian group is cocompact if and only if the associated algebra A is a division algebra of quaternions ([30] Chap. 4). Thus any non-cocompact arithmetic Fuchsian group is conjugate in $\mathrm{PSL}_2(\mathbb{C})$ to a group commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ since the only quaternion algebra giving rise to arithmetic Fuchsian groups which is not a division algebra is $M_2(\mathbb{Q})$.

Arithmetic Kleinian groups are obtained in a similar way and in analogy with the above, any non-cocompact arithmetic Kleinian group is conjugate in $\mathrm{PSL}_2(\mathbb{C})$ to a group commensurable with some $\mathrm{PSL}_2(O_d)$.

2.3. The arithmetic Fuchsian groups described in this paper all arise from indefinite quaternion algebras over $\mathbb{Q}$ (see §3.1 below). As such they are classified up to isomorphism by a finite set, of even cardinality, of primes at which the algebra is ramified ([30] Chap. 3). They arise here naturally in the form $\left(\frac{a,b}{\mathbb{Q}}\right)$ with $a, b \in \mathbb{Z} \setminus \{0\}$

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which describes the quaternion algebra over $\mathbb{Q}$ whose standard basis $\{1, i, j, ij\}$ satisfies $i^2 = a, j^2 = b, ij = -ji$.

3. **Finiteness results.**

3.1. Let F be a non-elementary Fuchsian subgroup of $\mathrm{PSL}_2(O_d)$ stabilizing a circle or straight line $\mathcal{C}$ in $\mathbb{C}$. Then F is a subgroup of

$$\mathrm{Stab}(\mathcal{C}, \mathrm{PSL}_2(O_d)) = \{\gamma \in \mathrm{PSL}_2(O_d) \mid \gamma(\mathcal{C}) = \mathcal{C} \text{ and } \gamma \text{ preserves the components of } \mathbb{C} \setminus \mathcal{C}\}.$$

which is a maximal arithmetic Fuchsian subgroup of $\mathrm{PSL}_2(O_d)$[18]. Thus every finite covolume Fuchsian subgroup of $\mathrm{PSL}_2(O_d)$ is arithmetic.

NOTATION 1. For brevity, denote $\mathrm{PSL}_2(O_2)$ by Γ_d.

NOTATION 2. Except briefly in § 6.6 we will only consider finite covolume subgroups. Thus without further mention, we reserve the term "F is a Fuchsian group" to mean that F is a finite covolume Fuchsian group.

In [18] it is shown that $\mathcal{C}$ has an equation of the form

$$a|z|^2 + Bz + \bar{B}\bar{z} + c = 0 \tag{3}$$

where $a, c \in \mathbb{Z}$ and $B \in O_d$, to which we associate the triple (a, B, c).

DEFINITION 3.1. Let $B = \frac{1}{2}(b_1 + b_2\sqrt{-d})$ with $b_i \in \mathbb{Z}\,(i = 1, 2)$ and $b_1 \equiv b_2 \pmod 2$(and $\equiv 0 \pmod 2$ unless $d \equiv -1 \pmod 4$). The triple (a, B, c) is called *primitive* if

$$g.c.d.\,(a, \frac{b_1}{2}, \frac{b_2}{2}, c) = 1 \quad \text{for } b_1 \equiv b_2 \equiv 0 \pmod 2$$
$$g.c.d.\,(a, b_1, b_2, c) = 1 \quad \text{for } b_1 \equiv b_2 \not\equiv 0 \pmod 2$$

Clearly a circle or straight line can be represented uniquely by a primitive triple up to sign.

DEFINITION 3.2. The *discriminant* of $\mathcal{C}$ is defined to be

$$D = D(\mathcal{C}) + |B|^2 - ac = \frac{1}{4}(b_1^2 + db_2^2) - ac$$

where (a, B, c) is the primitive triple representing $\mathcal{C}$.

We also refer to this as the discriminant of F where F is commensurable with $\mathrm{Stab}(\mathcal{C}, \Gamma_d)$.

Note that D is a positive integer. Moreover for every positive integer D let $\mathcal{C}_D$ denote the circle $\{z \in \mathcal{C} \mid |z|^2 = D\}$. Then $\mathrm{Stab}(\mathcal{C}_D, \Gamma_d)$ is an arithmetic Fuchsian subgroup of $\mathrm{PSL}_2(O_d)$ and moreover: ([18])

THEOREM M. *The quaternion algebra associated to any Fuchsian subgroup of* Γ_d *of discriminant D is isomorphic to* $\left(\frac{-d,D}{\mathbb{Q}}\right)$.

Two circles $\mathcal{C}, \mathcal{C}'$ stabilized by Fuchsian subgroups of Γ_d are Γ_d-equivalent if and only if there exists $T \in \Gamma_d$ with $T\mathcal{C} = \mathcal{C}'$. The following elementary result is a straight forward calculation ([21]).

LEMMA 1. *Let* $\mathcal{C}, \mathcal{C}'$ *be represented by triples* (a, B, c) *and* (a', B', c') *and let* $T\mathcal{C} = \mathcal{C}'$ *with* $T \in \Gamma_d$.

(i) (a, B, c) is primitive if and only if (a', B', c') is primitive
(ii) $D(\mathcal{C}) = D(\mathcal{C}')$.

LEMMA 2. *Let* $\mathcal{C}$ *be a circle represented by a primitive triple* (a, B, c). *Then* $\mathrm{Stab}(\mathcal{C}, \Gamma_d)$ *is an arithmetic Fuchsian subgroup of* Γ_d.

PROOF. We can assume that $a \neq 0$ [18]. Let $V = \begin{pmatrix} a & \bar{B} \\ 0 & 1 \end{pmatrix}$ so that $V\mathcal{C} = \mathcal{C}_D$ where $D = |B|^2 - ac$. Since $V \in GL_2(\mathbb{Q}(\sqrt{-d}))$, it follows that $P(V)\Gamma_d P(V)^{-1}$ is commensurable with Γ_d ([30] Chap. 4). Thus $P(V)\,\mathrm{Stab}(\mathcal{C}, \Gamma_d)P(V)^{-1}$ is commensurable with $\mathrm{Stab}(\mathcal{C}_D, \Gamma_d)$, which as indicated in (3.1) is arithmetic. ■

Let $\Sigma_d = \{$circles $\mathcal{C}$ represented by primitive triples in $O_d\}$.

3.2. THEOREM 1. *There are finitely many* $\mathrm{PSL}_2(O_d)$*-conjugacy classes of maximal Fuchsian subgroups of* $\mathrm{PSL}_2(O_d)$ *of fixed discriminant.*

PROOF. The conjugacy classes of maximal Fuchsian subgroups of Γ_d are in one-to-one correspondence with the Γ_d-equivalence classes of circles $\mathcal{C} \in \Sigma_d$. Let $\mathcal{C}$ be represented by a primitive triple (a, B, c) and define $\Phi(a, B, c) = \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix}$ which represents a binary hermitian form. Note that $\det \Phi(a, B, c) = -D$. Thus Φ defines a bijection from Σ_d to $\mathcal{H}_d$ where

$$\mathcal{H}_d = \left\{ \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} a, c \in \mathbb{Z} \quad B \in O_d, ac - |B|^2 < 0, \text{ and } (a, B, c) \text{ primitive} \right\}.$$

Now if $T \in \mathrm{GL}_2(O_d)$ then T acts on $\mathcal{H}_d$ by

$$\begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} \mapsto T \begin{pmatrix} a & B \\ \bar{B} & c \end{pmatrix} T^*$$

Under this action determinants are preserved and the centre acts trivially so that it can be considered as a $\mathrm{PGL}_2(O_d)$ action. Now $\mathrm{PGL}_2(O_d)$ also acts on Σ_d. Furthermore if $T\mathcal{C} = \mathcal{C}'$ and $\mathcal{C}'$ is represented by (a', B', c') then $V\Phi(a, B, c)V^* = \Phi(a', B', c')$ where $V = (T^{-1})^t$, and conversely. Thus the $\mathrm{PGL}_2(O_d)$-equivalence classes of circles $\mathcal{C}$ are in one-to-one correspondence with the $\mathrm{PGL}_2(O_d)$-equivalence classes of binary Hermitian forms as described above. But by a result of P. Humbert [16] there are finitely many $\mathrm{PGL}_2(O_d)$-equivalence classes of such binary Hermitian forms of determinant $-D$. Thus for any group commensurable with $\mathrm{PGL}_2(O_d)$ and so in particular for Γ_d, there are finitely many conjugacy classes of maximal Fuchsian subgroups of fixed discriminant. ∎

REMARK. In the case $d = 1$, Theorem 1 was proved by Harding [13] using a geometric argument.

3.3. In this section we investigate the dependence of $\mathrm{vol}(\mathrm{Stab}(C, \Gamma_d))$ on the discriminant D.

From Theorem M in §3.1, the quaternion algebra associated to $\mathrm{Stab}(C, \Gamma_d)$ is isomorphic to $\left(\frac{-d,D}{\mathbf{Q}}\right)$ so that there is a representation ρ of this algebra into $M_2(\mathbb{C})$ and a maximal order O such that $\mathrm{Stab}(C, \Gamma_d)$ is commensurable with $P\rho(O^1)$. But by the criteria of Takeuchi [27] we can assume that $\mathrm{Stab}(C, \Gamma_d) \subseteq P\rho(O^1)$. The covolume of the groups $P\rho(O^1)$ depends only on the isomorphism class of the quaternion algebra $\left(\frac{-d,D}{\mathbf{Q}}\right)$ ([30] Chap. 4). The isomorphism class does not uniquely determine the integer D and indeed for each D there are infinitely many D' such that $\left(\frac{-d,D}{\mathbf{Q}}\right) \cong \left(\frac{-d,D'}{\mathbf{Q}}\right)$. Nonetheless we prove,

THEOREM 2. *Let* $\{D_n\}$ *be a sequence of positive integers such that* $\lim_{n\to\infty} D_n = \infty$. *Then* $\lim_{n\to\infty} \mathrm{vol}(C_n, \Gamma_d) = \infty$ *where* $D(C_n) = D_n$.

PROOF. Let Γ be a torsion-free subgroup of Γ_d of finite index. Suppose as $D_n \longrightarrow \infty$, the volumes remain bounded. Then $\mathrm{vol}(\mathrm{Stab}(C_n, \Gamma))$ remains bounded since

$$[\mathrm{Stab}(C_n, \Gamma_d) : \mathrm{Stab}(C_n, \Gamma)] \leq [\Gamma_d : \Gamma]$$

Thus there exists a subsequence $\{D_n\}$ such that the groups $\mathrm{Stab}(C_n, \Gamma)$ all have the same signature. But by a result of Thurston ([29] Corollary 8.8.6) infinitely many of these are conjugate in $\mathrm{PSL}_2(\mathbb{C})$. An adaptation of an argument of Greenberg ([12] Theorem 2) now implies that infinitely many of them are conjugate in Γ. This is a contradiction since the C_n have distinct discriminants. ∎

4. **Γ_d and quadratic forms.** In [20] the relationship between those arithmetic Kleinian groups which contain non-elementary Fuchsian subgroups and discrete arithmetic subgroups of integer points in orthogonal groups of certain quaternary quadratic forms was exhibited. We now make this explicit for the groups Γ_d and subsequently fully exploit it in the case $d = 1$.

4.1. Let V be a 4-dimensional vector space over $\mathbb{Q}$, f a nondegenerate quadratic form on V with integer coefficients and S the associated symmetric matrix. Let $O(f)$ (resp. $SO(f)$) denote the orthogonal (resp. special orthogonal) group of f so that

$$O(f) = \{X \in GL_4 | X^t S X = S\}$$

For a subring R of $\mathbb{C}$, let $O(f, R)$, and $SO(f, R)$ denote the R-points of $O(f)$, and $SO(f)$ respectively.

If f has signature (3,1) then $SO^o(f, \mathbb{R})$, the identity component of $SO(f, \mathbb{R})$, is isomorphic to $\mathrm{PSL}_2(\mathbb{C})$. In which case $SO^o(f, \mathbb{Z})$ is a discrete arithmetic subgroup of $SO^o(f, \mathbb{R})$ and therefore is finitely-generated, finitely-presented and of finite covolume in $SO^o(f, \mathbb{R})$([3]).

Let $V(\mathbb{R}) = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathcal{H}$ denote the projective image of $\{\omega \in V(\mathbb{R}) | f(\omega) < 0\}$. Equipped with the metric induced by the restriciton of the indefinite metric f, $\mathcal{H}$ is a model of hyperbolic 3-space. In this model, a Fuchsian group will stabilize the projection of a 3-dimensional linear subspace of $V(\mathbb{R})$ and fix the 1-dimensional subspace orthogonal to it (orthogonal with respect to f).

4.2. Let $A = \left(\frac{-1,1}{\mathbb{Q}(\sqrt{-d})}\right)$ so that A is isomorphic to $M_2(\mathbb{Q}(\sqrt{-d}))$ and the order $O_d[1, i, \frac{1+j}{2}, \frac{i+ij}{2}]$ is isomorphic to $M_2(O_d)$ ([30] Chaps. 1 and 3). Now A admits the conjugate-linear involution τ given by

$$\tau(a_0 + a_1 i + a_2 j + a_3 ij) = \bar{a}_0 - \bar{a}_1 i - \bar{a}_2 j - \bar{a}_3 ij \text{ where } a_i \in \mathbb{Q}(\sqrt{-d}).$$

The fixed-point set of τ is a $\mathbb{Q}$-subspace with basis $\{1, \sqrt{-d}, \frac{\sqrt{-d}}{2}(-i + ij), \frac{\sqrt{-d}}{2}(i + ij)\}$ and the reduced norm of A restricted to V defines a quadratic form f_d given by

$$f_d(x_1, x_2, x_3, x_4) = x_1^2 + dx_2^2 + dx_3x_4 \tag{4}$$

For $y \in A^1$, the elements of norm 1 in A, define the automorphism ϕ_y of V by

$$\phi_y(x) = yx\tau(y)$$

so ϕ_y induces a homomorphism

$$\Phi_d : A^1 \longrightarrow O(f_d, \mathbb{Q})$$

which may be extended to $(A \otimes_{\mathbb{Q}(\sqrt{-d})} \mathbb{C})^1 \cong \mathrm{SL}_2(\mathbb{C})$. This then defines an isomorphism, also denoted by Φ_d,

$$\Phi_d : \mathrm{PSL}_2(\mathbb{C}) \longrightarrow SO^o(f_d, \mathbb{R})$$

such that $\Phi_d(\mathrm{PSL}_2(O_d))$ is commensurable with $SO^o(f_d, \mathbb{Z})$ (see e.g. [20] for details).

REMARK. $\Phi_d(\mathrm{PSL}_2(O_d))$is not necessarily a subgroup of $SO^o(f_d, \mathbb{Z})$.

Under Φ_d, Fuchsian subgroups of Γ_d give rise to Fuchsian subgroups of $SO^o(f_d, \mathbb{Z})$ (by dropping to a subgroup of finite index if necessary) as discussed in §4.1. It is easy to check that every Fuchsian subgroup of $SO^o(f_d, \mathbb{Z})$ is a subgroup of a maximal Fuchsian subgroup which has the form:

$$\mathrm{Stab}(\omega_0, SO^o(f_d, \mathbb{Z})) = \{T \in SO^o(f_d, \mathbb{Z}) | T\omega_0 = \omega_0\}$$

where ω_o is a fixed normal to the plane stabilized by the Fuchsian subgroup and satisfies $f_d(\omega_0) > 0$.

We now describe the vectors ω_0 explicitly

4.3. Let $C \in \Sigma_d$ be a circle of discriminant D represented by the primitive triple (a, B, c) with $B = \frac{1}{2}(b_1 + b_2\sqrt{-d})$ as before. Let $F(C)$ denote the group $\mathrm{Stab}(C, \Gamma_d)$

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LEMMA 3. *A normal to the plane stabilized by* $\Phi_d(F(\mathcal{C}))$ *is given by*

$$(-b_2 d, -b_1, -2c, 2a)$$

PROOF. We deal first with the case $\mathcal{C} = \mathcal{C}_D$. Let $B = \left(\frac{-d,D}{\mathbb{Q}}\right)$ with standard basis $\{1, i_B, j_B, i_B j_B\}$ and ρ the natural representation of B into $M_2(\mathbb{Q}(\sqrt{-d}))$ given by

$$\rho(i_B) = \begin{pmatrix} \sqrt{-d} & 0 \\ 0 & -\sqrt{-d} \end{pmatrix} \quad \rho(j_B) = \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix}$$

These matrices act on H^3 and both preserve the hyperbolic plane $H(\mathcal{C}_D)$ with $\rho(i_B)$ preserving the orientation of $H(\mathcal{C}_D)$ and $\rho(j_B)$ reversing it.

Now $P\rho(i_B) = P\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $P\rho(j_B) = P\begin{pmatrix} 0 & i\sqrt{D} \\ i/\sqrt{D} & 0 \end{pmatrix}$. By direct calculation, the images of these elements under Φ_d are

$$I_B = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad J_B = \begin{bmatrix} -1 & 0 & & \\ 0 & 1 & & \\ & & 0 & -D \\ & & -\frac{1}{D} & 0 \end{bmatrix}$$

respectively. Thus $\Phi_d(F(\mathcal{C}))$ stabilizes a plane Π_D with normal vector ω_0 satisfying $I_B(\omega_0) = \omega_0$ and $J_B(\omega_0) = -\omega_0$. Solving these equations yields $\omega_0 = (0, 0, D, 1)$.

As noted in the proof of Lemma 2,

$$PV = P\begin{pmatrix} \sqrt{a} & \frac{\bar{B}}{\sqrt{a}} \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix}$$

maps $\mathcal{C}$ to $\mathcal{C}_D$ so that $\Phi_d(PV)$ maps the plane stabilized by $\Phi_d(F(\mathcal{C}))$ to Π_D. Again a direct calculation yields that $\Phi_d(PV^{-1})(\omega_0) = (-b_2 d, -b_1, -2c, 2a)$. ∎

Note that the vector $\omega = (-b_2 d, -b_1, -2c, 2a)$ need not be primitive, in the obvious sense, but by dividing by the greatest common divisor we can obtain a primitive vector representing the same one-dimensional subspace.

DEFINITION 4.1. The *form discriminant* of a maximal Fuchsian subgroup $F(\mathcal{C})$ in Γ_d (or of $\Phi_d(F(\mathcal{C}))$ in $\Phi_d(\Gamma_d)$) is defined to be $f_d(\omega_0)$ where ω_0 is a primitive vector orthogonal to the plane stabilized by $\Phi_d(F(\mathcal{C}))$.

If G is a subgroup of finite index in $O(f_d, \mathbb{Z})$ and c is a primitive representation of an integer Δ i.e., $f_d(c) = \Delta$, then for $g \in G$, gc is primitive and represents Δ. The number of G-equivalence classes of such representatives is finite ([4] Chap. 9). Clearly ω_0 in the definition is unique up to sign and so we have

THEOREM 3. *Let $G = \Phi_d(\Gamma_d) \cap O(f_d, \mathbb{Z})$. The number of G-conjugacy classes of maximal Fuchsian subgroups of $\Phi_d(\Gamma_d)$ of fixed form discriminant is finite.*

REMARK. The form discriminant and the discriminant D described in § 3 of a Fuchsian subgroup need not be the same if $(d, D) > 1$. Nonetheless the relationship between the two sets of discriminants is finite to finite and thus an alternative proof of Theorem 1 may be obtained.

5. **Siegel's methods applied to f_1.**

5.1. We now concentrate on the case $d = 1$ and denote f_1 by f and Φ_1 by Φ.

In this case $\Phi(\Gamma_1)$ is a subgroup of $SO^o(f, \mathbb{Z})$. The group $\mathrm{PGL}_2(O_1)$ (or strictly its image in $\mathrm{PSL}_2(\mathbb{C})$) is also mapped by Φ into $SO^o(f, \mathbb{Z})$and since $\mathrm{PGL}_2(O_1)$ is a maximal arithmetic Kleinian group ([2],[14]) it follows that Φ maps $\mathrm{PGL}_2(O_1)$ isomorphically onto $SO^o(f, \mathbb{Z})$.

Let $\mathcal{C} \in \Sigma_1$ be represented by the primitive triple (a, B, c) where now, with a slight change of notation, $B = b_1 + b_2 i$ and $D = b_1^2 + b_2^2 - ac$. By Lemma 3, a primitive normal to the plane stabilized by $\Phi(F(\mathcal{C}))$ is $\omega_0 = (-b_2, -b_1, -c, a)$ and $f(\omega_o) = D$. Note that since ω_0 is only determined up to sign, we must consider ω_0 and $-\omega_0$ as equivalent.

THEOREM 4. *The following are equal*

(i) The number of Γ_1-conjugacy classes of maximal Fuchsian subgroups of Γ_1 of discriminant D.

(ii) The number of Γ_1-conjugacy classes of maximal Fuchsian groups of Γ_1 of form discriminant D.

(iii) The number of $\Phi(\Gamma_1)$-equivalence classes of primitive representations of D by f.

PROOF. It remains to show that each primitive representative $(\alpha, \beta, \gamma, \delta) \in \mathbb{Z}^4$ of D by f gives rise to a Fuchsian subgroup of Γ_1. This follows by taking $\mathcal{C}$ represented by the primitive triple $(\delta, -(\beta + \alpha i), \gamma)$ and applying lemma 2. ■

5.2. To count primitive representations of integers by integral quadratic forms we use methods due to Siegel [25]. In this section, we describe these results in the form we require following the notation and description used by Elstrodt, Grunewald and Mennicke in their notes [6] where some related calculations are carried out.

It will be convenient to replace the form f by $-2f$ which has signature (1,3) so that the corresponding matrix S is integral and in the sequel we shall denote what was previously $-2f$ by f. Siegel's methods are simplified in this case as f lies in a genus of one class (e.g. [4] Chap. 11).

Let X, X' be primitive solutions of $f(x) = -2D$. These are equivalent if there exists $g \in O(f, \mathbb{Z})$ such that $gX = X'$ (Note the difference with equivalence as defined in Theorem 4 above). Let the n equivalence classes have representatives $C_1, C_2, \dots, C_n$, so that $C_i^t S C_i = -2D$ for $i = 1, 2, \dots, n$.

For $p \in \mathbb{Z}$, a prime, let

$$A_{p^e}(S,D) = \text{ number of incongruent solutions of } f(x) \equiv -2D \pmod{p^e}$$

and define the *local density*

$$\alpha_p(S,D) = \lim_{e \to \infty} \frac{A_{p^e}(S,D)}{p^{3e}}.$$

This limit exists and indeed the expression $p^{-3e}A_{p^e}(S,D)$ is independent of e for sufficiently large e.

Now for each C_i choose a unimodular complement B_i for $i = 1,2,\dots,n$ i.e. an integral 4×3 matrix B_i such that $K_i = [C_i : B_i]$ is unimodular. Let $T = [-2D]$, $Q_i = C_i^t SB_i$ and

$$H_i = B_i^t SB_i - Q_i^t T^{-1} Q_i \qquad i = 1,2,\dots,n. \tag{5}$$

Then H_i is a 3×3 matrix with associated ternary quadratic form h_i. Let λ_i denote the number of matrices $T^{-1}Q_iW$ which are pairwise incongruent mod 1 where W runs through the elements of $O(h_i, \mathbb{Z})$. Define

$$\rho(S, C_i) = \lambda_i (2D)^3 \frac{\pi}{2} ((\det(2DH_i))^{-2} \operatorname{vol}(O(h_i, \mathbb{Z})). \tag{6}$$

REMARK. In our case, the forms h_i are indefinite ternary quadratic forms with rational coefficients so that $\operatorname{vol}(O(h_i, \mathbb{Z}))$ denotes the covolume of the arithmetic group $O(h_i, \mathbb{Z})$ acting on the 2-dimensional hyperbolic space defined in the obvious way by h_i. Define

$$\mu(S) = 2^{-5}\pi^2 \operatorname{vol}(O(f, \mathbb{Z})) \tag{7}$$

SIEGEL'S THEOREM [25].

$$\sum_{i=1}^{n} \rho(S, C_i) = \mu(S) \prod_p \alpha_p(S,D) \tag{8}$$

Part of the infinite product on the right-hand side of (8) can be evaluated as follows ([6])

$$\prod_{p \nmid 2D} \alpha_p(S,D) = \frac{\zeta(2)}{\zeta_{\mathbb{Q}(i)}(2)} \prod_{\substack{p \mid D \\ p \text{ odd}}} \left(1 - \left(\frac{-1}{p}\right) p^{-2}\right)^{-1}$$

where $\left(\frac{\cdot}{p}\right)$ is the Legende symbol, $\zeta(s)$ the Riemann Zeta function and $\zeta_{\mathbb{Q}(i)}(s)$ the Dedekind zeta function of $\mathbb{Q}(i)$. Thus to determine the infinite product, it remains to evaluate $\alpha_p(S,D)$ for $p \mid 2D$.

LEMMA 4. *For $p|2D$ we have*

$$\alpha_p(S,D)\begin{cases} p^{-3}(p-1)\left(p^2-\left(\frac{-1}{p}\right)\right) & p \text{ odd} \\ \frac{3}{2} & p=2 \text{ and } D\equiv 0,3 \pmod 4 \\ \frac{5}{2} & p=2 \text{ and } D\equiv 1 \pmod 4 \\ \frac{9}{4} & p=2 \text{ and } D\equiv 2 \pmod 8 \\ \frac{7}{4} & p=2 \text{ and } D\equiv 6 \pmod 8 \end{cases}$$

PROOF. We give the proof for p odd; $p = 2$ is similar but more complicated (cf. [6]). Consider the number of solutions $x^2+y^2-uv\equiv 0 \pmod p$ excluding $(0,0,0,0)$ by primitivity. Then $x^2 \equiv -(y^2+uv) \pmod p$ has a solution if and only if y^2+uv is a quadratic residue (resp. quadratic non-residue) mod p if $p\equiv 1 \pmod 4$ (resp. $\equiv 3 \pmod 4$)) or zero. For any choice of y we can choose uv in $\left(\frac{p-1}{2}\right)$ ways so that y^2+uv is a quadratic residue or non-residue. For each non-zero value of uv we can choose the pair (u,v) in $p-1$ ways. Thus if x is nonzero we get $p(p-1)\left(p+\left(\frac{-1}{p}\right)\right)$ solutions. If $x=0$, for each $y\neq 0$, there are $p-1$ pairs (u,v) such that $y^2+uv\equiv 0 \pmod p$. If $y=0$, there are $2(p-1)$ pairs (u,v) such that $uv\equiv 0 \pmod p$. This implies $A_p(S,D)=(p-1)\left(p^2-\left(\frac{-1}{p}\right)\right)$.

Now suppose $\underline{a}^e$ is a solution of $x^2+y^2+uv\equiv D \pmod{p^e}$. Put $\underline{a}^{e+1}=\underline{a}^e+p^e\underline{z}$. Then $\underline{a}^{e+1}$ is a solution mod p^{e+1} if and only if $v+2az_1+2bz_2+cz_6+dz_3\equiv 0 \pmod p$ where $(\underline{a}^e)^t=(a,b,c,d)$ and $a^2+b^2+cd-D=p^e v$. Thus a solution mod p^e extends to p^3 solutions mod p^{e+1} since at least one of $a,b,c,d\not\equiv 0 \pmod p$. This gives the desired value of $\alpha_p(S,D)$ ■

We now consider the terms λ_i for $i=1,2,\ldots,n$ defined above. Conjugating the group

$$\mathrm{Stab}(C_i,O(f,\mathbb{Z}))=\{g\in O(f,\mathbb{Z}) \mid gC_i=C_i\}$$

by K_i gives the group $\mathrm{Stab}(E_1,O(f_i',\mathbb{Z}))$ where $E_1^t=(1,0,0,0)$ and f_i' is represented by the matrix

$$\begin{bmatrix} -2D & Q_i \\ Q_i^t & H_i+Q_i^tT^{-1}Q_i \end{bmatrix}.$$

If $X\in\mathrm{Stab}(E_1,O(f_i',\mathbb{Z}))$ then X has the form

$$\begin{bmatrix} 1 & P \\ 0 & Y_X \end{bmatrix}$$

where P and Y_X are integral 1×3 and 3×3 matrices respectively. Moreover $Y_X^tH_iY_X=H_i$. Thus the mapping θ defined by

$$\theta(X)=Y_{K_iXK_i^{-1}}$$

defines a monomorphism

(10) $$\theta : \mathrm{Stab}(C_i, O(f, \mathbb{Z})) \longrightarrow O(h_i, \mathbb{Z})$$

LEMMA 5. $\mathrm{vol}(\theta(\mathrm{Stab}(C_i, O(f, \mathbb{Z})))) = \lambda_i \,\mathrm{vol}\, O(h_i, \mathbb{Z})$ for $i = 1, \ldots, n$.

PROOF. Let us determine the image of θ as at (10). If $Y \in O(h_i, \mathbb{Z})$ there exists $X \in \mathrm{Stab}(C_i, O(f, \mathbb{Z})$ such that $\theta(X) = Y$ if and only if the matrix $T^{-1}Q_i[(\det Y)I - Y]$ is integral. Thus $[O(h_i, \mathbb{Z}) : \theta(\mathrm{Stab}(C_i, O(f, \mathbb{Z})))]$ equals the number of matrices $T^{-1}Q_iY$ which are pairwise incongruent mod 1 as Y runs through the elements of $O(h_i, \mathbb{Z})$. ∎

Now $[O(f, \mathbb{Z}) : SO^o(f, \mathbb{Z})] = 4$ and $SO^o(f, \mathbb{Z}) = \Phi(\mathrm{PGL}_2(O_1))$. Thus,

$$\mathrm{vol}(O(f, \mathbb{Z})) = 8\,\mathrm{vol}(\mathrm{PSL}_2(O_1)) = \frac{2\zeta_{Q(i)}(2)}{\pi^2}. \qquad \text{([30] Chap. 4)}$$

Therefore from (7), (9) and Lemma 4 we can simplify the right-hand side of (8) and then re-express Siegel's Theorem as

(11) $$\sum_{i=1}^{n} \rho(S, C_i) = 2^{-7}\zeta(2)\alpha_2(S, D) \prod_{\substack{p \mid D \\ p \text{ odd}}} \left(1 + \left(\frac{-1}{p}\right)p^{-1}\right)$$

where the terms $\alpha_2(S, D)$ are defined in Lemma 6.

6. **Conjugacy classes of maximal Fuchsian subgroups of** $\mathrm{PSL}_2(O_1)$. In this section we use the results of § 5 to prove

THEOREM 5. *Let n_D denote the number of conjugacy classes of maximal Fuchsian subgroups of Γ_1 of discriminant D. Then*

$$n_D = \begin{cases} 1 & \text{if } D \equiv 0, 3 \pmod 4 \\ 3 & \text{if } D \equiv 1 \pmod 4 \\ 2 & \text{if } D \equiv 2 \pmod 4 \end{cases}$$

6.1. A key role is played by the circle C_D. Now

$$F_D = \left\{ \mathrm{Stab}(C_D, \Gamma_1) = P\begin{pmatrix} \alpha & D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SL_2(O_1) \mid \alpha, \beta \in O_1 \right\}$$

If B_D denotes the quaternion algebra $\left(\frac{-1, D}{\mathbb{Q}}\right)$ with standard basis $\{1, i, j, ij\}$, consider the order $O = \mathbb{Z}[1, i, j, ij]$. With the representation ρ induced by

$$\rho(i) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \qquad \rho(j) = \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix}$$

it follows that $F_D = P\rho(O^1)$.

There is a formula, known as Humbert's formula, for the covolumes of these groups given by

(12) $$\mathrm{vol}(F_D) = \eta\pi D \prod_{\substack{p \mid D \\ p \text{ odd}}} \left(1 + \left(\frac{-1}{p}\right)p^{-1}\right)$$

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where $\eta = \frac{1}{2}$ if $4|D$ and $\eta = 1$ otherwise (correcting a small error in [30] p. 120).

The primitive solution of $f(C) = -2D$ corresponding to the circle $\mathcal{C}_D$ is $C = [0\,0\,D\,1]^t$ and we first compute the contribution to Siegel's formula from this class, i.e. $\rho(S, C)$ as given at (6). The indefinite ternary form h (as at (5)) in this case is

$$h(\underline{x}) = \frac{1}{2D}x_1^2 - 2x_2^2 - 2x_3^2$$

and so we obtain

$$\rho(S, C) = \lambda \text{ vol } O(h, \mathbb{Z})\frac{\pi}{2}(2D)^3(16D^2)^{-2} \tag{13}$$

By Lemma 5, $\lambda \text{ vol } O(h, \mathbb{Z}) = \text{vol}(\theta(\text{Stab}(C, O(f, \mathbb{Z}))$. In addition $\text{vol}(F_D) = \text{vol}(\Phi(\text{Stab}(\mathcal{C}_D, \Gamma_1)) = \text{vol}(\text{Stab}(C, \Phi(\Gamma_1))$ so that we require to compute the index

$$I_1 = [\text{Stab}(C, O(f_1\mathbb{Z})) : \text{Stab}(C, \Phi((\Gamma_1))]$$

The group $\Phi(\Gamma_1)$ has index 8 in $O(f, \mathbb{Z})$, and g_1, g_2, g_3 below are the coset representives of $SO(f, \mathbb{Z})$ in $O(f, \mathbb{Z}), SO^o(f, \mathbb{Z})$ in $SO(f, \mathbb{Z})$ and $\Phi(\Gamma_1)$ in $SO^o(f, \mathbb{Z})$ respectively.

$$g_1 = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad g_2 = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \quad g_3 = \begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Clearly $g_1, g_3 \in \text{Stab}(C, O(f, \mathbb{Z}))$ while $g_2(C) = -C$. Now there exists an element $g \in \Phi(\Gamma_1)$ such that $g(C) = -C$ if and only if there is an element γ of Γ_1 which leaves $\mathcal{C}_D$ invariant but interchanges the components of $\mathbb{C} \setminus \mathcal{C}_D$. Such an element exists if and only if the Diophantine equation $x^2 + y^2 - D(z^2 + w^2) = -1$ has a solution, which occurs precisely when $4|D$ (e.g. [4] Chap. 9). Thus

$$I_1 = \begin{cases} 4 & \text{if } 4|D \\ 8 & \text{if } 4 \not| D. \end{cases}$$

It now follows from (12) and (13) that

$$\rho(S, C) = \frac{\pi^2}{2^9} \prod_{\substack{p|D \\ p \text{ odd}}} \left(1 + \left(\frac{-1}{p}\right)p^{-1}\right) \tag{14}$$

6.2.

PROPOSITION 1. *If* $D \equiv 0, 3 \pmod 4$ *then* $n_D = 1$.

PROOF. When $D \equiv 0, 3 \pmod 4$, $\alpha_2(S, D) = \frac{3}{2}$ and so from (14) and (11) there is just one $O(f, \mathbb{Z})$ equivalence class of primitive solutions of $f(C) = -2D$. But the coset representatives g_1, g_2, g_3 above map C to C or $-C$. Thus modulo the $\Phi(\Gamma_1)$-equivalence as defined in 5.1, there is again only one equivalence class and so from Theorem 4 just one conjugacy class of maximal Fuchsian subgroup of discriminant D. ∎

6.3. Since the values of $\alpha_2(S,D)$ in the other cases exceed $\frac{3}{2}$, it also follows from Siegel's Theorem, that there is more than one $O(f,\mathbb{Z})$-equivalence class, and hence more than one $\Phi(\Gamma_1)$-equivalence class. However, this can be shown more directly using the following Parity Lemma which we prove following [13].

LEMMA 6. *Let C, $C' \in \Sigma_1$ be represented by the primitive triples (a,B,c), (a',B',c') with $B = b_1 + b_2 i, B' = b'_1 + b'_2 i$. Let C, C' be Γ_1-equivalent.*
Then

(i) if at least one of a, c is odd, then at least one of a', c' is odd.
(ii) if both a, c are even, then both a', c' are even and $b_i \equiv b'_i \pmod 2$ $i = 1, 2$.

PROOF. The group Γ_1 is generated by the images of the following matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad u = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \quad \ell = \begin{pmatrix} i & o \\ 0 & -i \end{pmatrix}$$

and has the presentation (c.f. [7])

$$(15) \qquad \langle s,t,\ell,u \mid s^2 = \ell^2 = (s\ell)^2 = (t\ell)^2 = (u\ell)^2 = (st)^3 = (us\ell)^3 = 1,\ tu = ut \rangle$$

The effect of s, t, l, u on C can then be given:

$$s\colon (a,B,c) \longmapsto (c, -b_1 + b_2 i, a)$$
$$t\colon (a,B,c) \longmapsto (a, b_1 - a + b_2 i, c + a - 2b_1)$$
$$u\colon (a,B,c) \longmapsto (a, b_1 + (b_2 + a)i, c + a + 2b_2)$$
$$l\colon (a,B,c) \longmapsto (a, -b_1 - b_2 i, c).$$

The Parity lemma follows by inspection. ∎

6.4. We now consider the case $D \equiv 1 \bmod 4$.

By the Parity lemma, the circle C_D and the two circles

$$C_{D,1}\colon \quad 2|z|^2 + z + \bar{z} - \left(\frac{D-1}{2}\right) = 0$$
$$C_{D,2}\colon \quad 2|z|^2 + iz - i\bar{z} - \left(\frac{D-1}{2}\right) = 0$$

are not pairwise equivalent under Γ_1. There are thus at least three Γ_1-conjugacy classes in this case. Note however that $C_{D,1}$ is equivalent to $C_{D,2}$ under $P\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}_2(O_1)$.

As described in § 3.1, the element $T = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ maps $C_{D,1}$ to C_D and from the simple description of $\mathrm{Stab}(C_D, \mathrm{PSL}_2(\mathbb{C}))$ we obtain a simple description of $\mathrm{Stab}(C_{D,1}, \mathrm{PSL}_2(\mathbb{C}))$ and hence of $\mathrm{Stab}(C_{D,1}, \mathrm{PSL}_2(O_1))$.

On the other hand, conjugating the representation in ρ in § 6.1 by T gives a representation ρ' of B_D given by

$$\rho'(i) = \begin{pmatrix} \sqrt{-1} & \sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix} \quad \rho'(j) = \begin{pmatrix} -1 & \frac{1}{2}(D-1) \\ 2 & 1 \end{pmatrix}$$

Then taking $\mathcal{M}$ to be the order $\mathcal{M} = \mathbb{Z}\left[1, i, \frac{1+j}{2}, \frac{i+ij}{2}\right]$ in B_D

$$\mathrm{Stab}(C_{D,1}, \Gamma_1) = P\rho'(\mathcal{M}^1)$$

follows almost immediately. Now the discriminant of $O = \mathbb{Z}[1, i, j, ij]$ and $\mathcal{M}$ differs only at the prime 2 so that

$$[\mathcal{M}^1 : O^1] = [\mathcal{M}_2^1 : O_2^1]$$

where $\mathcal{M}_2$, O_2 are the localizations at 2.

LEMMA 7. $[\mathcal{M}_2^1 : O_2^1] = 3$.

PROOF. Since $D \equiv 1 \pmod 4$, B_D is unramified at 2, and $\mathcal{M}$, having discriminant $D\mathbb{Z}$ is maximal at the prime 2. Let

$$\sigma(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma(j) = \begin{pmatrix} \zeta & \eta \\ \eta & -\zeta \end{pmatrix}$$

where $\zeta^2 + \eta^2 = D$ in the 2-adic integers $\mathbb{Z}_2$, with ζ chosen so that $\zeta \equiv 1 \pmod 2$. This induces an isomorphism between the localization of B_D at 2 with $M_2(\mathbb{Q}_2)$ under which $\mathcal{M}_2$ is mapped isomorphically on $M_2(\mathbb{Z}_2)$. Furthermore we obtain the image of O_2 to be

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathbb{Z}_2) \middle| \alpha \equiv \delta \pmod 2, \quad \beta \equiv \gamma \pmod 2 \right\}.$$

By reducing to the residue class field of two elements we see that $[\mathcal{M}_2^1 : O_2^1] = 3$. ∎

COROLLARY 1. $\mathrm{vol}(\mathrm{Stab}(C_{D,i}, \Gamma_1) = \frac{\pi}{3} D \prod_{p|D} \left(1 + \left(\frac{-1}{p}\right) p^{-1}\right), i = 1, 2$.

PROOF. From (12) we obtain the volume of $P\rho(O^1)$, from which the results follow by Lemma 7 and the preceding remarks.

Now $C_2 = {}^t[1\ 0 - \frac{1}{2}(D-1) - 2]$ is the solution of $f(x) = -2D$ corresponding to the circle $C_{D,2}$ (Lemma 3) and we now compute the contribution $\rho(S, C_2)$ to Siegel's formula.

The indefinite ternary form h_2 in this case is given by the matrix

$$H_2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2/D & -\left(\frac{D+1}{2D}\right) \\ 0 & -\left(\frac{D+1}{2D}\right) & \frac{(D-1)^2}{8D} \end{bmatrix}$$

so that

$$\rho(S, C_2) = \lambda_2 \,\mathrm{vol}(O(h_2, \mathbb{Z}))\frac{\pi}{2}(2D^3)(16D^2)^{-2}$$

So from Lemma 5 we require to compute

$$[\mathrm{Stab}(C_2, O(f, \mathbb{Z})) : \mathrm{Stab}(C_2, \Phi(\Gamma_1))] = I_2.$$

As in §6.1, one can show that there are coset representatives of $SO(f,\mathbb{Z})$ in $O(f,\mathbb{Z})$ and $SO^o(f,\mathbb{Z})$ in $SO(f,\mathbb{Z})$ which fix C_2. Since $\Phi P\begin{pmatrix} i & 0 \\ 0 & 1\end{pmatrix} \in SO^o(f,\mathbb{Z})\backslash \Phi(\Gamma_1)$ and $P\begin{pmatrix} i & 0 \\ 0 & 1\end{pmatrix}$ maps $\mathcal{C}_{D,1} to \mathcal{C}_{D,2}$ it follows from the parity conditions that $I_2 = 4$. From this and the corollary above, we calculate $\rho(S, C_2)$, and with $C = {}^t[0\,0\,D\,1]$ as before we obtain

$$\rho(S,C_2) + \rho(S,C) = \mu(S)\prod_p \alpha_p(S,D)$$

as computed at (11) by Siegel's Theorem. Thus we obtain two $O(f,\mathbb{Z})$-equivalence classes of primitive solutions of $f(x) = -2D$. Analysing the effects of the coset representatives g_1, g_2, g_3 as before, it follows that there are 3 conjugacy classes of maximal Fuchsian subgroups of discriminant $D \equiv 1 \pmod 4$.

6.5. It remains to consider the case $D \equiv 2 \pmod 4$. The techniques are as above and we merely state the salient facts.

For $D \equiv 2 \pmod 4$ define the circle

$$\mathcal{C}_{D,3}\colon \quad 2|z|^2 + (1+i)z + (1-i)\bar{z} - \left(\frac{D-2}{2}\right) = 0$$

From the Parity Lemma $\mathcal{C}_{D,3}$ is not equivalent to $\mathcal{C}_D$. By conjugating by the relevant element we obtain a representation of B_D given by

$$p''(i) = \begin{pmatrix} \sqrt{-1} & 1+\sqrt{-1} \\ 0 & -\sqrt{-1}\end{pmatrix} \quad p''(j) = \begin{pmatrix} -1+\sqrt{-1} & \frac{D}{2}+\sqrt{-1} \\ 2 & 1-\sqrt{-1}\end{pmatrix}$$

Then if $\mathcal{N}$ is the order of B_D defined by $\mathcal{N} = \mathbb{Z}[1, i, \frac{1+i+j}{2}, \frac{1-i+ij}{2}]$ it follows as in §6.4 that

$$P\rho''(\mathcal{N}^1) = \mathrm{Stab}(\mathcal{C}_{D,3}, \Gamma_1)$$

With, as before $O = \mathbb{Z}[1, i, j, ij]$, we obtain

$$[\mathcal{N}^1 : O^1] = \begin{cases} 6 & \text{if } D \equiv 6 \pmod 8 \\ 2 & \text{if } D \equiv 2 \pmod 8\end{cases}$$

Now $C_3 = {}^t[1\,1 - \left(\frac{D-2}{2}\right) - 2]$ and $[\mathrm{Stab}(C_3, O(f,\mathbb{Z})) : \mathrm{Stab}(C_3 : \Phi(\Gamma_1))] = 8$. Hence $\rho(S, C_3) = \varepsilon 2^{-7}\pi^2 \prod_{\substack{p|D \\ p\text{ odd}}} \left(1 + \left(\frac{-1}{p}\right)p^{-1}\right)$ where $\varepsilon = \frac{1}{2}$ if $D \equiv 2 \pmod 8$ and $\varepsilon = \frac{1}{6}$ if $D \equiv 6 \pmod 8$. Thus from (11) there are two $O(f,\mathbb{Z})$ equivalence classes of solutions and analyzing the coset representatives gives two $\Phi(\Gamma_1)$-equivalence classes.

This concludes the proof of Theorem 5. The results of §6.1–5 yield the following classification of maximal Fuchsian subgroups of Γ_1.

COROLLARY 2. *Every maximal Fuchsian subgroup of* Γ_1 *of discriminant D is conjugate in* Γ_1 *to one of* $\mathrm{Stab}(\mathcal{C}_D, \Gamma_1)$ *or* $\mathrm{Stab}(\mathcal{C}_{D,i}, \Gamma_1)$ $i = 1, 2, 3$.

6.6. The fact that Γ_1 is a free product amalgamated over the subgroup $M = \mathrm{PSL}_2(\mathbb{Z})$ enabled Fine in [9] to investigate the relationship of Fuchsian subgroups of Γ_1 to M. In particular he proved the following two results.

THEOREM F1. *Let F be a torsion-free Fuchsian subgroup of* Γ_1. *Then F is free unless*
(a) F has a most cyclic intersection with all conjugates of M in Γ_1 *and*
(b) F has non-trivial intersection with at least one conjugate of M.

THEOREM F2. *Let F be a finitely-generated Fuchsian subgroup of* Γ_1.
(a) If F has trivial intersection with all conjugates of M then either F is finite or a free product of cyclics.
(b) If F has non-cyclic intersection with some conjugates of M, then F is a non-trivial free product of cyclics.

In these statements F is a geometric Fuchsian group in the sense of this paper, but is not necessarily non-elementary or of finite covolume. However, as we shall see, we can quickly reduce to that case. Using Corollary 2 we will expand on and clarify these results, showing in particular that the conditions of Theorem F1 fall short of giving a classification of free subgroups as both free and non-free groups satisfy (a) and (b). We also show that case (a) of Theorem F2 can arise for finite covolume groups.

Let F be a Fuchsian subgroup of Γ_1. If F is elementary, it is either cyclic generated by an elliptic, parabolic or hyperbolic element or is infinite dihedral generated by an elliptic element of order 2 and a hyperbolic element. If additionally it is a subgroup of M, it must be cyclic. Thus the terms "cyclic intersection" and "non-cyclic intersection" can be replaced by "elementary intersection" and "non-elementary intersection" in the theorems above.

The theorems are trivial if F is elementary so assume that F is non-elementary. It then has an invariant circle $\mathcal{C} \in \Sigma_1$ and is a subgroup of the maximal Fuchsian group $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ which has finite covolume. If F is of infinite index in $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ then it is a free product of cyclics [15]. We therefore assume, as before, that F has finite covolume.

LEMMA 8. *Let* $\mathcal{C} \in \Sigma_1$ *have discriminant D. Then* $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ *is non-cocompact and so a free product of cyclics if and only if D is not divisible by an odd power of a prime* $\equiv 3 \pmod 4$.

PROOF. Recall that $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ is an arithmetic Fuchsian group whose associated quaternion algebra is isomorphic to $\left(\frac{-1,D}{\mathbb{Q}}\right)$, so that $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ is non-cocompact if and only if $\left(\frac{-1,D}{\mathbb{Q}}\right) \cong M_2(\mathbb{Q})$. But $\left(\frac{-1,D}{\mathbb{Q}}\right)$ has no finite ramification if and only if $D = n^2 D_0$ where D_0 is square free and for every prime p, $p | D_0$ then $p = 2$ or $p \equiv 1 \pmod 4$. When $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ is non-cocompact it is conjugate to a subgroup commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ and so a free product of cyclics. ■

DEFINITION 6.1. Two Fuchsian subgroups F, F' of Γ_1 are *commensurable in the wide sense* if F and some conjugate of F' in Γ_1 are commensurable.

LEMMA 9. *Let* F_0, F *be Fuchsian subgroups of* Γ_1. *Then F has non-elementary intersection with a conjugate of* F_0 *if and only if F is in the wide commensurability class of* F_0.

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PROOF. If F and $\gamma F_0 \gamma^{-1}$ have non-elementary intersection then their invariant circles coincide and the groups are commensurable. ∎

In the case we are interested in $F_0 = \mathrm{PSL}_2(\mathbb{Z})$, and parts (a) of Theorem F1 and (b) of Theorem F2 follow.

Using the notation adopted in the previous sections, we let $F_D = \mathrm{Stab}(\mathcal{C}_D, \Gamma_1)$ and $F_{D,i} = \mathrm{Stab}(\mathcal{C}_{D,i}, \Gamma_1)$ $i = 1, 2, 3$. The rest of this section is devoted to proving the following result:

THEOREM 6. *Let F be a (finite covolume) Fuchsian subgroup of Γ_1. Then either:*

(i) a conjugate of F is commensurable with M or

(ii) every conugate of F has trivial intersection with M or

(iii) every conjugate of F has at most finite cyclic intersection with M or

(iv) some conjugate of F has infinite cyclic intersection with M.

Case (i) occurs if F belongs to the wide commensurability class of $F_{1,2}$. Case (ii) occurs if F belongs to the wide commensurability class of $F_{2,3}$ or $F_{10,3}$. Case (iii) occurs if F belongs to the wide commensurability class of $F_{1,1}$ or $F_{5,2}$ and case (iv) occurs in all other situations.

PROOF. Note that $M = \gamma F_{1,2} \gamma^{-1}$ where $\gamma(z) = \frac{iz}{z-i}$ so that (i) follows. Now for (iv); note that every group in the commensurability class of some maximal group F_0 will have non-trivial intersection with a conjugate of M if and only if a conjugate of F_0 intersects M in a hyperbolic or parabolic subgroup. In that case, the intersection of the circle corresponding to the conjugate of F_0 and the real axis will be the fixed points (resp. fixed point) of a hyperbolic (resp. parabolic) element of M. Conversely if the circle meets the real axis in a pair of points (resp. a single point) which are (resp. is) the fixed points (resp. point) of a hyperbolic (resp. parabolic) element of M, then that circle is also invariant under the element and we have infinite cyclic intersection.

If $\mathcal{C}$, given by a primitive triple (a, B, c) with discriminant D, meets the real axis, then it does so in the points $\frac{-b_1 \pm \sqrt{D - b_2^2}}{a}$ or $\left\{\infty, \frac{-c\sqrt{D-b_2^2}}{2b_1^2}\right\}$ or the single point $\frac{-b_1}{a}$ or ∞.

Recall that Pell's equation $x_0^2 - Dy_0^2 = 1$ has a solution for which $y_0 \neq 0$ when D is not a perfect square. We use this in each of the cases described by Corollary 2 to construct hyperbolic or parabolic elements of M with the required intersection with the real axis.

(A) F_D, D not a perfect square. The hyperbolic element $P\begin{pmatrix} x_0 & Dy_0 \\ y_0 & x_0 \end{pmatrix} \in M$ has fixed points $\{\pm\sqrt{D}\} = \mathcal{C}_D \cap \mathbb{R}$.

(B) $F_D, D = D_1^2$. The parabolic element $P\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in M$ has fixed point $\{0\} = \gamma\,\mathcal{C}_D \cap \mathbb{R}$ where $\gamma(z) = z + D_1 i$.

(C) $F_{D,1}, D$ not a perfect square. The hyperbolic element $P\begin{pmatrix} x_0 - y_0 & \frac{D-1}{2} y_0 \\ 2y_0 & x_0 + y_0 \end{pmatrix} \in M$ has fixed points $\left\{\frac{1}{2}(-1 \pm \sqrt{D})\right\} = \mathcal{C}_{D,1} \cap \mathbb{R}$.

(D) $F_{D,1}, D-4$ not a perfect square and positive. The element $P\begin{pmatrix} x_0-y_0 & \frac{D-5}{2}y_0 \\ 2y_0 & x_0+y_0 \end{pmatrix} \in M$ has fixed points $\left\{\frac{1}{2}(-1 \pm \sqrt{D-4})\right\} = \gamma\, C_{D,1} \cap \mathbb{R}$ where $\gamma(z) = z+i$.

(E) $F_{D,2}, D-1$ not a perfect square. The element $P\begin{pmatrix} x_0 & \frac{D-1}{2}y_0 \\ 2y_0 & x_0 \end{pmatrix} \in M$ has fixed points $\left\{\pm\frac{\sqrt{D-1}}{2}\right\} = C_{D,2} \cap \mathbb{R}$.

(F) $F_{D,2}, D-9$ not a perfect square and positive. Then $P\begin{pmatrix} x_0 & \frac{D-9}{2}y_0 \\ 2y_0 & x_0 \end{pmatrix} \in M$ has fixed points $\left\{\pm\frac{\sqrt{D-9}}{2}\right\} = \gamma\, C_{D,2} \cap \mathbb{R}$ where $\gamma(z) = z+i$.

(G) $F_{D,3}, D-1$ not a perfect square. The element $P\begin{pmatrix} x_0-y_0 & \frac{D-2}{2}y_0 \\ 2y_0 & x_0-y_0 \end{pmatrix} \in M$ has fixed points $\left\{\frac{1}{2}(-1 \pm \sqrt{D-1})\right\} = C_{D,3} \cap \mathbb{R}$.

(H) $F_{D,3}, D-9$ not a perfect square and positive. Then $P\begin{pmatrix} x_0-y_0 & \frac{D-10}{2}y_0 \\ 2y_0 & x_0-y_0 \end{pmatrix} \in M$ has fixed points $\left\{\frac{1}{2}(-1 \pm \sqrt{D-9})\right\} = \gamma\, C_{D,3} \cap \mathbb{R}$ with $\gamma(z) = z+i$.

To complete the proof of Theorem 6 we need to consider the cases not covered by any of (A)–(H) i.e. $F_{1,1}, F_{5,2}, F_{2,3}, F_{10,3}$. We first show that no conjugate of these has hyperbolic or parabolic intersection with M. The argument is the same in each case so we give it only for the first group.

From the Parity lemma any image of $C_{1,1}$ is represented by a primitive triple $(a, b_1 + b_2 i, c)$ with a, c even, $b_1 \equiv 1 \pmod 2$ and $b_2 \equiv 0 \pmod 2$, and $b_1^2 + b_2^2 - ac = 1$. If $a \neq 0$, this meets the real axis in the points $\frac{-b_1 \pm \sqrt{1-b_2^2}}{a}$ whose only solution occurs for $b_2 = 0$. But no hyperbolic element of M fixes the pair $\frac{-b_1 \pm 1}{a}$. When $a = 0$, the points of intersection are $\frac{c}{2}, \infty$ and the same remark applies.

This completes (iv) of the Theorem and so there only remains the possibility of finite cyclic intersection between a conjugate of the group and M. If that were so then the invariant circle of the conjugate would necessarily be an invariant circle of either $\omega_1 = P\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $\omega_2 = P\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. The element ω_1 has invariant circles with primitive triples (a, bi, a) and discriminant $b^2 - a^2$ while ω_2 has invariant circles with primitive triples $(2b_i, b_1 + b_2 i, 2b_1)$ and discriminant $b_2^2 - 3b_1^2$. From this it follows that the conjugate of $F_{1,1}$ with circle given by $(2, 1+2i, 2)$ intersects M in a subgroup of order 3, while the conjugate of $F_{5,2}$ with circle $(2, 3i, 2)$ intersects M in a subgroup of order 2. Thus (ii) follows immediately and also (iii) since the parity conditions show that all conjugates of $F_{2,3}$ and $F_{10,3}$ intersect M trivially. ∎

REMARKS. 1. Note that we obtain free groups which satisfy (a) and (b) of Theorem F1 by taking torsion-free subgroups of finite index in any of the maximal groups with discriminant $D \neq 1, 2, 5, 10$ and D not divisible by a prime $\equiv 3 \pmod 4$.

2. Groups in the wide commensurability classes of $F_{2,3}$ and $F_{10,3}$ and torsion-free groups in the wide commensurability classes of $F_{1,1}$ and $F_{5,2}$ give the only finite covolume examples of groups satisfying (a) of Theorem F2.

7. Signatures of maximal Fuchsian subgroups of Γ_1.

7.1. From Corollary 2 and the observation in § 6.4 that $F_{D,1}$ and $F_{D,2}$ are conjugate by an element of $\mathrm{PGL}_2(O_1)$, the signatures of the groups F_D for all $D, F_{D,1}$ for $D \equiv 1 \pmod 4$, and $F_{D,3}$ for $D \equiv 2 \pmod 4$ give the signatures of all maximal Fuchsian subgroups of Γ_1.

From § 6 we know the covolumes of these groups, so that to determine the signature we need to know in particular if the group is cocompact and its periods. Cocompactness depends only on the discriminant and necessary and sufficient criteria are given in Lemma 8. Since the traces of elements in any $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ are rational, elliptic elements will have order 2 or 3.

LEMMA 10. *Every maximal Fuchsian subgroup of Γ_1 contains elliptic elements.*

PROOF. Clearly $P\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in F_D$ for every D and $P\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ has order 2. It follows from the results in § 6.4 and § 6.5 that F_D is conjugate in $\mathrm{PSL}_2(\mathbb{C})$ to a subgroup of $F_{D,1}$ or $F_{D,3}$ when these are defined. ∎

Note that in [8] it is shown that any normal subgroup of Γ_1 containing elliptic elements has finite index in Γ_1. Thus

COROLLARY 3. *If $\mathcal{C} \in \Sigma_1$ then the normal closure of* $\mathrm{Stab}(\mathcal{C}, \Gamma_1)$ *in Γ_1 is an arithmetic Kleinian group.*

REMARK. Although elliptic elements of $\mathrm{Stab}(\mathcal{C}, \Gamma_d)$ for any d will also necessarily have order 2 or 3, the corresponding result in Lemma 10 is not true in this generality. For example, if $d \not\equiv -1 \pmod 4$ $d \neq 1$ then the group $\mathrm{Stab}(\mathcal{C}_p, \Gamma_d)$ where p is a prime such that $\left(\frac{d}{p}\right) = -1$ is torsion-free.

7.2. We will not compute the signatures of all maximal Fuchsian subgroups of Γ_1 as many of these are conjugates in $\mathrm{PSL}_2(\mathbb{C})$ to subgroups of finite index in others as we will show in this section. The signatures of the finite index subgroups may then be deduced by methods stemming from the structure theorem for Fuchsian groups [26].

Recall that $F_D = P\rho(O^1)$ where O is the order $\mathbb{Z}[1, i, j, ij]$ in B_D. Thus

$$F_D = P\left\{\begin{pmatrix} \alpha & D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \middle| \alpha, \beta \in O_1\right\}$$

From § 6.4 and § 6.5, the groups $F_{D,1}$ and $F_{D,3}$ can be conjugated in $\mathrm{PSL}_2(\mathbb{C})$ to give supergroups of F_D which we denote by G_D, H_D respectively. Thus $G_D = P\rho(\mathcal{M}^1)$ and $H_D = P\rho(\mathcal{N}^1)$ where $\mathcal{M}, \mathcal{N}$ are the orders defined in § 6.4 and § 6.5.

$$G_D = P\left\{\begin{pmatrix} \frac{\alpha}{2} & \frac{D\beta}{2} \\ \frac{\bar{\beta}}{2} & \frac{\bar{\alpha}}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \mid \alpha, \beta \in O_1, \alpha \equiv \beta \pmod 2\right\}$$

$$H_D = P\left\{\begin{pmatrix} \frac{\alpha}{2} & \frac{D\beta}{2} \\ \frac{\bar{\beta}}{2} & \frac{\bar{\alpha}}{2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}) \mid \alpha, \beta \in O_1, \alpha \equiv (1+i)\beta \pmod 2\right\}$$

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Of course G_D, H_D are only defined in the cases $D \equiv 1 \pmod 4$ and $D \equiv 2 \pmod 4$ respectively, but in these cases F_D is a subgroup fo G_D or H_D. We can reduce still further.

Let $D = n^2 D_0 D_1$ where $D_0 D_1$ is square-free and if p is a prime such that $p|D_0$ (resp. $p|D_1$) then $p = 2$ or $p \equiv 1 \pmod 4$ (resp. $p \equiv 3 \pmod 4$).

THEOREM 7. *The group F_D (resp. G_D) is conjugate in* $\mathrm{PSL}_2(\mathbb{C})$ *to a subgroup of finite index in F_{D_1} (resp. G_{D_1}), while for $D \equiv 6 \pmod 8$, H_D is conjugate to a subgroup of finite index in H_{2D_1}. Finally if $D \equiv 3 \pmod 4$, F_D is conjugate to a subgroup of finite index in H_{2D_1}.*

PROOF. By the definition of D_0 we can find integers a, b such that $a^2 + b^2 = D_0$ where if D_0 is odd, a is chosen to be odd and b even. Then

$$\begin{pmatrix} \frac{1}{n(a+bi)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & D\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} n(a+bi) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & D_1[n(a-bi)\beta \\ n(a+bi)\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

which proves the first part in the case of F_D. A similar argument works for G_D and in the case of H_D, using $\frac{D_0}{2}$, yields a subgroup of H_{2D_1}. In the case where $D \equiv 2 \pmod 8$, then $D_1 \equiv 1 \pmod 4$ and a further conjugation of H_{2D_1} by $\begin{pmatrix} (1+i)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ gives a subgroup of G_{D_1}. Likewise when $D \equiv 3 \pmod 4$, then $D_1 \equiv 3 \pmod 4$ and a further conjugation of F_{D_1} by $\begin{pmatrix} 1+i & 0 \\ 0 & 1 \end{pmatrix}$ gives a subgroup of H_{2D_1}.

The outcome is that we restrict ourselves to computing the signatures of the following collection of groups as all others are conjugate to subgroups of these.

(A) $G_D = P\rho(\mathcal{M}^1)$ where $D = p_1 p_2 \dots p_{2r} \quad p_i \equiv 3 \pmod 4$.

(B) $H_D = P\rho(\mathcal{N}^1)$ where $D = 2p_2 \dots p_{2r} \quad p_i \equiv 3 \pmod 4$.

Note that this collection cannot be reduced any further as the wide commensurability classes of these groups are in one-to-one correspondence with the isomorphism classes of the corresponding quaterion algebras [28] and distinct values of D as described at (A) and (B) give non-isomorphic quaternion algebras B_D. In particular, only the value $D = 1$ gives rise to a non-cocompact group.

7.3. The number of conjugacy classes of cyclic subgroups of order 2 and 3 in the arithmetic groups described at (A) and (B) above can be measured in terms of embeddings of the maximal order O_1, O_3 of $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ in B_D. Details of these measurements can be found in [30] pp. 92–98 (cf. [5] where an error in [30], not affecting computations here, is corrected.). Let m_2, m_3 denote the number of periods of orders 2 and 3 respectively. Note that for $p \equiv 3 \pmod 4$ we have $\left(\frac{O_1}{p}\right) = \left(\frac{-1}{p}\right) = -1$ for the computation of elements of order 2 and $\left(\frac{O_3}{p}\right) = \left(\frac{-3}{p}\right)$ for the computation of order 3 periods (see [30] p. 94).

(A) $D \equiv 1 \pmod 4 \quad D = p_1 p_2 \dots p_{2r} \quad D > 1$

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In this case B_D is ramified at these $2r$ primes and since $\mathcal{M}$ has discriminant $D\mathbb{Z}$ it is maximal. Thus immediately $m_2 = 2^{2r}$ and

$$m_3 = \begin{cases} 0 & \text{if some } p_i \equiv 1 \pmod 3 \\ 2^{2r} & \text{if all } p_i \equiv -1 \pmod 3 \\ 2^{2r-1} & \text{if } p_1 = 3 \text{ and all other } p_i \equiv -1 \pmod 3 \end{cases}$$

Of course, when $D = 1, P\rho(\mathcal{M}^1) \cong \mathrm{PSL}_2(\mathbb{Z})$ with signature $(0; 2, 3; 1)$

(B) $D \equiv 6 \pmod 8$ $\quad D = 2p_2 \dots p_{2r}$.

Again $\mathcal{N}$ is a maximal order in B_D and $m_2 = 2^{2r-1}$ while m_3 is as above in case (A).

From the covolumes obtained for these groups in (6), the complete signature can now be deduced. To express our result we introduce some notation.

Let $W = \{n \in \mathbb{Z} \mid n > 1, n = 2^\varepsilon 3^\eta p_1 p_2 \dots p_k$ where the p_i are distinct primes with $p_i \equiv 3 \pmod 4), \varepsilon, \eta \in \{0, 1\},\ \varepsilon + \eta + k \equiv 0 \pmod 2\}$

For $n \in W$, define $m_2(n) = 2^{\eta+k}$

$$m_3(n) = \begin{cases} 0 & \text{if some } p_i \equiv 1 \pmod 3 \\ 2^{\epsilon+k} & \text{otherwise} \end{cases}$$

$$g(n) = \left(\frac{2^\eta}{12}\prod_{i=1}^{k}(p_i - 1)\right) - \frac{m_2(n)}{4} - \frac{m_3(n)}{3} + 1$$

THEOREM 8. *Every (finite covolume) Fuchsian subgroup of Γ_1 is a subgroup of finite index in a group with signature $(0; 2, 3; 1)$ if it is non-cocompact or $(g(n); 2^{(m_2(n))}, 3^{m_3(n))}; 0)$ for some $n \in W$ if it is cocompact. (Where in the signature $2^{(N)}$ or $3^{(N)}$ means that there are N periods of order 2 or 3.)*

8. Totally geodesic surfaces immersed in arithmetic link complements obtained from torsion-free subgroups of $\mathrm{PSL}_2(O_1)$.

Let Γ be a torsion-free subgroup of finite index in Γ_d so that H^3/Γ is a complete orientable hyperbolic 3-manifold of finite volume with a finite number of toral ends. If F is a Fuchsian subgroup of Γ_d stabilizing $C \in \Sigma_d$, then we obtain a totallly geodesic immersion of the surface $H(C)/(F \cap \Gamma)$ in H^3/Γ. There is a plentiful supply of such immersions since there are infinitely many commensurability classes of cocompact Fuchsian subgroups in Γ_d and for one particular example we address the problem of determining the minimal genus of a closed immersed totally geodesic surface in such manifolds.

The example we consider is the complement of the Borromean rings B in S^3. It is well-known that $S^3 \setminus B$ admits a unique hyperbolic structure as H^3/Γ where Γ is a subgroup of index 24 in Γ_1 ([29] Chap. 6, 7 and [31]).

THEOREM 9. *The minimal genus of a closed immersed totally geodesic surface in $S^3 \setminus B$ is three. This surface corresponds to a subgroup of index 12 in the stabilizer of the circle $C_{6,3}$, i.e.* $2|z|^2 + (1+i)z + (1-i)\bar{z} - 2 = 0$.

PROOF. Let $F = F_{6,3}$ so that F has signature $(0; 2, 2, 3, 3; 0)$ by Theorem 8. We again make use of the presentation of Γ_1 given in [7] which we now recall. The group Γ_1 is generated by s, ℓ, t, u where

$$s(z) = \frac{-1}{z} \quad t(z) = z+1 \quad u(z) = z+i \quad \ell(z) = -z$$

and has defining relations:

$$s^2 = \ell^2 = (s\ell)^2 = (t\ell)^2 = (u\ell)^2 = (st)^3 = (us\ell)^3 = [t,u] = 1$$

The fundamental group of the Borromean rings complement is isomorphic to Γ, the normal closure of $\{t^4, tu^{-1}\}$ in Γ_1 which has index 24 in Γ_1 [11]. The quotient is isomorphic to S_4, the symmetric group on 4 letters, with Γ being the kernel of the homomorphism π defined by $\pi(s) = (12)$, $\pi(t) = \pi(u) = (1\,2\,3\,4)$ and $\pi(\ell) = (12)(34)$.

Now $F = P\rho''(\mathcal{N}^1)$ where $\mathcal{N} = \mathbb{Z}[1, i, \frac{1+i+j}{2}, \frac{1-i+ij}{2}]$ is maximal order in B_6. Note that $i \in \mathcal{N}^1$ and from the definition of ρ'', $P\rho''(i) = \ell tu^{-1}$. Thus the image of one of the generators of order 2 in F under π is a conjugate of $(12)(34)$ in S_4. Since the kernel of π is torsion-free, the image of the elements of order 3 in F under π must be non-trivial even permutations and hence $\pi(F) = A_4$. Thus from the volume formula $F \cap \Gamma$ has genus 3.

To complete the proof of Theorem 9 we need to prove that there is no closed totally geodesic surface of genus 2 immersed in $S^3 \setminus B$. Suppose that such a surface exists and let F' be the corresponding surface group so that $F' = \mathrm{Stab}(\mathcal{C}, \Gamma_1) \cap \Gamma$ for some $\mathcal{C} \in \Sigma_1$. Since F' is torsion-free $[\mathrm{Stab}(\mathcal{C}, \Gamma_1) : F'] \geq 2$, by Lemma 10. Thus by Theorem 7 we have $\mathrm{vol}(G_D) \leq 2\pi$ or $\mathrm{vol}\, H_D \leq 2\pi$ for D as defined at (A) and (B) in (7.2). But from the colvolumes of these groups there are only two possibilities, namely $D = 6$ or 14.

From Theorem 8, $H_{14} \cong F_{14,3}$ has signature $(1; 2, 2; 0)$ and $F_{14,3} = P\rho''(\mathcal{N}^1)$ where $\mathcal{N}$ is defined as above but in B_{14}. Now $\frac{1}{2}(3 + 3i + j) \in \mathcal{N}^1$ and $P\rho''(\frac{1}{2}(3 + 3i + j)) = u^2tst^2u^{-2}$. But $\pi(u^2tst^2u^{-2}) = (143)$ so $[F_{14,3} : F_{14,3} \cap \Gamma] \geq 3$ and $F_{14,3} \cap \Gamma$ then cannot have genus 2, and since Γ is normal in Γ_1, the same will hold for a conjugate of $F_{14,3}$. The case of $D = 6$ was dealt with above and the theorem follows. ∎

REMARKS. 1. By results of Lozano [17], $S^3 \setminus B$ does not contain a closed embedded incompressible surface, so in particular none of the closed totally geodesic surfaces in $S^3 \setminus B$ are embedded. However results of Scott [24] imply that every totally geodesic surface in $S^3 \setminus B$ will embed in a finite cover.

2. In comparison with these arithmetic examples, the second author [22] exhibits (necessarily non-arithmetic) knot complements in S^3 with hyperbolic structures which contain no closed totally geodesic surfaces but do contain one commensurability class of non-compact totally geodesic surfaces.

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