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A THEOREM ON ISOMETRIES AND THE APPLICATION OF IT TO THE ISOMETRIES OF $H^p(S)$ FOR $2 < p < \infty$

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1. Introduction. 1.1. Let X and Y be sets, let λ be a bounded positive measure on X, and let μ be a bounded positive measure on Y. Furthermore let M be a subalgebra of $L^\infty(\lambda)$, let $p \in (0, \infty)$, and let A be a linear transformation of M into $L^p(\mu)$ such that

$$\int |Af|^p d\mu = \int |f|^p d\lambda$$

for all f in M.

In § 2 of this paper we will prove the following theorem.

1.2. THEOREM. *If* (a) $p > 2$, *if* (b) $(Af)(y) \neq 0$ *for μ-almost all y in Y whenever $f \in M$ and $f \neq 0$, and if* (c) $A1 = 1$, *then*

$$A(fg) = AfAg$$

for all f and g in M and

$$\int Af\overline{Ag}d\mu = \int f\bar{g}d\lambda$$

for all f and g in M.

1.3. If the hypotheses (b) and (c) of Theorem 1.2 hold and if instead of (a) we have $p < 2$, then we do not know if the conclusion of Theorem 1.2 holds. We will denote by U the class of all f in M such that $f\bar{f} = 1$. It was proved in [**1**] that if $M = \mathbf{C}[U]$ and if the hypothesis (c) of Theorem 1.2 holds, then the conclusion of Theorem 1.2 holds for p in $(0, \infty)$. Furthermore it was proved in [**1**] that if the hypothesis (c) of Theorem 1.2 holds and if instead of (a) we have $p \geqq 4$, then the conclusion of Theorem 1.2 holds.

1.4. Let V be a vector space over $\mathbf{C}$ of complex dimension n with an inner product. If x and y are in V, then we will denote by $\langle x, y\rangle$ the inner product of x and y. We will denote by B the class of all x in V such that $\langle x, x\rangle < 1$, by $\bar{B}$ the class of all x in V such that $\langle x, x\rangle \leqq 1$, and by S the class of all x in V such that $\langle x, x\rangle = 1$. Thus S may be regarded as the Euclidean sphere of real dimension $2n - 1$. We will denote by σ the positive Radon measure on S which assigns to each open subset of S its Euclidean volume. We define $\alpha : \bar{B} \times B \to \mathbf{C}$ by

$$\alpha(x, y) = [\sqrt{(1 - \langle y, y\rangle)}]/(1 - \langle x, y\rangle)$$

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and we define $\beta : \bar{B} \times B \to (0, \infty)$ by $\beta = (\alpha\bar{\alpha})^n$. We recall that if ϕ is a function which is defined on the Cartesian product $E \times F$ of sets E and F and if $(x, y) \in E \times F$, then ϕ_x and ϕ^y are the functions which are defined on F and E respectively by $\phi_x(t) = \phi(x, t)$ and $\phi^y(s) = \phi(s, y)$. If $f \in L^1(\sigma)$, then we define $f^\# : B \to \mathbf{C}$ by

$$f^\#(y) = (1/\sigma(S)) \int f\beta^y d\sigma.$$

We remark that if $f \in L^1(\sigma)$, then $f^\#$ is of differentiability class C^∞. If $1 \leqq p \leqq \infty$, then we will denote by $H^p(S)$ the class of all f in $L^p(\sigma)$ such that $f^\#$ is holomorphic on B. It follows that $H^p(S)$ is a closed subspace of the Banach space $L^p(\sigma)$, and hence that $H^p(S)$ is a Banach space with respect to the norm of $L^p(\sigma)$. The definition of $H^p(S)$ is motivated by the change of variables formula with regard to holomorphic homeomorphisms of B that is expressed in Lemma 3.4. If $n = 1$, then $H^p(S)$ is the familiar Hardy class H^p (if we regard S as the unit circle in the complex plane).

As an application of Theorem 1.2 we will prove the following theorem.

1.5. THEOREM. *If* (a) *T is a linear isometry of the Banach space $H^p(S)$ onto itself and if* (b) $2 < p < \infty$, *then there is a holomorphic homeomorphism Z of B and a unimodular complex number θ such that for every f in $H^p(S)$ we have*

$$Tf = \theta(\alpha^z)^{2n/p} f \circ Z \tag{1.1}$$

where z in B is defined by $Z(z) = 0$.

1.6. The proof of Theorem 1.5 is in § 3. We remark that if Z is any holomorphic homeomorphism of B and if $p \in [1, \infty)$, then the expression (1.1) defines a linear isometry of $H^p(S)$ onto itself. (This follows from Lemma 3.4. The holomorphic homeomorphisms of B are described in Lemma 3.2.) If $n \geqq 2$, if the hypothesis (a) of Theorem 1.5 holds, and if instead of (b) we have $1 \leqq p < 2$, then we do not know if the conclusion of Theorem 1.5 holds. Furthermore if $n \geqq 2$, if $p \in [1, \infty)$, and if $p \neq 2$, then it is not known if there are any linear isometries of $H^p(S)$ into itself which are not onto.

2. The proof of Theorem 1.2. 2.1. If $w \in \mathbf{C}$ and if $r \in (0, \infty)$, then we will denote by $D(w, r)$ the open disc in $\mathbf{C}$ whose center is w and whose radius is r. The proof of the following lemma is in [**1**].

2.2. LEMMA. *Let ρ be a bounded positive measure on X, let τ be a bounded positive measure on Y, let $s \in (0, \infty)$, let $f \in L^s(\rho)$, and let $g \in L^s(\tau)$. If for some r in* $(0, \infty)$ *we have*

$$\int |1 + zf|^s d\rho = \int |1 + zg|^s d\tau$$

for all z in $D(0, r)$, then

$$\int |f|^2 d\rho = \int |g|^2 d\tau.$$

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2.3. We will now prove Theorem 1.2. We will break the proof up into several statements.

2.3.1. If $f \in M$ and $f \neq 0$, then

$$\int |A(fg)|^2 |Af|^{p-2} d\mu = \int |g|^2 |f|^p d\lambda \tag{2.1}$$

for all g in M.

For the purpose of proving statement 2.3.1 we let $d\rho = |f|^p d\lambda$ and $d\tau = |Af|^p d\mu$. If $g \in M$ and $z \in \mathbf{C}$, then

$$\begin{aligned}\int |1 + zg|^p d\rho &= \int |f + zfg|^p d\lambda \\ &= \int |Af + zA(fg)|^p d\mu \\ &= \int |1 + zA(fg)/Af|^p d\tau,\end{aligned}$$

and hence by Lemma 2.2 we have

$$\int |g|^2 d\rho = \int |A(fg)/Af|^2 d\tau$$

which completes the proof of statement 2.3.1.

We remark that the proof of statement 2.3.1 did not use either the fact that $A1 = 1$ or the fact that $p > 2$.

We will denote by M^{-1} the collection of all invertible elements of M.

2.3.2. If $f \in M^{-1}$, then

$$\int |Af|^{p-2} |Ag|^2 d\mu = \int |f|^{p-2} |g|^2 d\lambda \tag{2.2}$$

for all g in M.

Statement 2.3.2 follows from statement 2.3.1 upon replacing g in the identity (2.1) by g/f.

2.3.3. If $f \in M$ and $g \in M$, then

$$\int |1 + zAf|^{p-2} |Ag|^2 d\mu = \int |1 + zf|^{p-2} |g|^2 d\lambda$$

for all z in $D(0, 1/\|f\|_\infty)$.

For the purpose of proving statement 2.3.3 we may assume that M is a closed subalgebra of $L^\infty(\lambda)$. Since $1 + zf \in M^{-1}$ if $z \in D(0, 1/\|f\|_\infty)$, statement 2.3.3 follows from statement 2.3.2 upon replacing f in the identity (2.2) by $1 + zf$.

We remark that the proof of statement 2.3.3 did not use the fact that $p > 2$.

2.3.4. If $f \in M$ and $g \in M$, then

$$\int |Af|^2 |Ag|^2 d\mu = \int |f|^2 |g|^2 d\lambda.$$

Statement 2.3.4 follows from statement 2.3.3 and Lemma 2.2 (with $d\rho = |g|^2 d\lambda$, $d\tau = |Ag|^2 d\mu$, and $s = p - 2$).

It follows from statement 2.3.4 that if $f \in M$, then $Af \in L^4(\mu)$.

2.3.5. If a, b, c, and d are in M, then

$$\int Aa\overline{Ab}Ac\overline{Ad}d\mu = \int a\bar{b}c\bar{d}d\lambda.$$

Statement 2.3.5 follows from statement 2.3.4 by the method of polarization.

Statement 2.3.5 includes the second assertion of Theorem 1.2. Furthermore it follows from statement 2.3.5 that if $f \in M$ and $g \in M$, then

$$\int |A(fg) - AfAg|^2 d\mu = 0,$$

which completes the proof of Theorem 1.2.

2.4. We will denote by $\mathbf{Z}_+$ the class of all positive integers.

2.5. COROLLARY (of Theorem 1.2). *If $f \in M$, then* $||Af||_\infty = ||f||_\infty$.

Proof. If $k \in \mathbf{Z}_+$, then

$$\begin{aligned}\left(\int |Af|^{2k} d\mu\right)^{1/2k} &= \left(\int A(f^k)\overline{A(f^k)}d\mu\right)^{1/2k} \\ &= \left(\int |f|^{2k} d\lambda\right)^{1/2k},\end{aligned}$$

from which the desired conclusion follows upon letting k increase to ∞.

3. The proof of Theorem 1.5. 3.1. We will denote by $U(V)$ the class of all unitary transformations of V, and we will regard $\mathrm{SL}(2, \mathbf{R})$ as the class of all 2×2 matrices L of the form

$$\mathrm{L} = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

where a and b are in $\mathbf{C}$ and $\det(L) = a\bar{a} - b\bar{b} = 1$. We define $\gamma : \mathrm{SL}(2, \mathbf{R}) \times S \times \bar{B} \to \bar{B}$ by

$$\gamma(L, x, y) = [1/(\bar{b}\langle y, x\rangle + \bar{a})](y - \langle y, x\rangle x) + [(a\langle y, x\rangle + b)/(\bar{b}\langle y, x\rangle + \bar{a})]x$$

and we define $\delta : U(V) \times \mathrm{SL}(2, \mathbf{R}) \times S \times \bar{B} \to \bar{B}$ by

$$\delta(W, L, x, y) = W\gamma(L, x, y) = \gamma(L, Wx, Wy).$$

With regard to the definition of γ we remark that if $x \in S$ and if $y \in V$, then $y - \langle y, x\rangle x$ is the orthogonal projection of y into $V \ominus \mathbf{C}x$. Furthermore we remark that $\delta_{(W,L,x)}$ is a holomorphic homeomorphism of B for every triple (W, L, x) in $U(V) \times SL(2, \mathbf{R}) \times S$. We recall the following fact of the theory of functions on B.

3.2. LEMMA. *If Z is a holomorphic homeomorphism of B, then there is a triple (W, L, x) in $U(V) \times SL(2, \mathbf{R}) \times S$ such that*

$$Z(y) = \delta(W, L, x, y)$$

for all y in B.

3.3. The following lemma (which is well-known) follows from Lemma 3.2.

3.4. LEMMA. *If Z is a holomorphic homeomorphism of B, then*

$$\int f \circ Z d\sigma = \int f\beta^{Z(0)} d\sigma$$

for every f in $L^1(\sigma)$.

3.5. The following lemma is due to R. Schneider [**2**] who stated it and proved it in terms of the Hardy spaces of torii. His proof applies as well to $H^p(S)$.

3.6. LEMMA. *If $p \in [1, \infty]$, if $g \in H^p(S)$ and $g \neq 0$, if $h \in L^\infty(\sigma)$, and if $gh^k \in H^p(S)$ for all k in $\mathbf{Z}_+$, then $h \in H^\infty(S)$.*

3.7. We will now prove Theorem 1.5. For this purpose we recall that if $g \in H^p(S)$ and $g \neq 0$, then $g(y) \neq 0$ for σ almost all y in S. We let $a = T1$, $d\mu = |a|^p d\sigma$, and define $A : H^p(S) \to L^p(\mu)$ by $Af = Tf/a$. Since $H^\infty(S)$ is a subalgebra of $L^\infty(\sigma)$, it follows from Theorem 1.2 and Corollary 2.5 that if f and g are in $H^\infty(S)$, then $Af \in L^\infty(\sigma)$ and $A(fg) = AfAg$. It follows from this and Lemma 3.6 that if $f \in H^\infty(S)$, then $Af \in H^\infty(S)$ since $a(Af)^k = aA(f^k) = T(f^k)$ and $T(f^k) \in H^p(S)$ for all k in $\mathbf{Z}_+$. Thus if A is restricted to $H^\infty(S)$, then A is an algebra homomorphism of $H^\infty(S)$ into $H^\infty(S)$. Furthermore we have $||Af||_\infty = ||f||_\infty$ for all f in $H^\infty(S)$.

We define $\chi : S \times V \to \mathbf{C}$ by $\chi(x, y) = \langle x, y \rangle$, we let F be an orthonormal basis of V, and we define $Z : B \to V$ by

$$Z(x) = \sum_{y \in F} [(A\chi^y)^\#(x)]y.$$

It follows that if $(x, y) \in B \times V$, then $\langle Z(x), y \rangle = (A\chi^y)^\#(x)$. Hence Z (which is holomorphic) maps B into itself, and $(A\chi^y)^\# = (\chi^y)^\# \circ Z$ for all y in V. Thus if g is in the ring $\mathbf{C}[\chi^y : y \in V]$, then $(Tg)^\# = a^\#(Ag)^\# = a^\# g^\# \circ Z$, from which it follows that if $f \in H^p(S)$, then

$$(Tf)^\# = a^\# f^\# \circ Z \tag{3.1}$$

since $\mathbf{C}[\chi^y : y \in V]$ is dense in $H^p(S)$.

We now consider T^{-1}. It follows that there is a function b in $H^p(S)$ and a holomorphic transformation W of B into itself such that if $f \in H^p(S)$, then

$$(T^{-1}f)^\# = b^\# f^\# \circ W. \tag{3.2}$$

From (3.1) and (3.2) it follows that if $f \in H^p(S)$, then $f^\# \circ W \circ Z = f^\# = f^\# \circ Z \circ W$, and hence Z is a holomorphic homeomorphism of B (whose inverse is W). Thus (by Lemma 3.2) Z is defined on $\bar{B}$ as well as on B, Z maps S onto itself, and we have

$$Tf = af \circ Z \tag{3.3}$$

for all f in $H^p(S)$.

We will now prove that for σ-almost all x in S we have

$$|a(x)|^p = \beta(x, z) \tag{3.4}$$

where $z = W(0)$. If $f \in H^p(S)$, then by (3.3) and Lemma 3.4 we have

$$\int |f|^p|a|^p d\sigma = \int |f \circ W \circ Z|^p|a|^p d\sigma$$

$$= \int |f \circ W|^p d\sigma = \int |f|^p\beta^z d\sigma.$$

From this and Theorem 1.2 it follows that if f and g are in $\mathbf{C}[\chi^y : y \in V]$, then

$$\int f\bar{g}|a|^p d\sigma = \int f\bar{g}\beta^z d\sigma,$$

from which it follows by the Stone-Weierstrass theorem that (3.4) holds for σ-almost all x in S. We will denote by $A(S)$ the class of all f in $C(S)$ such that $f^\#$ is holomorphic on B. With regard to the proof of (3.4) we remark that if $n \geqq 2$, then $\{|f| : f \in A(S)\}$ is not dense in $\{|f| : f \in C(S)\}$.

We let $\theta = a/[(\alpha^z)^{2n/p}]$. Then $\theta\bar{\theta} = 1$, $\theta \in H^\infty(S)$, and if $f \in H^p(S)$, then $Tf = \theta(\alpha^z)^{2n/p}f \circ Z$. Thus if $f = T^{-1}1$, then $f \in H^\infty(S)$ and $\bar{\theta} = (\alpha^z)^{2n/p}f \circ Z$, and hence θ is a constant. This completes the proof of Theorem 1.5.

References

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