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FINITE SOLVABLE TIDY GROUPS ARE DETERMINED BY HALL SUBGROUPS WITH TWO PRIMES

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Abstract

In this paper, we investigate finite solvable tidy groups. We prove that a solvable group with order divisible by at least two primes is tidy if all of its Hall subgroups that are divisible by only two primes are tidy.

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1. Introduction

In this paper, all groups are finite. We will refer to [8] for standard group theory results but nearly all of the results we refer to will be in any standard group theory text. Let G be a group and let $x \in G$. Define $\mathrm{Cyc}_G(x) = \{g \in G \mid \langle x, g\rangle \text{ is cyclic}\}$. It is not difficult to find examples of a group G and an element x where $\mathrm{Cyc}_G(x)$ is not a subgroup.

Following [11], a group G is said to be *tidy* if $\mathrm{Cyc}_G(x)$ is a subgroup of G for every element $x \in G$. As far as we can determine, tidy groups were introduced in [11] and in a second paper [10]. In [11], the authors study tidy groups that satisfy additional hypotheses. Perhaps the most interesting result is that if G is any group where all the Sylow subgroups are cyclic or generalised quaternion, then G is tidy. In [10], they prove that if G and H are groups of coprime order, then $G \times H$ is tidy if and only if both G and H are tidy. Applying this to nilpotent groups, it follows that a nilpotent group is tidy if and only if all of its Sylow subgroups are tidy.

In this paper, we focus on tidy solvable groups. It is not difficult to see that the tidy condition is inherited by subgroups; so the Sylow subgroups of tidy groups are tidy. Unlike the nilpotent case, we will present an example of a solvable group where all the Sylow subgroups are tidy, but the group is not tidy. However, recall that a solvable

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group G has a Hall π-subgroup for every set of primes π. It turns out that the Hall π-subgroups of G, where π has size 2, play a critical role in establishing the tidiness of the whole group G. In particular, we prove the following theorem.

THEOREM 1.1. *Suppose G is a solvable group, let $\sigma(G)$ be the set of primes dividing $|G|$ and assume $|\sigma(G)| \geq 2$. If G has a tidy Hall ρ-subgroup for each subset $\rho \subseteq \sigma(G)$ of size 2, then G is tidy.*

As the theorem shows, one of our methods of analysing tidy groups is in terms of their Hall subgroups. To obtain many of our results, we also need a second method of analysis. This method involves studying the elements of prime power order. We will show in Section 2 that it is sufficient to check whether the elements of prime power order satisfy the condition of the definition to determine if a group is tidy.

Tidy groups have also been studied in [6] where it is shown that a partitioned group is tidy if and only if the subgroups in the partition are tidy. The study of tidy groups with a partition is continued in [1–3]. Both [3, 6] consider infinite tidy groups. However, [1, 2] consider a specialisation of tidy groups. We note that the term 'tidy group' appears to have been used to refer to a very different concept in [13].

It appears that tidy groups were initially studied in their own right. However, we believe it is helpful to think of tidy groups in the context of their cyclic graphs, often called enhanced power graphs. Given a group G, the *cyclic graph* of G is the graph whose vertex set is $G \setminus \{1\}$ and there is an edge between $x, y \in G \setminus \{1\}$ if $\langle x, y\rangle$ is cyclic. It follows that $\mathrm{Cyc}_G(x) \setminus \{1\}$ is x and its set of neighbours in the cyclic graph. Hence, tidy groups are the groups where x, its neighbours and the identity form a subgroup of G for all $x \in G$. We note that [9] considers the relationship between the sets $\mathrm{Cyc}_G(x)$ and the cyclic graph of G.

2. Elements of prime power order

In this section, we explore the relationship between tidiness and the elements of prime power order. We begin by showing that we only need to consider the elements of prime power order to demonstrate tidiness.

Before presenting our preliminary lemmas, we mention an elementary result that lies at the heart of our arguments. Let G be a group and let $g \in G$. Write $o(g) = nm$, where the positive integers n and m are coprime. Then, there exist commuting elements x and y such that $g = xy$, $o(x) = n$ and $o(y) = m$. Moreover, this factorisation of g is unique; in particular, the elements x and y are powers of g. This fact is easy to prove and appears in [12, Exercise 7].

In the first lemma, we examine a central element.

LEMMA 2.1. *Suppose G is a group. Let $x \in Z(G)$ and write $p_1, \ldots, p_n$ for the prime divisors of $o(x)$. If $x = x_1 \cdots x_n$, where each x_i is a power of x and has p_i-power order, then* $\mathrm{Cyc}_G(x) = \bigcap_{i=1}^{n} \mathrm{Cyc}_G(x_i)$.

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PROOF. Consider $c \in \mathrm{Cyc}_G(x)$. This implies that $\langle x, c\rangle$ is cyclic. For each i, x_i is a power of x, so $x_i \in \langle x, c\rangle$. It follows that $\langle x_i, c\rangle$ is a subgroup of $\langle x, c\rangle$. We deduce that $\langle x_i, c\rangle$ is cyclic, so $c \in \mathrm{Cyc}_G(x_i)$. We conclude that $\mathrm{Cyc}_G(x) \subseteq \bigcap_{i=1}^n \mathrm{Cyc}_G(x_i)$.

Conversely, suppose $c \in \bigcap_{i=1}^n \mathrm{Cyc}_G(x_i)$. We can write $c = c_1 \cdots c_n \cdot c'$, where $c_1, \ldots, c_n, c'$ are powers of c and each c_i has p_i-power order and c' has π'-order where $\pi = \{p_1, \ldots, p_n\}$. Since they are all powers of c, we see that $c_1, \ldots, c_n, c'$ all commute. For each i, by hypothesis, $c \in \mathrm{Cyc}_G(x_i)$ and so $\langle c, x_i\rangle$ is cyclic. Notice that $c_i \in \langle c, x_i\rangle$, so $\langle c_i, x_i\rangle$ is cyclic. Now, c_i and x_i commute and have p_i-power order; then we have $c_i^* \in \{x_i, c_i\}$ so that $\langle c_i^*\rangle = \langle c_i, x_i\rangle$. Since the x_i are central and the c_i and c' commute with each other, we conclude that the c_i^* and c' commute with each other. Let $c^* = c_1^* \cdots c_n^* \cdot c'$. Observe that each c_i^* will be a power of c^*. Because x_i and c_i are both powers of c_i^*, we obtain $x_i, c_i \in \langle c^*\rangle$ for $i = 1, \ldots, n$. Also, c' is a power of c^*. This implies that as $x = x_1 \cdots x_n$ and $c = c_1 \cdots c_n \cdot c'$, we have both $x, c \in \langle c^*\rangle$ and $\langle x, c\rangle \leq \langle c^*\rangle$. We conclude that $c \in \mathrm{Cyc}_G(x)$. This proves the result that $\mathrm{Cyc}_G(x) = \bigcap_{i=1}^n \mathrm{Cyc}_G(x_i)$. □

We now show that if the elements of prime power order in the group satisfy the tidy condition, then all elements do. Note that if $x \in G$, then $\mathrm{Cyc}_G(x) \subseteq C_G(x)$. Replacing G by $C_G(x)$, we can think of x as being central in G and apply Lemma 2.1.

LEMMA 2.2. *Suppose G is a group. If every element $1 \neq x \in G$ having prime power order satisfies the condition that $\mathrm{Cyc}_G(x)$ is a subgroup of G, then G is a tidy group.*

PROOF. Fix the elements $g \in G$ and $x \in C_G(g)$. We obtain the equation $\mathrm{Cyc}_{C_G(g)}(x) = \mathrm{Cyc}_G(x) \cap C_G(g)$. Applying this observation to the nonidentity elements of prime power order in $C_G(g)$, we see that $C_G(g)$ satisfies the hypotheses of our lemma. Thus, we may proceed in the group $H = C_G(g)$, and so $g \in Z(H)$. Let $p_1, \ldots, p_n$ be the prime divisors of $o(g)$ and write $g = g_1 \cdots g_n$, where $g_1, \ldots, g_n$ are powers of g and g_i has p_i-power order for all i. By Lemma 2.1, $\mathrm{Cyc}_H(g) = \bigcap_{i=1}^n \mathrm{Cyc}_H(g_i)$ and since each $\mathrm{Cyc}_H(g_i)$ is a subgroup, we conclude that $\mathrm{Cyc}_H(g) = \mathrm{Cyc}_G(g)$ is a subgroup of H and thus a subgroup of G. Since g was arbitrarily chosen, we conclude G is tidy. □

With this in mind, we can focus on the elements of prime power order. Since subgroups of tidy groups are tidy, we need to focus on tidy Sylow p-subgroups. A characterisation of the tidy p-groups is given in [10, Theorem 14] gives. In [4], we use this characterisation to obtain a classification of the tidy p-groups.

THEOREM 2.3. *Let G be a p-group for some prime p. Then G is a tidy group if and only if one of the following occurs:*

(1) *G has exponent p;*
(2) *G is cyclic;*
(3) *$p = 2$ and G is dihedral or generalised quaternion.*

We first consider central elements of order p when the Sylow subgroup has exponent p.

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LEMMA 2.4. *Let G be a group and let p be a prime. Suppose $x \in Z(G)$ has order p and a Sylow p-subgroup of G has exponent p. Then $\mathrm{Cyc}_G(x)$ is a subgroup of G if and only if G has a normal p-complement. In this case, $\mathrm{Cyc}_G(x) = \langle x \rangle K$, where K is the normal p-complement of G.*

PROOF. Suppose first that $\mathrm{Cyc}_G(x)$ is a subgroup of G. Let P be a Sylow p-subgroup so that $P \cap \mathrm{Cyc}_G(x)$ is a Sylow p-subgroup of $\mathrm{Cyc}_G(x)$. Observe that $P \cap \mathrm{Cyc}_G(x) = \mathrm{Cyc}_P(x) = \langle x \rangle$ since P has exponent p. Let q be a prime other than p and let Q be a Sylow q-subgroup of G, and consider $y \in Q$. Since x and y have coprime orders and commute, we see that $\langle x, y \rangle$ is cyclic. Thus, $y \in \mathrm{Cyc}_G(x)$, so $Q \leq \mathrm{Cyc}_G(x)$. Hence, $\mathrm{Cyc}_G(x)$ contains a Sylow q-subgroup of G for every prime q other than p. Hence, $|G : \mathrm{Cyc}_G(x)|$ is a p-power.

Observe that $\langle x \rangle$ is central in $\mathrm{Cyc}_G(x)$ and is a Sylow p-subgroup. By Burnside's normal p-complement theorem [8, Theorem 5.13], $\mathrm{Cyc}_G(x)$ has a normal p-complement K and, since $\langle x \rangle$ is central, $\mathrm{Cyc}_G(x) = \langle x \rangle \times K$. Now, K is characteristic in $\mathrm{Cyc}_G(x)$. Notice that $\mathrm{Cyc}_G(x)$ is uniquely determined by x in G. Since x is central in G, it follows that $\mathrm{Cyc}_G(x)$ is normal in G. Because $\mathrm{Cyc}_G(x)$ is normal in G and has p-power index in G, we deduce that K is a normal p-complement of G as desired.

Conversely, suppose that G has a normal p-complement K. Since x is central in G, we see that $\langle x \rangle K = \langle x \rangle \times K$. This implies that $\langle x \rangle$ is the unique subgroup of $\langle x \rangle K$ that has order p. We deduce that x is a power of every element in $\langle x \rangle K$ whose order is divisible by p. Consider an element $g \in \langle x \rangle K$. If $p \mid o(g)$, then x is a power of g and $\langle x, g \rangle = \langle g \rangle$. Otherwise, $o(g)$ is coprime to $p = o(x)$ and g commutes with x, so $\langle x, g \rangle$ is cyclic. In either case, $g \in \mathrm{Cyc}_G(x)$. Hence, $\langle x \rangle K \subseteq \mathrm{Cyc}_G(x)$.

Now, suppose $c \in \mathrm{Cyc}_G(x)$. We can write $c = hk$, where h and k are powers of c, and h has p-power order and k has order coprime to p. We know that $\langle x, c \rangle$ is cyclic and $h \in \langle x, c \rangle$. Thus, $\langle x, h \rangle$ is cyclic. Now, x and h commute and have p-power orders. Thus, $\langle x, h \rangle$ is a p-subgroup of G. Since a Sylow p-subgroup of G has exponent p and $\langle x, h \rangle$ is cyclic, we see that $\langle x, h \rangle$ has order p. This implies that $h \in \langle x \rangle$. Since k has p'-order and G has a normal p-complement K, we see that $k \in K$. We conclude that $c = hk \in \langle x \rangle K$. We deduce that $\mathrm{Cyc}_G(x) \subseteq \langle x \rangle K$ and we obtain the desired equality. □

We are now able to give examples of solvable groups where all the Sylow subgroups are tidy, but the groups are not tidy. A specific example is $G = S_3 \times Z_3$. Observe that the Sylow subgroups of G are elementary abelian. Also, 3 divides $|Z(G)|$ and G does not have a normal 3-complement, so by Lemma 2.4, G is not tidy. In fact, if p and q are primes so that p divides $q - 1$ and $G = F \times Z_q$, where F is a nonabelian group of order p times q, then the Sylow subgroups of G will be elementary abelian so the Sylow subgroup are tidy. Also, q divides $|Z(G)|$, but G does not have a normal q-complement, so again Lemma 2.4 implies that G is not tidy.

When G has a cyclic Sylow p-subgroup and a central element of p-power order, then we show that $G = \mathrm{Cyc}_G(x)$ and we use Burnside's normal p-complement theorem and Fitting's theorem [8, Theorem 4.34] to show that G has a normal p-complement.

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LEMMA 2.5. *Let G be a group and let p be a prime. If G has a cyclic Sylow p-subgroup and p divides $|Z(G)|$, then G has a normal p-complement.*

PROOF. Take P to be a Sylow p-subgroup of G and note that $P \leq C_G(P)$. Thus, $|N_G(P) : C_G(P)|$ is not divisible by p. Let q be a prime that is different than p and Q be a Sylow q-subgroup of $N_G(P)$. Consider the action of Q on P and note that Fitting's theorem applies. Hence, $P = [P, Q] \times C_P(Q)$. By our hypothesis, $Z(G)$ has an element x of order p, and so, $\langle x \rangle$ is a normal p-subgroup of G. In particular, $\langle x \rangle \leq C_P(Q)$. Since P is cyclic, $\langle x \rangle$ is the only subgroup of order p in P. This forces $[P, Q] = 1$, and so, $Q \leq C_G(P)$. We have now shown that $N_G(P) = C_G(P)$, so $P \leq Z(N_G(P))$. We may now appeal to Burnside's normal p-complement theorem to conclude that G has a normal p-complement. □

For the next result, the proof is nearly the same for generalised quaternion Sylow subgroups as for cyclic subgroups, so we include both here.

LEMMA 2.6. *Let G be a group and let p be a prime. Suppose $1 \neq x \in Z(G)$ has p-power order. If a Sylow p-subgroup of G is either cyclic or generalised quaternion, then $G = \mathrm{Cyc}_G(x)$.*

PROOF. Let $o(x) = p^n$. We know $\langle x \rangle$ is central in G. When a Sylow p-subgroup is cyclic, $\langle x \rangle$ is the unique subgroup of G having order p^n, and when a Sylow p-subgroup is generalised quaternion, the centre of a Sylow 2-subgroup has order 2 and is the unique subgroup of order 2 in the Sylow subgroup. It is not difficult to see that $\langle x \rangle$ will be the unique subgroup of order 2 in G.

Let $g \in G$. We can write $g = hk$, where h and k are powers of g, and h has p-power order and k has order coprime to p. Observe that h and x commute, so $\langle h, x \rangle$ is a p-group. If the Sylow subgroup is cyclic, it is obvious that $\langle h, x \rangle$ is cyclic. However, when we have a generalised quaternion Sylow subgroup and x is in the centre, it is not difficult to see that $\langle h, x \rangle$ must be cyclic. Now, $\langle h, x \rangle$ is a cyclic p-group in both cases, and it follows that either h or x must generate this subgroup. Thus, we choose $h^* \in \{h, x\}$ so that $\langle h^* \rangle = \langle h, x \rangle$. Notice that h^* commutes with k in either case. Since h^* and k have coprime orders and commute, we see that $\langle h^*, k \rangle$ is cyclic. Now, $x, h, k \in \langle h^*, k \rangle$ implies that $\langle x, g \rangle \leq \langle h^*, k \rangle$ which is cyclic. We conclude that $g \in \mathrm{Cyc}_G(x)$, so $G \subseteq \mathrm{Cyc}_G(x)$. Since the other containment is obvious, we have the desired conclusion. □

We next consider a central element of order 2 in a group whose Sylow 2-subgroups are dihedral and there is a normal 2-complement. In the next lemma, we make use of a result proved by Gorenstein and Walter in [7] regarding groups that have a dihedral subgroup as a Sylow 2-subgroup.

LEMMA 2.7. *Let G be a group and let $x \in Z(G)$ have order 2. Suppose a Sylow 2-subgroup T of G is dihedral and write D for the cyclic subgroup of index 2 in T. Then G has a normal 2-complement K and $\mathrm{Cyc}_G(x) = DK$. In particular, $\mathrm{Cyc}_G(x)$ is a subgroup of G.*

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PROOF. As in [7, Lemma 8], Burnside's normal p-complement theorem implies that any group that has a dihedral Sylow 2-subgroup and a central element of order 2 will have a normal 2-complement. Thus, G has a normal 2-complement K. Notice that DK has a cyclic Sylow 2-subgroup. By Lemma 2.6, $DK = \mathrm{Cyc}_{DK}(x) \subseteq \mathrm{Cyc}_G(x)$. Suppose $c \in \mathrm{Cyc}_G(x)$. We can write $c = hk$, where h and k are powers of c, h has 2-power order and k has $2'$-order. We know that $\langle x, c\rangle$ is cyclic. Since h is a power of c, we have $h \in \langle x, c\rangle$, and so $\langle x, h\rangle$ is cyclic. We see that x and h commute, so $\langle x, h\rangle$ is a cyclic 2-group. This implies that $\langle x, h\rangle$ must lie in a conjugate of D. Since D has index 2 in T, we see that DK has index 2 in G and so is normal in G. Thus, $h \in DK$. Since K is a normal 2-complement and k has $2'$-order, we see that $k \in K$. Hence, $c = hk \in DK$. We have shown that $\mathrm{Cyc}_G(x) \subseteq DK$. We conclude $\mathrm{Cyc}_G(x) = DK$, as desired. □

Groups where all Sylow subgroups are cyclic or generalised quaternion are tidy (see [11, Theorem 6]). We can extend this to include Sylow subgroups that are dihedral.

COROLLARY 2.8. *Suppose G is a group where all Sylow subgroups are cyclic, generalised quaternion or dihedral. Then G is tidy.*

PROOF. By Lemma 2.2, it suffices to prove that $\mathrm{Cyc}_G(x)$ is a subgroup whenever $1 \neq x \in G$ has prime power order. Suppose $1 \neq x$ has p-power order for some prime p. Then a Sylow p-subgroup of $C_G(x)$ is either cyclic, generalised quaternion or dihedral. By Lemmas 2.6 and 2.7, this implies that $\mathrm{Cyc}_G(x)$ is a subgroup, proving the result. □

We now come to the proof of Theorem 1.1. To prove this theorem, we use the idea of a Sylow system. Let G be a group. Following the terminology in [8], a Sylow system in G is a collection $\mathcal{S}$ consisting of exactly one Sylow p-subgroup of G for each prime p that divides $|G|$ such that $PQ = QP$ for every $P, Q \in \mathcal{S}$. By [8, Problem 3C.3(b)], every solvable group has a Sylow system. We shall use this fact in the following proof. (For a further discussion of Sylow systems and related concepts in finite solvable groups, see the comprehensive treatment in [5].)

PROOF OF THEOREM 1.1. By Lemma 2.2, it suffices to show that $\mathrm{Cyc}_G(x)$ is a subgroup of G for all $1 \neq x \in G$ such that x has prime power order. Suppose that $1 \neq x \in G$ has p-power order for some prime p. Notice that $C_G(x)$ has tidy Hall ρ-subgroups for each subset $\rho \subseteq \sigma(G)$ with $|\rho| = 2$. Since $\mathrm{Cyc}_G(x) = \mathrm{Cyc}_{C_G(x)}(x)$, we may assume that $G = C_G(x)$. In particular, we assume that x is central in G. Also, since we are assuming that all Hall ρ-subgroups of G are tidy when ρ is a two element subset of $\sigma(G)$, we see that G must have a tidy Sylow p-subgroup. If a Sylow p-subgroup of G is either cyclic or generalised quaternion, we have the result by Lemma 2.6. Next, if a Sylow p-subgroup of G is dihedral, then we may apply Lemma 2.7 to see that $\mathrm{Cyc}_G(x)$ is a subgroup. Thus, by Theorem 2.3, we may assume that a Sylow p-subgroup of G has exponent p. By Lemma 2.4, it suffices to prove that G has a normal p-complement. Write $\sigma(G) = \{p_1 = p, p_2, \ldots, p_n\}$ and let $\{P_i \mid i = 1, \ldots n\}$ be a Sylow system for G. That is, $P_i \in \mathrm{Syl}_{p_i}(G)$ and P_iP_j is a subgroup for all i, j. In particular, P_1P_i is a Hall

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$\{p, p_i\}$-subgroup of G; so by hypothesis, it is tidy. Applying Lemma 2.4, we see that P_1P_i must have a normal p-complement. That is, P must normalise P_i. This implies that P normalises $N = P_2 \cdots P_n$ and we see that N is a normal p-complement. This proves the desired conclusion. □

Since nonsolvable groups generally do not have Hall $\{p, q\}$-subgroups for all primes p and q, one cannot expect to remove the hypothesis that G is solvable from Theorem 1.1 and have any hope that the conclusion is still true. However, we wonder if the following might be true. Let G be a group. If all of the $\{p, q\}$-subgroups of G are tidy for all primes p and q, then will G be tidy?

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