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LOCAL UNIQUENESS OF SOLUTIONS OF THE CHARACTERISTIC CAUCHY PROBLEM

GERSON PETRONILHO

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Abstract

Local uniqueness of solutions of the characteristic Cauchy problem is shown for operators which are perturbations of operators which already have such a uniqueness.

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1. Introduction

This work is related to discrete phenomena in the local uniqueness of solutions of the characteristic Cauchy problem for operators with double characteristics at a point of the initial curve.

On this type of phenomena F. Treves [5] furnished an example, that is, the Cauchy problem for the equation

$$(\partial_t + t\partial_x)(\partial_t - t\partial_x)u - c\partial_x u = 0$$

with data given in $x = 0$, and proved that for the problem in the upper half-plane $x > 0$ there is uniqueness of solutions, if and only if $c \neq 0, 2, 4, \dots$.

A. P. Bergamasco and H. S. Ribeiro [1] and A. Menikoff [3], extended this study for operators of the form

$$(\partial_t + at^k\partial_x)(\partial_t - bt^k\partial_x) - ct^{k-1}\partial_x$$

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where k is an odd natural number. In [1], $a = b < 0$ and $c \in \mathbb{R}$, while in [3], $a, b > 0$ and $c \in \mathbb{C}$.

We extend for operators of the form

$$P(c_1, c_2) = (\partial_t + a_1 t^k \partial_x + a_2 t^k)(\partial_t - b_1 t^k \partial_x - b_2 t^k) - c_1 t^{k-1} \partial_x - c_2 t^{k-1}$$

where k is an odd natural number, $c_1, c_2 \in \mathbb{C}$, $a_1, b_1 > 0$, and $a_2, b_2 \leq 0$.

In Section 2 we prove, under suitable assumptions, the local uniqueness, in the class C^3, of solutions of the characteristic Cauchy problem for the operators $P(c_1, c_2)$. The essential point in the proof is getting an appropriate Carleman inequality.

In Section 3, by using the results of Section 2 and a variation of concatenations in O. R. B. Oliveira [4], we prove that there is an integer m depending only on $\mathrm{Re}(c_1)$ such that the local uniqueness of solutions in the class C^m holds in the characteristic Cauchy problem for the operators $P(c_1, c_2)$ if $\mathrm{Re}(c_1) \neq j(k+1)\delta_1$, $\mathrm{Re}(c_1) \neq j(k+1)\delta_1 + \delta_1$, $j = 0, 1, 2, \ldots$ and $\delta_1 = a_1 + b_1$.

If $a_2 = b_2 = c_2 = 0$ then $P(c_1, c_2)$ is the operator studied in [3].

We show also that the uniqueness for the operators $P(c_1, c_2)$ holds (when $a_1, b_1 < 0$ and $a_2, b_2 \geq 0$) if $\mathrm{Re}(c_1) \neq -\delta_1[k + j(k+1)]$, $\mathrm{Re}(c_1) \neq -\delta_1(j+1)(k+1)$. This result contains part of [1, Theorem 3.1].

We observe also that the local uniqueness of solutions in the class of distributions holds for these operators (see [1] and B. Birkeland and J. Persson [2]).

2. A Carleman inequality and uniqueness

In this section, we prove a Carleman inequality which is needed in order to prove local uniqueness of solutions of the characteristic Cauchy problem for operators of the form

$$P = P(c_1, c_2) = XY - c_1 t^{k-1} \partial_x - c_2 t^{k-1} \tag{2.1}$$

where

$$X = \partial_t + a_1 t^k \partial_x + a_2 t^k,$$
$$Y = \partial_t - b_1 t^k \partial_x - b_2 t^k,$$
$$a_1, b_1 > 0, a_2, b_2 \leq 0,$$

c_1, c_2 are complex numbers, k is an odd natural number and $(t, x) \in [-T, T] \times \mathbb{R}$, $T > 0$.

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Since

$$P(c_1, c_2) = \partial_t^2 - a_1 b_1 t^{2k}\partial_x^2 + (a_1 - b_1)t^k \partial_t \partial_x - (c_1 + b_1 k)t^{k-1}\partial_x - (a_1 b_2 + a_2 b_1)t^{2k}\partial_x - (c_2 + b_2 k)t^{k-1} - a_2 b_2 t^{2k} + (a_2 - b_2)t^k \partial_t,$$

if

$$f(t) = -\frac{1}{2}\int_0^t (a_2 - b_2)s^k \, ds$$

and

$$u(t, x) = \exp[f(t)] \cdot v(t, x)$$

we obtain

$$P(c_1, c_2)u = \exp[f(t)] \cdot Q(c_1, c_2)v \tag{2.2}$$

where

$$Q = Q(c_1, c_2) = \partial_t^2 - a_1 b_1 t^{2k}\partial_x^2 + (a_1 - b_1)t^k \partial_t \partial_x - (c_1 + b_1 k)t^{k-1}\partial_x - \frac{1}{2}(a_1 + b_1)(a_2 + b_2)t^{2k}\partial_x - \left[c_2 + \frac{k}{2}(a_2 + b_2)\right] t^{k-1} - \frac{1}{4}(a_2 + b_2)^2 t^{2k}. \tag{2.3}$$

The purpose of introducing the integrating factor $\exp[f(t)]$ was to obtain a new operator, Q, in which the term $(a_2 - b_2)t^k \partial_t$ is not present.

By using the formula (2.2), we can see that the local uniqueness of solutions of the characteristic Cauchy problem holds for P if and only if it holds for Q.

LEMMA 2.1. *Let* $Q(c_1, c_2)$ *be given by* (2.3). *If* $\mathrm{Re}[c_1 + (k/2)(a_1 + b_1)] \leq 0$, $\mathrm{Re}[c_2 + (k/2)(a_2 + b_2)] \geq 0$ *and* Q_τ *is the operator*

$$Q_\tau = \exp(\tau x) \cdot Q \cdot \exp(-\tau x)$$

then the following inequalities hold for all $v \in C_c^3(\mathbb{O}; \mathbb{R})$ *and for* $\tau \geq 1$, *where* $\mathbb{O} \subset \{(t, x) \in \mathbb{R}^2 : |t| \leq T\}$ *is a nonempty bounded open subset of* $\mathbb{R}^2$.

$$\text{I} - 1: \qquad \mathrm{Re} < Q_\tau v, (\partial_x - \tau)v > \geq a_1 b_1 \iint_{\mathbb{O}} t^{2k}|v|^2 \, dt \, dx$$

where $\langle \ , \ \rangle$ *is the usual inner product of* $\mathbb{L}^2(\mathbb{O}; \mathbb{R})$.

$$\text{I} - 2: \qquad \iint_{\mathbb{O}} a_1 b_1 t^{2k} \exp(2\tau x)|v|^2 \, dt \, dx \leq \iint_{\mathbb{O}} \exp(2\tau x)|Qv \cdot \partial_x v| \, dt \, dx.$$

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PROOF. Since

$$\begin{aligned} Q_\tau v = \partial_t^2 v - a_1 b_1 t^{2k}(\partial_x - \tau)^2 + (a_1 - b_1)t^k \partial_t(\partial_x - \tau) - (c_1 + b_1 k)t^{k-1}(\partial_x - \tau) \\ - \frac{1}{2}(a_1 + b_1)(a_2 + b_2)t^{2k}(\partial_x - \tau) - \left[c_2 + \frac{k}{2}(a_2 + b_2)\right] t^{k-1} \\ - \frac{1}{4}(a_2 + b_2)^2 t^{2k} \end{aligned}$$

integrations by parts show that

$$\begin{aligned} \mathrm{Re}\langle Q_\tau v, (\partial_x - \tau)v\rangle = \tau \iint_{\mathbb{O}} |\partial_t v|^2\, dt\, dx + a_1 b_1 \tau \iint_{\mathbb{O}} t^{2k}(|\partial_x v|^2 + \tau^2 |v|^2)\, dt\, dx \\ - \mathrm{Re}\left[c_1 + \frac{k}{2}(a_1 + b_1)\right] \iint_{\mathbb{O}} t^{k-1}(|\partial_x v|^2 + \tau^2|v|^2)\, dt\, dx \\ - \frac{1}{2}(a_1 + b_1)(a_2 + b_2) \iint_{\mathbb{O}} t^{2k}(|\partial_x v|^2 + \tau^2 |v|^2)\, dt\, dx \\ + \mathrm{Re}\left[c_2 + \frac{k}{2}(a_2 + b_2)\right] \tau \iint_{\mathbb{O}} t^{k-1}|v|^2\, dt\, dx \\ + \frac{1}{4}(a_2 + b_2)^2 \tau \iint_{\mathbb{O}} t^{2k}|v|^2\, dt\, dx \\ \geq a_1 b_1 \iint_{\mathbb{O}} t^{2k}|v|^2\, dt\, dx. \end{aligned}$$

The proof of $\mathrm{I}-1$ is complete.

We prove $\mathrm{I}-2$, as follows:

$$\begin{aligned} \iint_{\mathbb{O}} a_1 b_1 t^{2k} \exp(2\tau x)|v|^2\, dt\, dx &\leq \mathrm{Re}\langle Q_\tau[\exp(\tau x)v], (\partial_x - \tau)[\exp(\tau x)v]\rangle \\ &= \mathrm{Re}\langle \exp(\tau x) Q v, \exp(\tau x)\partial_x v\rangle \\ &\leq \iint_{\mathbb{O}} \exp(2\tau x)|Qv \cdot \partial_x v|\, dt\, dx. \end{aligned}$$

By using inequality $\mathrm{I}-2$ and following the proof of [6, Theorem 2.3], we can prove

THEOREM 2.1. *Let Q be given by* (2.3) *and assume that*

$$\mathrm{Re}[c_1 + (k/2)(a_1 + b_1)] \leq 0$$

and

$$\mathrm{Re}[c_2 + (k/2)(a_2 + b_2)] \geq 0.$$

Let $\mathbb{F} \subset \mathbb{O}$ be a closed subset such that

$$\mathbb{K} = \mathbb{F} \cap \{(t, x) \in \mathbb{O} : x \geq 0\}$$

is compact. Then there is an open neighborhood $\mathbb{U}$ *of* $\mathbb{K}$ *such that any function* $u \in C^3(\mathbb{O}; \mathbb{R})$ *satisfying*

$$Qu = 0 \text{ in } \mathbb{O}; \operatorname{supp} u \subset \mathbb{F}$$

must vanish in $\mathbb{U}$.

3. Concatenation and uniqueness

Theorem 2.1 guarantees local uniqueness for the operator $Q(c_1, c_2)$ and therefore for $P(c_1, c_2)$ if $\operatorname{Re}(c_1 + (k/2)\delta_1) \leq 0$ and $\operatorname{Re}(c_2 + (k/2)\delta_2) \geq 0$, where $\delta_1 = a_1 + b_1$ and $\delta_2 = a_2 + b_2$, that is, if $\operatorname{Re}(c_1)$ is small and $\operatorname{Re}(c_2)$ is large.

To prove uniqueness when $\operatorname{Re}(c_1)$ is large and any c_2, we shall use the method of concatenations.

Note that we have

$$[X, Y] = -kt^{k-1}(\delta_1\partial_x + \delta_2) \tag{3.1}$$

$$[t(\delta_1\partial_x + \delta_2), YX] = -(\delta_1\partial_x + \delta_2)(X + Y) \tag{3.2}$$

$$\begin{aligned}[t(\delta_1\partial_x + \delta_2)Y, t^{k-1}(c_1\partial_x + c_2)] \\ = (k-1)t^{k-1}(\delta_1\partial_x + \delta_2)(c_1\partial_x + c_2)\end{aligned} \tag{3.3}$$

Since $Y = X - t^k(\delta_1\partial_x + \delta_2)$, we have

$$Y^2 = XY - t^k(\delta_1\partial_x + \delta_2)Y. \tag{3.4}$$

We shall try to find operators

$$T = t(\delta_1\partial_x + \delta_2)Y + (A_1\partial_x + B_1), \quad S = t(\delta_1\partial_x + \delta_2)Y + (A_2\partial_x + B_2)$$

so that

$$TP(c_1, c_2) = P(c_1', c_2')S. \tag{3.5}$$

By using (3.1)–(3.4) we have

$$\begin{aligned}
TP(c_1, c_2) &= [t(\delta_1\partial_x + \delta_2)Y + (A_1\partial_x + B_1)](XY - c_1t^{k-1}\partial_x - c_2t^{k-1}) \\
&= t(\delta_1\partial_x + \delta_2)YXY - t(\delta_1\partial_x + \delta_2)Yt^{k-1}(c_1\partial_x + c_2) \\
&\quad + (A_1\partial_x + B_1)XY - (A_1\partial_x + B_1)t^{k-1}(c_1\partial_x + c_2) \\
&= -(\delta_1\partial_x + \delta_2)XY - (\delta_1\partial_x + \delta_2)Y^2 + YXt(\delta_1\partial_x + \delta_2)Y \\
&\quad - [t(\delta_1\partial_x + \delta_2)Y, t^{k-1}(c_1\partial_x + c_2)] \\
&\quad - t^{k-1}(c_1\partial_x + c_2)t(\delta_1\partial_x + \delta_2)Y \\
&\quad + (A_1\partial_x + B_1)XY - (A_1\partial_x + B_1)t^{k-1}(c_1\partial_x + c_2) \\
&= [-2(\delta_1\partial_x + \delta_2) + (A_1\partial_x + B_1)]XY + (\delta_1\partial_x + \delta_2)t^k(\delta_1\partial_x + \delta_2)Y \\
&\quad + (XY - [X, Y])t(\delta_1\partial_x + \delta_2)Y \\
&\quad - (k-1)t^{k-1}(\delta_1\partial_x + \delta_2)(c_1\partial_x + c_2) \\
&\quad - t^{k-1}(c_1\partial_x + c_2)t(\delta_1\partial_x + \delta_2)Y - (A_1\partial_x + B_1)t^{k-1}(c_1\partial_x + c_2) \\
&= XYt(\delta_1\partial_x + \delta_2)Y + kt^{k-1}(\delta_1\partial_x + \delta_2)t(\delta_1\partial_x + \delta_2)Y \\
&\quad - [-2(\delta_1\partial_x + \delta_2) + (A_1\partial_x + B_1)]XY \\
&\quad + t^{k-1}(\delta_1\partial_x + \delta_2)t(\delta_1\partial_x + \delta_2)Y \\
&\quad - (k-1)t^{k-1}(\delta_1\partial_x + \delta_2)(c_1\partial_x + c_2) \\
&\quad - t^{k-1}(c_1\partial_x + c_2)t(\delta_1\partial_x + \delta_2)Y - (A_1\partial_x + B_1)t^{k-1}(c_1\partial_x + c_2) \\
&= \{XY - t^{k-1}[(c_1\partial_x + c_2) - (k-1)(\delta_1\partial_x + \delta_2)]\} \\
&\quad \circ [t(\delta_1\partial_x + \delta_2)Y - 2(\delta_1\partial_x + \delta_2) + (A_1\partial_x + B_1)] \\
&\quad + t^{k-1}[(c_1\partial_x + c_2) - (k+1)(\delta_1\partial_x + \delta_2)] \\
&\quad - [-2(\delta_1\partial_x + \delta_2) + (A_1\partial_x + B_1)] \\
&\quad - (k-1)t^{k-1}(\delta_1\partial_x + \delta_2)(c_1\partial_x + c_2) - (A_1\partial_x B_1)t^{k-1}(c_1\partial_x + c_2).
\end{aligned}$$

If we choose $A_1 = 2\delta_1 - c_1$ and $B_1 = 2\delta_2 - c_2$, we obtain (3.5), that is,

$$\begin{aligned}
&[t(\delta_1\partial_x + \delta_2)Y + 2(\delta_1\partial_x + \delta_2) - (c_1\partial_x + c_2)][XY - t^{k-1}(c_1\partial_x + c_2)] \\
&\quad = \{XY - t^{k-1}[(c_1\partial_x + c_2) - (k+1)(\delta_1\partial_x + \delta_2)]\} \\
&\quad\quad \circ [t(\delta_1\partial_x + \delta_2)Y - (c_1\partial_x + c_2)].
\end{aligned}$$

Now, we shall try to find operators Q and R so that

$$QS + RP(c_1, c_2) = (c_1\partial_x + c_2)[(\delta_1\partial_x + \delta_2) - (c_1\partial_x + c_2)]. \tag{3.6}$$

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To obtain (3.6) write

$$XS = X[t(\delta_1\partial_x+\delta_2)Y-(c_1\partial_x+c_2)] = Xt(\delta_1\partial_x+\delta_2)Y - X(c_1\partial_x+c_2)$$
$$= (\delta_1\partial_x+\delta_2)Y + t(\delta_1\partial_x+\delta_2)XY - (c_1\partial_x+c_2)X.$$

Then

$$XS - t(\delta_1\partial_x+\delta_2)P(c_1, c_2)$$
$$= XS - t(\delta_1\partial_x+\delta_2)XY + t(\delta_1\partial_x+\delta_2)t^{k-1}(c_1\partial_x+c_2)$$
$$= (\delta_1\partial_x+\delta_2)Y - (c_1\partial_x+c_2)X + t^k(\delta_1\partial_x+\delta_2)(c_1\partial_x+c_2).$$

Since $X - Y = t^k(\delta_1\partial_x+\delta_2)$ we have

$$XS - t(\delta_1\partial_x+\delta_2)P(c_1, c_2) = [(\delta_1\partial_x+\delta_2)-(c_1\partial_x+c_2)]Y$$

and therefore

$$[t(\delta_1\partial_x+\delta_2)X + (c_1\partial_x+c_2) - (\delta_1\partial_x+\delta_2)]S - t^2(\delta_1\partial_x+\delta_2)^2P(c_1, c_2)$$
$$= (c_1\partial_x+c_2)[(\delta_1\partial_x+\delta_2)-(c_1\partial_x+c_2)].$$

Then (3.6) holds with

$$Q = t(\delta_1\partial_x+\delta_2)X + (c_1\partial_x+c_2) - (\delta_1\partial_x+\delta_2)$$

and

$$R = -t^2(\delta_1\partial_x+\delta_2)^2.$$

The computations above yield the following result.

Lemma 3.1. *For $j, l = 0, 1, 2, \ldots$, let*

$$c_{1j} = c_1 - j(k+1)\delta_1, \qquad c_{2l} = c_2 - l(k+1)\delta_2$$
$$T_{j,l} = t(\delta_1\partial_x+\delta_2)Y + 2(\delta_1\partial_x+\delta_2) - (c_{1j}\partial_x + c_{2l})$$
$$S_{j,l} = t(\delta_1\partial_x+\delta_2)Y - (c_{1j}\partial_x+c_{2l})$$
$$P(c_{1j}, c_{21}) = XY - t^{k-1}(c_{1j}\partial_x + c_{2l})$$
$$Q_{j,l} = t(\delta_1\partial_x+\delta_2)X + (c_{1j}\partial_x+c_{2l}) - (\delta_1\partial_x+\delta_2).$$

Then

$$T_{j,l}P(c_{1j}, c_{2l}) = P(c_{1,j+1}, c_{2,l+1})S_{j,l} \tag{3.7}$$

and we can find operators $Q_{j,l}$ such that

$$Q_{j,l}S_{j,l} + RP(c_{1j}, c_{2l}) = (c_{1j}\partial_x+c_{2l})[(\delta_1\partial_x+\delta_2)-(c_{1j}\partial_x+c_{2l})]. \tag{3.8}$$

Consider the statement,

$\mathbf{A}^m_{j,l}$: Every $u \in C^m(\mathbb{O})$ which satisfies $P(c_{1j}, c_{2l})u = 0$ in $\mathbb{O}$ with $\operatorname{supp} u \subset \mathbb{F}$ must vanish in Ω, where Ω is an open neighborhood of $\mathbb{K}$. (Here, $\mathbb{O}$, $\mathbb{F}$ and $\mathbb{K}$ are as in Theorem 2.1 and $m \geq 3$).

Lemma 3.2. *If* $c_{1j} \neq 0$ *and* $c_{1j} \neq \delta_1$ *then* $\mathbb{A}^m_{j+1,l+1}$ *implies* $\mathbb{A}^{m+2}_{j,l}$.

Proof. Let $u \in C^{m+2}(\mathbb{O})$ so that $P(c_{1j}, c_{2l}) = 0$ in $\mathbb{O}$ and $\operatorname{supp} u \subset \mathbb{F}$. Since

$$T_{j,l}P(c_{1j}, c_{2l}) = P(c_{1,j+1}, c_{2,l+1})S_{j,l} (\text{see } (3.7))$$

we have

$$P(c_{1,j+1}, c_{2,l+1})(S_{j,l}u) = 0 \text{ in } \mathbb{O}.$$

Since $S_{j,l}$ is a second order linear operator we have $S_{j,l}u \in C^m(\mathbb{O})$ and therefore by hypothesis

$$S_{j,l}u = 0 \text{ in } \Omega.$$

By (3.8) there are operators $Q_{j,l}$ and R such that

$$Q_{j,l}S_{j,l}u + RP(c_{1j}, c_{2l})u = (c_{1j}\partial_x + c_{2l})[(\delta_1 - c_{1j})\partial_x + (\delta_2 - c_{2l})]u$$

and therefore

$$(c_{1j}\partial_x + c_{2l})[(\delta_1 - c_{1j})\partial_x + (\delta_2 - c_{2l})]u = 0 \quad \text{in } \Omega.$$

Since $c_{1j} \neq 0$ we have local uniqueness of solutions of the noncharacteristic Cauchy problem for the operator $c_{1j}\partial_x + c_{2l}$ and therefore

$$[(\delta_1 - c_{1j})\partial_x + (\delta_2 - c_{2l})]u = 0 \quad \text{in } \mathbb{W},$$

where $\mathbb{W}$ is an open neighborhood of $\mathbb{K}$.

Now, since by the hypothesis $c_{1j} \neq \delta_1$, the uniqueness for the operator $(\delta_1 - c_{1j})\partial_x + (\delta_2 - c_{2l})$ implies $u = 0$ in $\mathbb{U}$, where $\mathbb{U}$ is an open neighborhood of $\mathbb{K}$.

Theorem 3.1. *Suppose that* $c_1 \neq j(k+1)\delta_1$ *and* $c_1 \neq j(k+1)\delta_1 + \delta_1$, $j = 0, 1, 2, \ldots$ *and* $c_2 \in \mathbb{C}$. *Let* $\mathbb{O}, \mathbb{F}$ *and* $\mathbb{K}$ *be as in Theorem* 2.1. *Then there is an integer* m *depending only on* $\operatorname{Re}(c_1)$ *and an open neighborhood* $\mathbb{U}$ *of* $\mathbb{K}$ *such that every* $u \in C^m(\mathbb{O}, \mathbb{R})$ *with support in* $\mathbb{F}$ *which satisfies*

$$P(c_1, c_2)u = 0 \quad \text{in } \mathbb{O}$$

must vanish in $\mathbb{U}$.

Proof. Let $c_1, c_2 \in \mathbb{C}$ and $c_1' = c_1 + (k/2)\delta_1$, $c_2' = c_2 + (k/2)\delta_2$.

If $\operatorname{Re}(c_1') \leq 0$ and $\operatorname{Re}(c_2') \geq 0$ then Theorem 2.1 guarantees for the operator $Q(c_1, c_2)$, and therefore for $P(c_1, c_2)$, the uniqueness result stated in Theorem 3.1 with $m = 3$.

If $\operatorname{Re}(c_1') \leq 0$ and $\operatorname{Re}(c_2') < 0$, let l_0 be the smallest natural number such that $\operatorname{Re}[c_{2l} + (k/2)\delta_2] \geq 0$ where $c_{2l_0} = c_2 - l_0(k+1)\delta_2$. Thus,

$\mathrm{Re}[c_{1l_0} + (k/2)\delta_1] < 0$ where $c_{1l_0} = c_1 - l_0(k+1)\delta_1$ and therefore as in the preceding case, $A^3_{l_0,l_0}$ holds.

We apply Lemma 3.2 l_0 times and conclude that $A^{3+2l_0}_{0,0}$ holds; in this case $m = 3 + 2l_0$.

If $\mathrm{Re}(c_1') > 0$ and $\mathrm{Re}(c_2') \leq 0$, let j_0 and l_0 be the smallest natural numbers such that $\mathrm{Re}[c_{1j_0} + (k/2)\delta_1] \leq 0$ and $\mathrm{Re}[c_{2l_0} + (k/2)\delta_2] \geq 0$.

If $j_0 < l_0$ we have $\mathrm{Re}[c_{1l_0} + (k/2)\delta_1] \leq 0$ and therefore as in the first case, $A^3_{l_0,l_0}$ holds. We apply Lemma 3.2, l_0 times and conclude that $A^{3+2l_0}_{0,0}$ holds; in this case $m = 3 + 2l_0$.

If $j_0 > l_0$ we have $\mathrm{Re}[c_{2j_0} + (k/2)\delta_2] \geq 0$ and as in the preceding case, $A^{3+2j_0}_{0,0}$ holds; in this case $m = 3 + 2j_0$.

If $\mathrm{Re}(c_1') > 0$ and $\mathrm{Re}(c_2') > 0$, let j_0 be the smallest natural number such that $\mathrm{Re}[c_{1j_0} + (k/2)\delta_1] \leq 0$. Thus, $\mathrm{Re}[c_{2j_0} + (k/2)\delta_2] > 0$ and therefore as in the preceding case, $A^{3+2j_0}_{0,0}$ holds; in this case $m = 3 + 2j_0$. The proof is complete.

REMARK 3.1. Now consider the operator $P(c_1, c_2)$ with $a_1, b_1 < 0$ and $a_2, b_2 \geq 0$. We can write

$$\begin{aligned} P(c_1, c_2) &= (\partial_t + a_1 t^k \partial_x + a_2 t^k)(\partial_t - b_1 t^k \partial_x - b_2 t^k) - c_1 t^{k-1}\partial_x - c_2 t^{k-1} \\ &= [\partial_t + (-b_1)t^k \partial_x + (-b_2)t^k][\partial_t - (-a_1)t^k\partial_x - (-a_2)t^k] \\ &\quad - (c_1 + k\delta_1)t^{k-1}\partial_x - (c_2 + k\delta_2)t^{k-1}. \end{aligned}$$

If $\mathrm{Re}(c_1) \neq -\delta_1[k + j(k+1)]$ and $\mathrm{Re}(c_1) \neq -\delta_1(j+1)(k+1)$, $j = 0, 1, 2, \ldots,$ then Theorem 3.1 ensures local uniqueness for the operator $P(c_1, c_2)$.

In the case that $a_2 = b_2 = c_2 = 0$ and $b_1 = a_1$ we obtain part of [1, Theorem 3.1].

REMARK 3.2. As the operator $P(c_1, c_2)$ belongs to the class of operators considered in [1, Theorem 4.1], this ensures its uniqueness in the class of distributions.

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Universidade Federal de São Carlos (UFSCar)
Departamento de Matemática
Caixa Postal 676
13560 - São Carlos - SP - Brazil