Canad. J. Math. Vol. **63** (4), 2011 pp. 755–797
doi:10.4153/CJM-2011-026-1
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On the Geometry of the Moduli Space of Real Binary Octics

Kenneth C. K. Chu

Abstract. The moduli space of smooth real binary octics has five connected components. They parametrize the real binary octics whose defining equations have $0, \dots, 4$ complex-conjugate pairs of roots respectively. We show that each of these five components has a real hyperbolic structure in the sense that each is isomorphic as a real-analytic manifold to the quotient of an open dense subset of 5-dimensional real hyperbolic space $\mathbb{RH}^5$ by the action of an arithmetic subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$. These subgroups are commensurable to discrete hyperbolic reflection groups, and the Vinberg diagrams of the latter are computed.

1 Introduction

A *(complex) binary octic* refers to a hypersurface of degree eight in the complex projective line $\mathbb{CP}^1$; in other words, it is the set of roots in $\mathbb{CP}^1$ of a homogeneous polynomial of degree eight in two variables with complex coefficients. One can think of a binary octic as an 8-point configuration in $\mathbb{CP}^1$, counting multiplicity. A binary octic is said to be *smooth* if it is smooth as a hypersurface in $\mathbb{CP}^1$; equivalently, it is smooth if the eight roots of any of its defining polynomials are pairwise distinct. The GIT-stable (or more briefly, stable) binary octics are those with at worst triple-point singularities. A *real binary octic* is a binary octic that is preserved by complex conjugation on $\mathbb{CP}^1$.

Using periods of certain branched covers of $\mathbb{CP}^1$, Deligne–Mostow [7], Terada [16], [15], Matsumoto–Yoshida [12] have described the arithmetic hyperbolic 5-ball quotient structure of the moduli space $\mathcal{M}_s$ of stable complex binary octics. The use of periods of curves is classical, for instance, in the construction of the moduli space of elliptic curves and Picard curves [14]. Kondō [11] produced the same description of $\mathcal{M}_s$ using periods of $K3$ surfaces.

Following the approach of Allcock–Carlson–Toledo in [3] for real cubic surfaces and [2] for real binary sextics, this article describes how the Deligne–Mostow construction of the moduli space of complex binary octics induces a real hyperbolic structure on each connected component of the moduli space of smooth real binary octics. This result is stated as Theorem 6.1. Unlike in [3] and [2], the scalar ring involved here is the Gaussian integers and the lattice involved is no longer unimodular. These lead to considerable added computational complexities, as well as the unforeseen semi-direct product structure of one of the monodromy groups.

Received by the editors August 7, 2007; revised January 3, 2010.
Published electronically April 30, 2011.
AMS subject classification: **32G13**, 32G20, 14D05, 14D20.
Keywords: real binary octics, moduli space, complex hyperbolic geometry, Vinberg algorithm.

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

2 The Moduli Space of Complex Binary Octics as an Arithmetic Quotient of $\mathbb{CH}^5$

2.1 The Fibration of Cyclic Covers Branched over Octics and the Hermitian Structure of the Cohomology of its Fiber

Define

$$\mathfrak{X} := \{(p, [x_0 : x_1 : y]) \in \mathcal{P} \times \mathbb{P}(1,1,2) \mid y^4 - p(x_0, x_1) = 0\},$$

where $\mathcal{P}$ is the space of all binary octic forms and $\mathbb{P}(1,1,2)$ is the weighted projective space of weights $(1,1,2)$, which is defined as follows:

$$\mathbb{P}(1,1,2) := (\mathbb{C}^3 - \{0\})/\sim,$$

where the equivalence relation $\sim$ is given by: $(x_0, x_1, y) \sim (x_0', x_1', y')$ if and only if there exists $\lambda \in \mathbb{C} - \{0\}$ such that $(x_0', x_1', y') = (\lambda x_0, \lambda x_1, \lambda^2 y)$. $\mathbb{P}(1,1,2)$ is a projective variety isomorphic to the cone in $\mathbb{CP}^3$ with apex $[0 : 0 : 0 : 1]$ over a conic plane curve in $\{[Z_0 : Z_1 : Z_2 : 0] \in \mathbb{CP}^3\} \cong \mathbb{CP}^2$. See [10], for example, for this isomorphism. The cone point of $\mathbb{P}(1,1,2)$ is $[0 : 0 : 1] \in \mathbb{P}(1,1,2)$ and it is the unique singular point of $\mathbb{P}(1,1,2)$. The hypersurface $\mathfrak{X}$ does not contain this singular point.

Define the maps

$$\sigma\colon \mathfrak{X} \to \mathfrak{X}\colon (p, [x_0 : x_1 : y]) \longmapsto (p, [x_0 : x_1 : \mathbf{i}y]),$$

$$\Pi\colon \mathfrak{X} \to \mathcal{P}\colon (p, [x_0 : x_1 : y]) \longmapsto p,$$

$$\pi\colon \mathfrak{X} \to \mathbb{CP}^1\colon (p, [x_0 : x_1 : y]) \longmapsto [x_0 : x_1].$$

Let $\mathcal{P}_0$ be the space of smooth binary octic forms (homogeneous binary polynomials of degree eight). Let $\mathfrak{X}_0 := \Pi^{-1}(\mathcal{P}_0)$. Then for each $p \in \mathcal{P}_0$, the fiber

$$X_p := \Pi^{-1}(p) = \{[x_0 : x_1 : y] \in \mathbb{P}(1,1,2) \mid y^4 - p(x_0, x_1) = 0\}$$

is a (smooth) compact Riemann surface. The map $\sigma\colon \mathfrak{X} \to \mathfrak{X}$ induces a cyclic action on $\mathfrak{X}$ of order 4. σ preserves every fiber of Π, hence restricts to a cyclic action of order 4 on each fiber $X_p := \Pi^{-1}(p)$, $p \in \mathcal{P}_0$. The map $\pi\colon \mathfrak{X} \to \mathbb{CP}^1$ is well-defined since $[0 : 0 : 1] \in \mathbb{P}(1,1,2) - \mathfrak{X}$. Observe that for each $p \in \mathcal{P}_0$, the restricted map $\pi|_{X_p}\colon X_p \to \mathbb{CP}^1$ is a cyclic cover of $\mathbb{CP}^1$ of degree 4 branched over the eight distinct roots of $p(x_0, x_1)$ in $\mathbb{CP}^1$, and it has exactly eight ramification points, each with ramification index 4. By the Riemann–Hurwitz theorem, $g(X_p) = h^{1,0}(X_p) = 9$, for each $p \in \mathcal{P}_0$. Thus, $\mathfrak{X}_0 \xrightarrow{\Pi} \mathcal{P}_0$ is a fibration whose fiber over each $p \in \mathcal{P}_0$ is the compact Riemann surface $X_p := \Pi^{-1}(p)$, which has genus 9 and is a cyclic covering of $\mathbb{CP}^1$ branched over the roots in $\mathbb{CP}^1$ of the polynomial $p(x_0, x_1)$.

For economy of notation, we denote also by σ the restriction of $\sigma\colon \mathfrak{X} \to \mathfrak{X}$ to the fiber $X_p = \Pi^{-1}(p)$, for any $p \in \mathcal{P}_0$. Similarly, we denote also by σ the induced map on the $H^1(X_p, \mathbb{Z})$ or $H^1(X_p, \mathbb{C})$.

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Lemma 2.1 *The map $\sigma\colon H^1(X_p,\mathbb{C})\to H^1(X_p,\mathbb{C})$ preserves the Hodge decomposition $H^1(X,\mathbb{C})=H^{1,0}(X_p)\oplus H^{0,1}(X_p)$, and*

$$H^{1,0}(X_p)=H^{1,0}_{\sigma=-1}(X_p)\oplus H^{1,0}_{\sigma=\mathbf{i}}(X_p)\oplus H^{1,0}_{\sigma=-\mathbf{i}}(X_p),$$

where $H^{1,0}_{\sigma=-1}(X_p)$, $H^{1,0}_{\sigma=\mathbf{i}}(X_p)$, and $H^{1,0}_{\sigma=-\mathbf{i}}(X_p)$ are the (-1)-, $(\mathbf{i})$-, and $(-\mathbf{i})$-eigenspaces of $\sigma\colon H^{1,0}(X_p)\to H^{1,0}(X_p)$, respectively. Furthermore,

$$\dim_{\mathbb{C}} H^{1,0}_{\sigma=-1}(X_p)=3,\quad \dim_{\mathbb{C}} H^{1,0}_{\sigma=\mathbf{i}}(X_p)=5,\quad \textit{and}\quad \dim_{\mathbb{C}} H^{1,0}_{\sigma=-\mathbf{i}}(X_p)=1.$$

Proof Since $X_p\xrightarrow{\sigma}X_p$ is holomorphic, $H^1(X_p,\mathbb{C})\xrightarrow{\sigma}H^1(X_p,\mathbb{C})$ preserves the Hodge decomposition $H^1(X_p,\mathbb{C})\cong H^{1,0}(X_p)\oplus H^{0,1}(X_p)$. So restriction of σ to $H^{1,0}(X_p)$ yields a map $H^{1,0}(X_p)\xrightarrow{\sigma}H^{1,0}(X_p)$. Since $\sigma|_{H^{1,0}(X_p)}$ still satisfies the identity $\sigma^4-1=0$, and 1 is not an eigenvalue of σ (because $H^1(\mathbb{CP}^1,\mathbb{C})=\{0\}$), it immediately follows that $H^{1,0}(X_p)=H^{1,0}_{\sigma=-1}(X_p)\oplus H^{1,0}_{\sigma=\mathbf{i}}(X_p)\oplus H^{1,0}_{\sigma=-\mathbf{i}}(X_p)$.

To compute the indicated dimensions, it is sufficient to compute it for a polynomial $p(x_0,x_1)$ whose roots do not include $0=[0:1],\infty=[1:0]\in\mathbb{CP}^1$ and such that $p(x_0,1)$ is a monic polynomial in x_0. Such an X_p is isomorphic to the completion X of the affine plane curve $\{(x,y)\in\mathbb{C}^2 \mid y^4-p(x,1)=0\}$. See [13] for this completion process.

The Riemann surface X also has an order-four cyclic action, and the induced action on the cohomology of X. We denote both these actions also by σ. Straightforward verifications show that $\frac{dx}{y^2}$, $\frac{x\,dx}{y^2}$, and $\frac{x^2\,dx}{y^2}$ define holomorphic 1-forms on X and belong to the (-1)-eigenspace of the cyclic action σ on the cohomology of X. Similarly, $\frac{dx}{y^3}$, $\frac{x\,dx}{y^3}$, $\frac{x^2\,dx}{y^3}$, $\frac{x^3\,dx}{y^3}$, $\frac{x^4\,dx}{y^3}$ define holomorphic 1-forms on X belonging to the $(+\mathbf{i})$-eigenspace of σ, and $\frac{dx}{y}$ defines a holomorphic 1-form on X belonging to the $(-\mathbf{i})$-eigenspace of σ. These nine holomorphic 1-forms are linearly independent over $\mathbb{C}$, and since $\dim_{\mathbb{C}} H^{1,0}(X)=\dim_{\mathbb{C}} H^{1,0}(X_p)=9$, we see that their cohomology classes form a basis for $H^{1,0}(X)$. Consequently, $\dim_{\mathbb{C}} H^{1,0}_{\sigma=-1}(X)=3$, $\dim_{\mathbb{C}} H^{1,0}_{\sigma=+\mathbf{i}}(X)=5$, and $\dim_{\mathbb{C}} H^{1,0}_{\sigma=-\mathbf{i}}(X)=1$. Since X_p and X are isomorphic as cyclic branched covers of $\mathbb{CP}^1$, the lemma follows. ∎

Next, for each $p\in\mathcal{P}_0$, define

$$\Lambda(X_p):=H^1_{\sigma^2=-1}(X_p,\mathbb{Z}):=\{\phi\in H^1(X_p,\mathbb{Z})\mid \sigma^2(\phi)=-\phi\}.$$

Then $\sigma|_{\Lambda(X_p)}$ satisfies $\sigma^2+1=0$. Consequently, if we define multiplication by $-\mathbf{i}$ in $\Lambda(X_p)$ by

$$-\mathbf{i}\cdot\phi:=\sigma(\phi),$$

then $\Lambda(X_p)$ becomes a $\mathbb{Z}[\mathbf{i}]$-module. We need to define the action of σ to be multiplication by $-\mathbf{i}$ because we want to embed $\Lambda(X_p)$ into $H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$. This is because the $(1,0)$-summand of $H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$ is 1-dimensional (see Lemma 2.1 or Proposition 2.3), which will eventually yield a holomorphic complex period map into complex hyperbolic space; see Section 2.3. If we defined the action of σ to be multiplication by $\mathbf{i}$, then $\Lambda(X_p)$ would be embedded into $H^1_{\sigma=\mathbf{i}}(X_p,\mathbb{C})$, whose 1-dimensional

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summand under Hodge decomposition is its $(0,1)$-summand. The resulting period map into complex hyperbolic space would then be anti-holomorphic instead.

Proposition 2.2 *With the above $\mathbb{Z}[\mathbf{i}]$-module structure, $\Lambda(X_p)$ becomes a free $\mathbb{Z}[\mathbf{i}]$-module of rank 6.*

Proof First, note that $\Lambda(X_p)$ is torsion-free as a $\mathbb{Z}$-module, being a $\mathbb{Z}$-submodule of the free $\mathbb{Z}$-module $H^1(X_p,\mathbb{Z}) \cong \mathbb{Z}^{18}$. And, elementary arguments show that, for any element $\phi \in \Lambda(X_p)$, $\sigma(\phi)$ cannot be a $\mathbb{Z}$-multiple of ϕ. These two observations together imply that $\Lambda(X_p)$ is torsion-free over $\mathbb{Z}[\mathbf{i}]$. Since $\mathbb{Z}[\mathbf{i}]$ is a PID, $\Lambda(X_p)$ is a free $\mathbb{Z}[\mathbf{i}]$-module. Next, note that

$$\begin{aligned}
\operatorname{rank}_{\mathbb{Z}[\mathbf{i}]} \Lambda(X_p) &= \frac{1}{2}\cdot \operatorname{rank}_{\mathbb{Z}} \Lambda(X_p) = \frac{1}{2}\cdot \operatorname{rank}_{\mathbb{Z}} H^1_{\sigma^2=-1}(X_p,\mathbb{Z}) \\
&= \frac{1}{2}\cdot \dim_{\mathbb{C}} H^1_{\sigma^2=-1}(X_p,\mathbb{C}) \\
&= \frac{1}{2}\cdot \{\dim_{\mathbb{C}} H^{1,0}_{\sigma^2=-1}(X_p) + \dim_{\mathbb{C}} H^{0,1}_{\sigma^2=-1}(X_p)\} \\
&= \frac{1}{2}\cdot 2\cdot \dim_{\mathbb{C}} H^{1,0}_{\sigma^2=-1}(X_p) = \dim_{\mathbb{C}} H^{1,0}_{\sigma=\mathbf{i}}(X_p) + \dim_{\mathbb{C}} H^{1,0}_{\sigma=-\mathbf{i}}(X_p) \\
&= 6,
\end{aligned}$$

where Lemma 2.1 was used in the second last equality. ∎

Consider the embedding $\Lambda(X_p) \hookrightarrow H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$ induced by

$$\begin{array}{ccccccc}
H^1_{\sigma^2=-1}(X_p,\mathbb{Z}) & \hookrightarrow & H^1_{\sigma^2=-1}(X_p)\otimes_{\mathbb{Z}}\mathbb{R} & \hookrightarrow & H^1_{\sigma^2=-1}(X_p,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} & \xrightarrow{\sim} & H^1_{\sigma^2=-1}(X_p,\mathbb{C}) \\
\| & & & & & & \| \\
\Lambda(X_p) & & & & & & H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C}) \oplus H^1_{\sigma=\mathbf{i}}(X_p,\mathbb{C}) \\
& \searrow & & & & & \downarrow \\
& & & & & & H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})
\end{array}$$

That the above composition $\Lambda(X_p) \to H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$ is indeed injective follows from the fact that if V is a real vector space with a complex structure J, then the projection map $V\otimes_{\mathbb{R}}\mathbb{C} \to V^{(0,1)}\colon v \longmapsto v + \mathbf{i}J(v)$ maps V bijectively onto $V^{(0,1)}$, where $V^{(0,1)}$ is the $(-\mathbf{i})$-eigenspace of the $\mathbb{C}$-linear extension of J to $V\otimes_{\mathbb{R}}\mathbb{C}$.

Let $h'\colon H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C}) \times H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C}) \to \mathbb{C}$ be the Hermitian form given by

$$(\alpha,\beta) \overset{h'}{\longmapsto} 2\mathbf{i}\int_{X_p} \alpha\wedge\overline{\beta}.$$

The above Hermitian form induces a $\mathbb{Z}[\mathbf{i}]$-lattice (*i.e.*, a $\mathbb{Z}[\mathbf{i}]$-module equipped with a $\mathbb{Z}[\mathbf{i}]$-valued Hermitian form) structure on $\Lambda(X_p)$, as the following Proposition shows:

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Proposition 2.3

(i) *h' is positive-definite on $H^{1,0}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$ and negative-definite $H^{0,1}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$. Consequently, $\big(H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C}),h'\big)$ is isometric to the standard Lorentzian–Hermitian space $\mathbb{C}^{1,5}=\mathbb{C}^{1+,5-}$.*

(ii) *Let h be the pullback to $\Lambda(X_p)$ of the Lorentzian–Hermitian form*

$$h'\colon H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})\times H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})\to\mathbb{C}$$

by the embedding $\Lambda(X_p)\hookrightarrow H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$. Then h is in fact $\mathbb{Z}[\mathbf{i}]$-valued on $\Lambda(X_p)\times\Lambda(X_p)$, and it is a $\mathbb{Z}[\mathbf{i}]$-valued Hermitian form on $\Lambda(X_p)$ given by the following formula:

$$h(\xi,\eta)=\Omega\big(\xi,\sigma(\eta)\big)+\mathbf{i}\Omega(\xi,\eta),\quad \textit{for any } \xi,\eta\in\Lambda(X_p),$$

where $\Omega\colon H^1(X_p,\mathbb{Z})\times H^1(X_p,\mathbb{Z})\to\mathbb{Z}$ is given by

$$\Omega(\alpha,\beta):=\langle\alpha\cup\beta,[X_p]\rangle.$$

(iii) *The Lorentzian $\mathbb{Z}[\mathbf{i}]$-valued Hermitian quadratic form on $\Lambda(X_p)$ constructed as in* (ii) *is abstractly isometric to the following $\mathbb{Z}[\mathbf{i}]$-lattice:*

$$\Lambda:=\left(\mathbb{Z}[\mathbf{i}]^6,\begin{bmatrix}-2&1+\mathbf{i}\\1-\mathbf{i}&-2\end{bmatrix}\oplus\begin{bmatrix}-2&1+\mathbf{i}\\1-\mathbf{i}&-2\end{bmatrix}\oplus\begin{bmatrix}0&1+\mathbf{i}\\1-\mathbf{i}&0\end{bmatrix}\right).$$

Proof

(i) Let $z=x+\mathbf{i}y$ be a local coordinate of the Riemann surface X_p, with real and imaginary parts x and y, respectively. Then

$$\mathrm{d}z\wedge\mathrm{d}\bar{z}=\mathrm{d}(x+\mathbf{i}\,y)\wedge\mathrm{d}(x-\mathbf{i}\,y)=-\mathbf{i}\,\mathrm{d}x\wedge\mathrm{d}y+\mathbf{i}\,\mathrm{d}y\wedge\mathrm{d}x=-2\mathbf{i}\,\mathrm{d}x\wedge\mathrm{d}y.$$

Hence, $2\mathbf{i}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}=4\,\mathrm{d}x\wedge\mathrm{d}y$, which immediately shows that h' is positive-definite on holomorphic 1-forms and negative-definite on antiholomorphic 1-forms.

(ii) Write Z for the embedding $\Lambda(X_p)\overset{Z}{\hookrightarrow}H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$. For an arbitrary $\xi\in\Lambda(X_p)\subset H^1_{\sigma^2=-1}(X_p,\mathbb{C})=H^1_{\sigma=\mathbf{i}}(X_p,\mathbb{C})\oplus H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$, write $\xi=\xi_\iota+\xi_{\bar\iota}$, where $\xi_\iota\in H^1_{\sigma=\mathbf{i}}(X_p,\mathbb{C})$ and $\xi_{\bar\iota}\in H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$. Of course, we then have $Z(\xi)=\xi_{\bar\iota}$. Now, note that

$$\begin{aligned}\int_{X_p}\xi_{\bar\iota}\wedge\overline{\xi_{\bar\iota}}&=\int_{X_p}\xi_{\bar\iota}\wedge\xi_\iota=\frac{1}{2}\int_{X_p}\xi_{\bar\iota}\wedge\xi_\iota-\frac{1}{2}\int_{X_p}\xi_\iota\wedge\xi_{\bar\iota}\\&=\frac{1}{2}\int_{X_p}\xi_{\bar\iota}\wedge\xi_\iota+\frac{1}{2}\int_{X_p}\xi_\iota\wedge\xi_\iota-\frac{1}{2}\int_{X_p}\xi_{\bar\iota}\wedge\xi_{\bar\iota}-\frac{1}{2}\int_{X_p}\xi_\iota\wedge\xi_{\bar\iota}\\&=\frac{1}{2}\int_{X_p}(\xi_{\bar\iota}+\xi_\iota)\wedge\xi_\iota-\frac{1}{2}\int_{X_p}(\xi_{\bar\iota}+\xi_\iota)\wedge\xi_{\bar\iota}\\&=\frac{1}{2}\int_{X_p}\xi\wedge\xi_\iota-\frac{1}{2}\int_{X_p}\xi\wedge\xi_{\bar\iota}\\&=\frac{1}{2}\int_{X_p}\xi\wedge(\xi_\iota-\xi_{\bar\iota}).\end{aligned}$$

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Hence,

$$h'\big(Z(\xi), Z(\xi)\big) = 2\mathbf{i}\int_{X_p} \xi_{\bar\iota}\wedge\overline{\xi_{\bar\iota}} = 2\mathbf{i}\cdot\frac{1}{2}\int_{X_p}\xi\wedge(\xi_\iota - \xi_{\bar\iota}) = \int_{X_p}\xi\wedge(\mathbf{i}\xi_\iota - \mathbf{i}\xi_{\bar\iota})$$
$$= \int_{X_p}\xi\wedge\big(\sigma(\xi_\iota)+\sigma(\xi_{\bar\iota})\big) = \int_{X_p}\xi\wedge\sigma(\xi) = \Omega\big(\xi,\sigma(\xi)\big).$$

So we now know that the quadratic form $Q\colon \Lambda(X_p) \to \mathbb{C}$ associated with the Hermitian form $\Lambda(X_p)\times\Lambda(X_p) \xrightarrow{h} \mathbb{C}$ is in fact $\mathbb{Z}$-valued and is given by $Q(\xi) = \Omega\big(\xi,\sigma(\xi)\big)$. This immediately implies that the real part $h_{\mathrm{symm}}(\cdot,\cdot)$ of $h(\cdot,\cdot) = h_{\mathrm{symm}}(\cdot,\cdot) + \mathbf{i}h_{\mathrm{skew}}(\cdot,\cdot)$ is also $\mathbb{Z}$-valued and is given by $h_{\mathrm{symm}}(\xi,\eta) = \Omega\big(\xi,\sigma(\eta)\big)$.

Next, recall that the real and imaginary parts of a general Hermitian form $H(\xi,\eta) = F(\xi,\eta) + \mathbf{i}G(\xi,\eta)$ are related by: $F\big(J(\xi),\eta\big) = -G(\xi,\eta)$, where J is the pertinent complex structure. See, for example, [5, Section 7.2]. In our context, $\sigma = -J$, since σ is multiplication by $-\mathbf{i}$, and $F(\xi,\eta) = h_{\mathrm{symm}}(\xi,\eta) = \Omega\big(\xi,\sigma(\eta)\big)$. Thus, we have $-h_{\mathrm{skew}}(\xi,\eta) = -G(\xi,\eta) = F\big(J(\xi),\eta\big) = \Omega\big(-\sigma(\xi),\sigma(\eta)\big) = -\Omega(\xi,\eta)$, which implies

$$h_{\mathrm{skew}}(\xi,\eta) = \Omega(\xi,\eta).$$

We may now conclude that $h\colon \Lambda(X_p)\times\Lambda(X_p)\to\mathbb{C}$ is in fact $\mathbb{Z}[\mathbf{i}]$-valued and it is given by the following formula:

$$h(\xi,\eta) = \Omega\big(\xi,\sigma(\eta)\big) + \mathbf{i}\Omega(\xi,\eta).$$

(iii) The inner product matrix of the $\mathbb{Z}[\mathbf{i}]$-lattice $\Lambda(X_p)$ with respect to the basis for $\Lambda(X_p)$ chosen in Section 6 of [12] (pp. 273–274 thereof) is

$$B := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1+\mathbf{i} \\ 0 & -2 & -1+\mathbf{i} & 0 & 0 & 0 \\ 0 & -1-\mathbf{i} & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1+\mathbf{i} & 0 \\ 0 & 0 & 0 & -1-\mathbf{i} & -2 & 0 \\ -1-\mathbf{i} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is readily seen to be equivalent over $\mathbb{Z}[\mathbf{i}]$ to

$$A := \begin{bmatrix} -2 & 1+\mathbf{i} & 0 & 0 & 0 & 0 \\ 1-\mathbf{i} & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1+\mathbf{i} & 0 & 0 \\ 0 & 0 & 1-\mathbf{i} & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\mathbf{i} \\ 0 & 0 & 0 & 0 & 1-\mathbf{i} & 0 \end{bmatrix}.$$

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More precisely, let

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{i} \end{bmatrix} \in \mathbb{Z}[\mathbf{i}]^{6\times 6}.$$

Then $\det(M) = \mathbf{i}$; hence, $M \in \mathrm{GL}(\mathbb{Z}[\mathbf{i}], 6)$. Furthermore, the following equality holds:

$$A = M^t B \overline{M}.$$

This proves statement (iii). ■

2.2 The Space of Framed Octic Forms

In this section, we describe the space of framed smooth octic forms and its Fox completion [8] over the stable octic forms; this Fox completion is called the space of framed stable octic forms. These spaces of framed octic forms are the domains of the period maps described in the subsequent sections. The complex ball quotient structure of $\mathcal{M}_s$ arises through these period maps. We omit all proofs, but refer to [1], which treats the analogous case of the complex cubic surfaces.

Definition 2.4 A *framed smooth octic form* over $p \in \mathcal{P}_0$ is a "projective equivalence class" of an (abstract) isometry of $\Lambda(X_p) \xrightarrow{\sim} \Lambda$, where two such isometries are said to be "projectively equivalent" if one is a $\mathbb{Z}[\mathbf{i}]$-unit scalar multiple of the other.

Let $\Lambda(\mathfrak{X}_0)$ be the sheaf over $\mathcal{P}_0$ associated with the presheaf

$$U \mapsto H^1_{\sigma^2=-1}\big(\Pi^{-1}(U), \mathbb{Z}\big).$$

Proposition 2.3 (iii) implies that $\Lambda(\mathfrak{X}_0)$ is a sheaf over $\mathcal{P}_0$ of $\mathbb{Z}[\mathbf{i}]$-valued Hermitian modules, with stalks isomorphic to the rank-six $\mathbb{Z}[\mathbf{i}]$-lattice Λ. Let $\mathbb{P}\mathrm{Hom}\big(\Lambda(\mathfrak{X}_0), \mathcal{P}_0 \times \Lambda\big)$ be the sheaf of projective equivalence classes of sheaf homomorphisms from $\Lambda(\mathfrak{X}_0)$ to $\mathcal{P}_0 \times \Lambda$.

Definition 2.5 The *space* $\mathcal{F}_0$ *of framed smooth octic forms* over $\mathcal{P}_0$ is the subsheaf of $\mathbb{P}\mathrm{Hom}\big(\Lambda(\mathfrak{X}_0), \mathcal{P}_0 \times \Lambda\big)$ consisting of projective equivalence classes of sheaf homomorphisms $\Lambda(\mathfrak{X}_0) \to \mathcal{P}_0 \times \Lambda$ that restrict to an isometry on each stalk.

$\mathcal{F}_0$ is a complex manifold and its stalks are the framed smooth octic forms, as defined in Definition 2.4. $\mathcal{F}_0$ can be alternatively described as the Galois covering of $\mathcal{P}_0$ associated with the kernel of the "projectivized monodromy representation"

$$\mathbb{P}\rho\colon \pi_1(\mathcal{P}_0, p_0) \to \mathbb{P}\,\mathrm{Isom}\big(\Lambda(X_{p_0})\big) \cong \mathbb{P}\,\mathrm{Isom}(\Lambda),$$

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which of course derives from the standard monodromy representation

$$\rho\colon \pi_1(\mathcal{P}_0, p_0) \to \operatorname{Isom}\big(\Lambda(X_{p_0})\big),$$

where $p_0 \in \mathcal{P}_0$ is an arbitrary but fixed smooth octic. It is clear from this description of $\mathcal{F}_0$ as a Galois covering over a path-connected base space that it is connected. The monodromy group—and the deck transformation group—$\rho\big(\pi_1(\mathcal{P}_0, p_0)\big) \subset \mathbb{P}\operatorname{Isom}(\Lambda)$ turns out to be all of $\mathbb{P}\operatorname{Isom}(\Lambda)$. So, $\mathbb{P}\Gamma := \mathbb{P}\operatorname{Isom}(\Lambda)$ acts on $\mathcal{F}_0$ as deck transformations, and $\mathbb{P}\Gamma\backslash\mathcal{F}_0 \cong \mathcal{P}_0$.

Let $G := \mathrm{GL}(2,\mathbb{C})/\langle\text{all eighth roots of unity}\rangle$. G acts naturally on $\mathcal{P}_0$ (by linear change of variables). This action extends to a free action on $\mathcal{F}_0$ via the induced action on cohomology (see Sections (2.10) and (3.10) in [1] for the analogous case of cubic surfaces).

Next, let $\mathcal{P}_s$ be the space of all stable binary octic forms and $\mathcal{F}_s \to \mathcal{P}_s$ be the Fox completion (see [8]) of the covering $\mathcal{F}_0 \to \mathcal{P}_0$ over $\mathcal{P}_s$. $\mathcal{F}_s$ is a branched covering of $\mathcal{P}_s$ with four-fold branching over $\Delta^1_s \subset \mathcal{P}_s$, the locus in $\mathcal{P}_s$ corresponding to octics with one double point and no other singularities. Intuitively, $\mathcal{F}_s$ coincides with $\mathcal{F}_0$ over $\mathcal{P}_0$, and, for a singular octic $p \in \Delta^1_s$, $\mathcal{F}_s$ retains information about the vanishing cohomology corresponding to the singularities of p. We call $\mathcal{F}_s$ the *space of framed stable octic forms*.

The actions of G and $\mathbb{P}\Gamma$ on $\mathcal{F}_0$ extend naturally to $\mathcal{F}_s$, and it can be shown that $\mathbb{P}\Gamma\backslash\mathcal{F}_s \cong \mathcal{P}_s$.

2.3 The Complex Period Map and the $\mathbb{CH}^5$ Quotient Structure of $\mathcal{M}_s$

The period map of interest to us is defined as follows:

$$\begin{array}{ccc} \mathcal{F}_0 & \xrightarrow{\mathfrak{p}} & \mathbb{CH}^5 = \mathbb{CH}(\Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C}) \\ \Big[\Lambda(X_p) \xrightarrow{i} \Lambda\Big] & \longmapsto & i\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p)\big). \end{array}$$

Note that $\mathbb{P}\Gamma = \mathbb{P}\operatorname{Isom}(\Lambda)$ naturally acts on $\mathbb{CH}^5 = \mathbb{CH}(\Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C})$. The period map $\mathfrak{p}$ is holomorphic because the Hodge filtration varies holomorphically (see, for example, [1, (2.16)]), invariant under the action of G on $\mathcal{F}_0$, and it is equivariant with respect to the actions of $\mathbb{P}\Gamma = \mathbb{P}\operatorname{Isom}(\Lambda)$ on $\mathcal{F}_0$ and $\mathbb{CH}^5$.

The period map $\mathfrak{p}$ extends holomorphically to $\mathcal{F}_s$ to a ($G \curvearrowright \mathcal{F}_s$)-invariant and $\mathbb{P}\Gamma$-equivariant map, also denoted by $\mathfrak{p}$. The map $\mathfrak{p}$ therefore descends to a map $\mathfrak{p}\colon \mathcal{F}_s/G \to \mathbb{CH}^5$, which turns out to be an isomorphism of complex manifolds. Furthermore, $\mathfrak{p}$ maps $\mathcal{F}_0$ bijectively to $(\mathbb{CH}^5 - \mathcal{H})$, where

$$\mathcal{H} := \bigcup \left\{ \mathbb{CH}(r^\perp) \subset \mathbb{CH}^5 \,\middle|\, \begin{array}{l} r \text{ is a vector in } \Lambda \text{ of} \\ \text{squared norm } -2 \end{array} \right\},$$

restricting also to an isomorphism of complex manifolds $\mathcal{F}_0/G \xrightarrow{\mathfrak{p}} (\mathbb{CH}^5 - \mathcal{H})$.

The results of Deligne–Mostow [7] and Matsumoto–Yoshida [12] show that $\mathcal{M}_s$ and $\mathbb{P}\Gamma\backslash\mathbb{CH}^5$ are isomorphic as complex analytic (quasi-projective) varieties via the

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

following series of isomorphisms:

$$\mathcal{M}_s := \mathbb{P}(\mathcal{P}_s)/\mathbb{P}\,\mathrm{GL}(2,\mathbb{C}) \cong \mathcal{P}_s/G \cong (\mathbb{P}\Gamma\backslash\mathcal{F}_s)/G \cong \mathbb{P}\Gamma\backslash(\mathcal{F}_s/G) \cong \mathbb{P}\Gamma\backslash\mathbb{CH}^5.$$

We remark that $\mathcal{M}_s$ and $\mathbb{P}\Gamma\backslash\mathbb{CH}^5$ are isomorphic only as complex analytic varieties, but not as complex analytic orbifolds. That their orbifold structures are distinct can be seen by the fact that the orbifold points of $\mathbb{P}(\mathcal{P}_s)/\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})$ correspond to octics with nontrivial authomorphisms, whereas the orbifold points of $\mathbb{P}\Gamma\backslash\mathbb{CH}^5$ correspond to singular octics.

3 The Allcock–Carlson–Toledo Construction of $\mathcal{M}_0^{\mathbb{R}}$

As shown in the last section, the moduli space $\mathcal{M}_s$ of stable binary octics is isomorphic as a complex analytic variety to the ball quotient $\mathbb{P}\Gamma\backslash\mathbb{CH}^5$. We shall show in Section 3.5 that periods in $\mathbb{CH}^5$ corresponding to real octics lie on a certain collection of copies of real hyperbolic 5-space $\mathbb{RH}^5$ inside $\mathbb{CH}^5$.

More precisely, complex conjugation κ on $\mathbb{CP}^1$ naturally induces a map on the space $\mathcal{P}$ of complex octic forms by conjugating coefficients (see Definition 3.2), and real binary octic forms (*i.e.*, those octic forms with only real coefficients) can be characterized as fixed points of this induced map. This implies that, for a smooth real binary octic form p, the complex conjugation $\kappa\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ induces an antiholomorphic involution $\kappa_p\colon X_p \to X_p$, which in turn induces an *involutive anti-isometry* (see Definition 3.10) κ_p' on $H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$, which likewise restricts to an involutive anti-isometry on $\Lambda(X_p)$. Any isometry $i\colon \Lambda(X_p) \xrightarrow{\sim} \Lambda$ will then induce an involutive anti-isometry χ_{κ_p} on Λ, and χ_{κ_p} extends to an involutive anti-isometry on $\mathbb{C}^{1,5} = \Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C}$. In Section 3.5, we will show that the complex period $i\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p)\big) \in \mathbb{CH}^5$ is a fixed point of the projective class $[\chi_{\kappa_p}]$ of χ_{κ_p}, and that the fixed point set of $[\chi_{\kappa_p}]$ is isomorphic to real hyperbolic 5-space $\mathbb{RH}^5$. A copy of $\mathbb{RH}^5$ within $\mathbb{CH}^5 = \mathbb{CH}(\Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C})$ will be called an *integral copy of* $\mathbb{RH}^5$ if it is the fixed point set of the projective class of an involutive anti-isometry Λ.

Thus, the complex periods of real binary octic forms all lie on integral copies of $\mathbb{RH}^5$ within the period domain $\mathbb{CH}^5$. Consequently, in order to locate all the periods in $\mathbb{CH}^5$ corresponding to real binary octic forms, we first determine all the involutive anti-isometries of Λ, and subsequently their fixed point sets. However, there is a slight complication due to the fact that $\mathbb{CP}^1$ admits two $\mathrm{PGL}(2,\mathbb{C})$-conjugacy classes of antiholomorphic involutions, represented by complex conjugation and the antipodal map on $\mathbb{CP}^1$ respectively.[1] The antipodal map will also induce involutive anti-isometries on Λ. We will therefore need to separate the two kinds of involutive anti-isometries of Λ and discard those induced by the antipodal map.

3.1 Complex Conjugation and the Antipodal Map on $\mathbb{CP}^1$ and Their Related Maps

Definition 3.1 Define the maps $\kappa\colon \mathbb{C}^2 \to \mathbb{C}^2$, and $\alpha\colon \mathbb{C}^2 \to \mathbb{C}^2$ respectively by $\kappa(x_0,x_1) := (\overline{x_0},\overline{x_1})$, and $\alpha(x_0,x_1) := (\overline{x_1},-\overline{x_0})$.

[1] János Kollár, *Real forms*, unpublished notes.

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Definition 3.2 Let $\nu\colon \mathbb{C}^2 \to \mathbb{C}^2$ be either κ or α as in Definition 3.1. We define the action of ν on the space $\mathcal{P}$ of complex binary octic forms as follows:

$$(\nu \cdot p)(x_0, x_1) := \overline{p\big(\nu(x_0, x_1)\big)}, \quad \text{for } p \in \mathcal{P}.$$

Definition 3.3 We define an *antilinear anti-involution* on a complex vector space V to be an antilinear map $V \xrightarrow{\nu} V$ such that $\nu^2 = -\operatorname{id}_V$.

Note that an antilinear anti-involution has order four. We will use this notion in the proof of Lemma 3.32.

Remark 3.4 The map $\kappa\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an antilinear involution, whereas $\alpha\colon \mathbb{C}^2 \to \mathbb{C}^2$ is an antilinear anti-involution. $\kappa\colon \mathbb{C}^2 \to \mathbb{C}^2$ descends to complex conjugation on $\mathbb{CP}^1$, whereas $\alpha\colon \mathbb{C}^2 \to \mathbb{C}^2$ descends to the antipodal map on $\mathbb{CP}^1$. We will also use κ to denote complex conjugation on $\mathbb{CP}^1$ and α the antipodal map on $\mathbb{CP}^1$. Which map is intended should be clear from the context.

Definition 3.5 A binary octic form is said to be *real* (respectively *antipodal*) if it is preserved by complex conjugation $\mathbb{C}^2 \xrightarrow{\kappa} \mathbb{C}^2$ (respectively the antipodal map $\mathbb{C}^2 \xrightarrow{\alpha} \mathbb{C}^2$) via the action as in Definition 3.2. We denote by $\mathcal{P}_0^{\mathbb{R}}$ the set of smooth real binary octic forms, and by $\mathcal{P}_0^{\text{antip}}$ the set of smooth antipodal binary octic forms. We denote by $\mathcal{F}_0^{\mathbb{R}}$ and $\mathcal{F}_0^{\text{antip}}$ the preimages of $\mathcal{P}_0^{\mathbb{R}}$ and $\mathcal{P}_0^{\text{antip}}$, respectively, under the covering map $\mathcal{F}_0 \to \mathcal{P}_0$.

Remark 3.6 There are smooth octics that are preserved by both complex conjugation and the antipodal map. In other words, $\mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\text{antip}} \neq \varnothing$. We also point out that, unlike their complex counterparts, $\mathcal{F}_0^{\mathbb{R}}$ and $\mathcal{F}_0^{\text{antip}}$ are not connected; in fact, they have infinitely many connected components. This will become clear in Lemma 3.32.

Notation 3.7 For $p \in \mathcal{P}_0^{\mathbb{R}}$, we denote by $\kappa_p\colon X_p \to X_p$ the antiholomorphic involution on X_p induced by complex conjugation $\mathbb{CP}^1 \xrightarrow{\kappa} \mathbb{CP}^1$. Similarly, for $p \in \mathcal{P}_0^{\text{antip}}$, we denote by $\alpha_p\colon X_p \to X_p$ the antiholomorphic involution on X_p induced by the antipodal map $\mathbb{CP}^1 \xrightarrow{\alpha} \mathbb{CP}^1$. Note that, for each octic $p \in \mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\text{antip}}$, both κ_p and α_p on X_p are defined.

Definition 3.8 Let $\mathrm{GL}(2,\mathbb{C})'$ be the group of all linear and antilinear automorphisms of $\mathbb{C}^2$; note that $\mathrm{GL}(2,\mathbb{C})' = \mathrm{GL}(2,\mathbb{C}) \rtimes \langle\kappa\rangle$. Let every linear element $g \in \mathrm{GL}(2,\mathbb{C})'$ and every antilinear element $\nu \in \mathrm{GL}(2,\mathbb{C})'$ act on $\mathbb{C}^3$ respectively by

$$g(x_0, x_1, y) := \big(g(x_0, x_1), y\big), \quad \text{and} \quad \nu(x_0, x_1, y) := \big(\nu(x_0, x_1), \overline{y}\big).$$

We will also consider elements of $\mathrm{GL}(2,\mathbb{C})'$ as automorphisms of $\mathbb{P}(1,1,2)$ via the representation $\mathrm{GL}(2,\mathbb{C})' \to \mathrm{Aut}'\,\mathbb{P}(1,1,2)$ corresponding to the action $\mathrm{GL}(2,\mathbb{C})' \curvearrowright \mathbb{C}^3$ above, where $\mathrm{Aut}'\,\mathbb{P}(1,1,2)$ is the automorphism group of $\mathbb{P}(1,1,2)$ induced by linear and antilinear automorphisms of $\mathbb{C}^3$.

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Definition 3.9 Let $G^{\mathbb{R}}$ be the stabilizer in G of the set $\mathcal{P}_0^{\mathbb{R}}$ with respect to the action $G \curvearrowright \mathcal{P}_0$. Similarly, let G^{antip} be the stabilizer in G of the set $\mathcal{P}_0^{\text{antip}}$ with respect to the same action $G \curvearrowright \mathcal{P}_0$.

Straightforward calculations show that $G^{\mathbb{R}} = \mathrm{GL}(2,\mathbb{R})/\langle \pm 1\rangle$ and

$$G^{\text{antip}} = \left\{ g \in \mathrm{GL}(2,\mathbb{C}) \;\middle|\; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \overline{g} = \pm g \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$
$$= \left\{ \begin{bmatrix} z_1 & z_2 \\ \pm\overline{z_2} & \mp\overline{z_1} \end{bmatrix} \in \mathbb{C}^{2\times 2} \;\middle|\; |z_1|^2 + |z_2|^2 \neq 0 \right\}.$$

It is obvious that $G^{\mathbb{R}}$ is also the stabilizer in G of the set $\mathcal{F}_0^{\mathbb{R}}$ with respect to the action $G \curvearrowright \mathcal{F}_0$; similarly, $G^{\mathbb{R}}$ is also the stabilizer in G of the set $\mathcal{F}_0^{\text{antip}}$ with respect to the action $G \curvearrowright \mathcal{F}_0$. The exact roles played by $G^{\mathbb{R}}$ and G^{antip} in constructing the moduli space of smooth real octics can be seen in Proposition 3.36.

Definition 3.10 An *anti-isometry* on a $\mathbb{Z}[\mathbf{i}]$-lattice $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ (or a complex vector space equipped with a Hermitian inner product) is a bijective antilinear map $\nu\colon V \to V$ such that $\langle \nu(x), \nu(y)\rangle = \overline{\langle x, y\rangle}$, for all $x, y \in V$.

Definition 3.11 Let $\mathcal{F}_0'$ be the space of all pairs $(p, [i])$, where $p \in \mathcal{P}_0$, $\Lambda(X_p) \xrightarrow{i} \Lambda$ is either an isometry or an anti-isometry, and $[i]$ is the projective equivalence class of i. Let every linear element $g \in \mathrm{GL}(2,\mathbb{C})'$ and every antilinear element $\nu \in \mathrm{GL}(2,\mathbb{C})'$ act on $\mathcal{F}_0'$ respectively by

$$(p,[i])\cdot g := \big(p\circ g, [i\circ (g^*)^{-1}]\big), \quad \text{and} \quad (p,[i])\cdot \nu := \big(\overline{p\circ\nu}, [i\circ(\nu^*)^{-1}]\big).$$

Note that, for $(x_0 : x_1 : y) \in \mathbb{P}(1,1,2)$, we have $(x_0 : x_1 : y) \in X_{\overline{p\circ\nu}} \iff y^4 = \overline{p\big(\nu(x_0,x_1)\big)} \iff (\overline{y})^4 = p\big(\nu(x_0,x_1)\big) \iff \nu\cdot(x_0,x_1,y) := \big(\nu(x_0,x_1),\overline{y}\big) \in X_p$. So, $\nu(X_{\overline{p\circ\nu}}) = X_p$. Hence, $\Lambda(X_p) \xrightarrow{\nu^*} \Lambda(X_{\overline{p\circ\nu}})$ and $\Lambda(X_{\overline{p\circ\nu}}) \xrightarrow{(\nu^*)^{-1}} \Lambda(X_p) \xrightarrow{i} \Lambda$.

3.2 The Deformation Types of Real and Antipodal Smooth Octics and Forms

There are five distinct deformation types of smooth real binary octics, in the sense that a real octic, of any fixed deformation type, cannot be deformed to a real octic of a different type through the space $\mathcal{O}_0^{\mathbb{R}} = \mathcal{P}_0^{\mathbb{R}}/\mathbb{R}^*$ of smooth real octics (where $\mathbb{R}^* := \mathbb{R}\backslash\{0\}$ acts by scalar multiplication on the real octic forms, as usual). In other words, $\mathcal{O}_0^{\mathbb{R}}$ has five connected components, *i.e.*,

$$\mathcal{O}_0^{\mathbb{R}} = \mathcal{O}_0^{\mathbb{R},0} \sqcup \mathcal{O}_0^{\mathbb{R},1} \sqcup \mathcal{O}_0^{\mathbb{R},2} \sqcup \mathcal{O}_0^{\mathbb{R},3} \sqcup \mathcal{O}_0^{\mathbb{R},4},$$

where $\mathcal{O}_0^{\mathbb{R},0}, \dots, \mathcal{O}_0^{\mathbb{R},4}$ parametrize the five types of real binary octics according to Table 3.1.

On the other hand, every smooth antipodal octic can be deformed to every other smooth antipodal octic through smooth antipodal octics. In other words, $\mathcal{O}_0^{\text{antip}}$ is connected and there is only one deformation type of smooth antipodal octics.

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

components of $\mathcal{O}_0^{\mathbb{R}}$	$\mathcal{O}_0^{\mathbb{R},0}$	$\mathcal{O}_0^{\mathbb{R},1}$	$\mathcal{O}_0^{\mathbb{R},2}$	$\mathcal{O}_0^{\mathbb{R},3}$	$\mathcal{O}_0^{\mathbb{R},4}$
# complex conjugate pairs	0	1	2	3	4
# real points	8	6	4	2	0

Table 3.1: Deformation types smooth real binary octics.

Definition 3.12 Let $\mathcal{M}_0^{\mathbb{R}}$ denote the moduli space of smooth real binary octics, *i.e.*, $\mathcal{M}_0^{\mathbb{R}} := \mathcal{O}_0^{\mathbb{R}}/\operatorname{PGL}(2,\mathbb{R})$, and $\mathcal{M}_0^{\mathbb{R},0}, \mathcal{M}_0^{\mathbb{R},1}, \dots, \mathcal{M}_0^{\mathbb{R},4}$ its five connected components of $\mathcal{M}_0^{\mathbb{R}}$, parametrizing octics in $\mathcal{O}_0^{\mathbb{R},0}, \mathcal{O}_0^{\mathbb{R},1}, \dots, \mathcal{O}_0^{\mathbb{R},4}$, respectively. (Therefore, $\mathcal{M}_0^{\mathbb{R}} = \bigsqcup_{i=0}^4 \mathcal{M}_0^{\mathbb{R},i}$.) Let $\mathcal{M}_0^{\text{antip}}$ denote the moduli space of smooth antipodal octics, *i.e.*, $\mathcal{M}_0^{\text{antip}} := \mathcal{O}_0^{\text{antip}}/\operatorname{Stab}_{\operatorname{PGL}(2,\mathbb{C})}(\mathcal{O}_0^{\text{antip}})$.

Remark 3.13 We do not claim that the definition for $\mathcal{M}_0^{\text{antip}}$ above is the "correct" or "natural" notion for the moduli space of smooth antipodal octics. We make this definition because the method we employ to give a description of each connected component of $\mathcal{M}_0^{\mathbb{R}} := \mathcal{O}_0^{\mathbb{R}}/\operatorname{PGL}(2,\mathbb{R})$ as a real hyperbolic quotient will simultaneously yield, as a byproduct, a similar description for $\mathcal{O}_0^{\text{antip}}/\operatorname{Stab}_{\operatorname{PGL}(2,\mathbb{C})}(\mathcal{O}_0^{\text{antip}})$; see Proposition 3.36.

On the other hand, the quotient $\mathcal{O}_0^{\text{antip}}/\operatorname{Stab}_{\operatorname{PGL}(2,\mathbb{C})}(\mathcal{O}_0^{\text{antip}})$ may be regarded in a sense as the "antipodal counterpart" of $\mathcal{O}_0^{\mathbb{R}}/\operatorname{PGL}(2,\mathbb{R})$ in light of the fact that $\operatorname{PGL}(2,\mathbb{R}) = \operatorname{Stab}_{\operatorname{PGL}(2,\mathbb{C})}(\mathcal{O}_0^{\mathbb{R}})$.

As we just observed, it is easy to count the number of connected components of $\mathcal{O}_0^{\mathbb{R}}$ or $\mathcal{M}_0^{\mathbb{R}}$. By contrast, in order to do the same for $\mathcal{P}_0^{\mathbb{R}}$, we need to take into account the fact that $\mathbb{R}^*$ has two connected components. Write $\mathcal{P}_0^{\mathbb{R},i}$ for the preimage of $\mathcal{O}_0^{\mathbb{R},i}$ under the projection $\mathcal{P}_0^{\mathbb{R}} \to \mathcal{O}_0^{\mathbb{R}} = \mathcal{P}_0^{\mathbb{R}}/\mathbb{R}^*$, $i = 0, \dots, 4$. Consider a smooth real binary octic in $\mathcal{O}_0^{\mathbb{R},i}$, determined by say the roots of an octic form $p(x_0, x_1) \in \mathcal{P}_0^{\mathbb{R},i}$. Then both $p(x_0, x_1)$ and $-p(x_0, x_1)$ descend to the same given octic (8-point configuration), but they may or may not belong to the same connected component of $\mathcal{P}_0^{\mathbb{R},i}$. It is now clear that each $\mathcal{P}_0^{\mathbb{R},i}$, $i = 0, \dots, 4$, has either one or two connected components, depending on whether or not any (hence every) element $p(x_0, x_1) \in \mathcal{P}_0^{\mathbb{R},i}$ can be deformed to its negative $-p(x_0, x_1)$ within $\mathcal{P}_0^{\mathbb{R},i}$. We now prove the following:

Lemma 3.14 *$\mathcal{P}_0^{\mathbb{R},4}$ has two connected components,*[2] *whereas each of $\mathcal{P}_0^{\mathbb{R},0}$, $\mathcal{P}_0^{\mathbb{R},1}$, $\mathcal{P}_0^{\mathbb{R},2}$, $\mathcal{P}_0^{\mathbb{R},3}$, and $\mathcal{P}_0^{\text{antip}}$ is connected.*

Proof If we regard x_0 and x_1 as real variables, then each pair $p(x_0, x_1), -p(x_0, x_1) \in \mathcal{P}_0^{\mathbb{R},4}$ can be regarded as continuous $\mathbb{R}$-valued nowhere vanishing functions of the real variables x_0, x_1 of opposite signs. Consequently, any continuous deformation from $p(x_0, x_1)$ to $-p(x_0, x_1)$ through the space of continuous $\mathbb{R}$-valued functions must pass through one that admits zeroes, thereby passing outside $\mathcal{P}_0^{\mathbb{R},4}$, since every

[2] The author wishes to express his gratitude to Dr. János Kollár for pointing out the author's earlier overlooking of this fact in a private communication.

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

smooth real binary octic form in $\mathcal{P}_0^{\mathbb{R},4}$ has no real roots. This proves that $\mathcal{P}_0^{\mathbb{R},4}$ has two connected components.

Next, consider the following 1-parameter family of binary polynomials:

$$q_3(x_0, x_1; \theta) := (x_0 \cos\theta - x_1 \sin\theta)(x_0 \sin\theta + x_1 \cos\theta), \quad \theta \in [0, \pi/2].$$

Then $q_3(x_0, x_1; 0) = x_0x_1$, whereas $q_3(x_0, x_1; \pi/2) = -x_0x_1$. Let $r(x_0, x_1)$ be any smooth real binary sextic form with no real roots. Then

$$p(x_0, x_1; \theta) := q_3(x_0, x_1; \theta) r(x_0, x_1), \quad \theta \in [0, \pi/2],$$

is a continuous path in $\mathcal{P}_0^{\mathbb{R},3}$ such that $p(x_0, x_1; 0) = x_0x_1 \cdot r(x_0, x_1)$, while

$$p(x_0, x_1; \pi/2) = -x_0x_1 \cdot r(x_0, x_1).$$

This proves that $\mathcal{P}_0^{\mathbb{R},3}$ is connected.

Similarly, we may define continuous paths in $\mathcal{P}_0^{\mathbb{R},i}$, $i = 0, 1, 2$, whose endpoints are negatives of each other by using the following three families in place of q_3:

$$\begin{aligned}
q_2(x_0, x_1; \theta_2) := {}& (x_0 \cos\theta_2 - x_1 \sin\theta_2)(x_0 \sin\theta_2 + x_1 \cos\theta_2) \\
&\times \big(x_0 \cos(\theta_2 + \pi/4) - x_1 \sin(\theta_2 + \pi/4)\big) \\
&\times \big(x_0 \sin(\theta_2 + \pi/4) + x_1 \cos(\theta_2 + \pi/4)\big), \\
q_1(x_0, x_1; \theta_1) := {}& \prod_{n=0}^{2} \big(x_0 \cos(\theta_1 + n\pi/6) - x_1 \sin(\theta_1 + n\pi/6)\big) \\
&\times \big(x_0 \sin(\theta_1 + n\pi/6) + x_1 \cos(\theta_1 + n\pi/6)\big), \\
q_0(x_0, x_1; \theta_0) := {}& \prod_{n=0}^{3} \big(x_0 \cos(\theta_0 + n\pi/8) - x_1 \sin(\theta_0 + n\pi/8)\big) \\
&\times \big(x_0 \sin(\theta_0 + n\pi/8) + x_1 \cos(\theta_0 + n\pi/8)\big),
\end{aligned}$$

where $\theta_2 \in [0, \pi/4]$, $\theta_1 \in [0, \pi/6]$, $\theta_0 \in [0, \pi/8]$. Thus, $\mathcal{P}_0^{\mathbb{R},0}$, $\mathcal{P}_0^{\mathbb{R},1}$, and $\mathcal{P}_0^{\mathbb{R},2}$ are connected. Lastly, we conclude that $\mathcal{P}_0^{\text{antip}}$ is also connected by noting that $q_0(x_0, x_1; \theta_0)$ is a family of antipodal octic forms (in addition to being real). ∎

In summary, $\mathcal{P}_0^{\mathbb{R}}$ has six connected components, *i.e.*,

$$\mathcal{P}_0^{\mathbb{R}} = \mathcal{P}_0^{\mathbb{R},0} \sqcup \mathcal{P}_0^{\mathbb{R},1} \sqcup \mathcal{P}_0^{\mathbb{R},2} \sqcup \mathcal{P}_0^{\mathbb{R},3} \sqcup \mathcal{P}_0^{\mathbb{R},4+} \sqcup \mathcal{P}_0^{\mathbb{R},4-},$$

where $\mathcal{P}_0^{\mathbb{R},4+}$ and $\mathcal{P}_0^{\mathbb{R},4-}$ are the two connected components of $\mathcal{P}_0^{\mathbb{R},4}$.

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3.3 Each $p \in \mathcal{P}_0^{\mathbb{R}} \sqcup \mathcal{P}_0^{\mathrm{antip}}$ Gives Rise to an Involutive Anti-isometry of $\Lambda(X_p)$

Let $p \in \mathcal{P}_0^{\mathbb{R}} \cup \mathcal{P}_0^{\mathrm{antip}}$, and let ν_p be κ_p or α_p, whichever is defined on X_p. Then the antiholomorphic involution $X_p \xrightarrow{\nu_p} X_p$ induces an antilinear involution on $H^1(X_p, \mathbb{C})$ via

$$\begin{aligned} H^1(X_p, \mathbb{C}) &\xrightarrow{\nu_p'} H^1(X_p, \mathbb{C}) \\ \phi &\longmapsto \overline{(\nu_p)^*(\phi)}. \end{aligned}$$

Lemma 3.15

(1) *The antilinear map ν_p' preserves both the Hodge decomposition and the σ-eigenspace decomposition of $H^1(X_p, \mathbb{C})$.*
(2) *The map ν_p' restricts to an involutive anti-isometry on $H^1_{\sigma=-\mathbf{i}}(X_p, \mathbb{C})$, which in turn restricts to an involutive anti-isometry on the $\mathbb{Z}[\mathbf{i}]$-lattice on $\Lambda(X_p)$.*
(3) *The usual pullback $\nu_p^* \colon H^1(X_p, \mathbb{Z}) \to H^1(X_p, \mathbb{Z})$ induced by ν_p preserves $\Lambda(X_p) = H^1_{\sigma^2=-1}(X_p, \mathbb{Z})$, and the restriction $\nu_p'|_{\Lambda(X_p)}$ agrees with $\nu_p^* \colon \Lambda(X_p) \to \Lambda(X_p)$.*

Outline of Proof

(1) Since ν_p is antiholomorphic, the pullback $\nu_p^* \colon H^1(X_p, \mathbb{C}) \to H^1(X_p, \mathbb{C})$ switches Hodge types of $\mathbb{C}$-valued differential forms; similarly, complex conjugation on $\mathbb{C}$-valued differential forms switches Hodge types. Hence, ν_p' preserves Hodge types. To prove that ν_p' preserves σ-eigenspaces, we first state two facts: that $\sigma \circ \nu_p = \nu_p \circ \sigma^3$, and that the action of σ^* on $\mathbb{C}$-valued differential forms commutes with complex conjugation of differential forms. Both of these facts can be verified with straightforward calculations. Using these two facts, another straightforward calculation will show that ν_p' preserves the σ-eigenspace decomposition of $H^1(X_p, \mathbb{C})$.
(2) The second statement also follows from a direct computation.
(3) The equality $\sigma \circ \nu_p = \nu_p \circ \sigma^3$ implies that $\nu_p^* \colon H^1(X_p, \mathbb{Z}) \to H^1(X_p, \mathbb{Z})$ preserves $\Lambda(X_p) = H^1_{\sigma^2=-1}(X_p, \mathbb{Z})$. It is now immediate that $\nu_p'|_{\Lambda(X_p)}$ agrees with $\nu_p^* \colon \Lambda(X_p) \to \Lambda(X_p)$, since complex conjugation on $H^1(X_p, \mathbb{C})$ acts identically on $H^1(X_p, \mathbb{Z})$. ■

Remark 3.16 We will use the notation ν_p^* for $\nu_p'|_{\Lambda(X_p)}$ to emphasize that the restriction $\nu_p'|_{\Lambda(X_p)}$ is an endomorphism of the submodule $\Lambda(X_p) = H^1_{\sigma^2=-1}(X_p, \mathbb{Z})$ of $H^1(X_p, \mathbb{Z})$, where complex conjugation on $H^1(X_p, \mathbb{C})$ acts identically.

Notation 3.17 We denote by $\mathrm{IAI}(\Lambda(X_p))$ and $\mathrm{IAI}(\Lambda)$ the sets of all involutive anti-isometries of $\Lambda(X_p)$ and Λ, respectively.

Definition 3.18 We define the map $\pi_0(\mathcal{F}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{F}_0^{\mathrm{antip}}) \to \mathbb{P}\,\mathrm{IAI}(\Lambda)$

$$\big[(p, [i])\big] \longmapsto \begin{cases} [i \circ \kappa_p^* \circ i^{-1}], & \text{if } p \in \mathcal{P}_0^{\mathbb{R}}, \\ [i \circ \alpha_p^* \circ i^{-1}], & \text{if } p \in \mathcal{P}_0^{\mathrm{antip}}, \end{cases}$$

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

where $p \in \mathcal{P}_0^{\mathbb{R}} \sqcup \mathcal{P}_0^{\text{antip}}$, $i\colon \Lambda(X_p) \to \Lambda$ is an isometry, $[i]$ stands for the projective equivalence class of i (see Definition 2.4), and $\big[(p,[i])\big]$ stands for the connected component of $\mathcal{F}_0^{\mathbb{R}}$ or $\mathcal{F}_0^{\text{antip}}$ containing $(p,[i])$.

Definition 3.19 We also define

$$\pi_0(\mathcal{P}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{P}_0^{\text{antip}}) \to \mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$$
$$[p] \longmapsto \begin{cases} [i \circ \kappa_p^* \circ i^{-1}], & \text{if } p \in \mathcal{P}_0^{\mathbb{R}}, \text{ where } i \text{ is any frame over } p, \\ [i \circ \alpha_p^* \circ i^{-1}], & \text{if } p \in \mathcal{P}_0^{\text{antip}}, \text{ where } i \text{ is any frame over } p. \end{cases}$$

Remark 3.20 The occurrences of κ_p^* in Definitions 3.18 and 3.19 can be replaced with κ_p', since they agree on $\Lambda(X_p)$; see Lemma 3.15 and Remark 3.16. Similarly, the occurrences of α_p^* can be replaced with α_p'.

The maps in Definitions 3.18 and 3.19 are well-defined because $i \circ \kappa_p^* \circ i^{-1}$ and $i \circ \alpha_p^* \circ i^{-1}$ lie in the discrete subset $\mathrm{IAI}(\Lambda)$ of $\mathrm{IAI}(\Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C}) \cong \mathrm{IAI}(\mathbb{C}^{1,5})$, and hence remain constant as p and $(p,[i])$ vary within each connected component of $\mathcal{P}_0^{\mathbb{R}} \sqcup \mathcal{P}_0^{\text{antip}}$ and $\mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\text{antip}}$ respectively.

Remark 3.21 Recall that $\mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\text{antip}} \neq \varnothing$. It may thus appear unmotivated at this point that we are working with the disjoint unions $\pi_0(\mathcal{F}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{F}_0^{\text{antip}})$ and $\pi_0(\mathcal{P}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{P}_0^{\text{antip}})$ in Definitions 3.18 and 3.19. The significance of working with the disjoint unions is that, for $p \in \mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\text{antip}}$, we would like to regard p as a real octic form as distinct from p as an antipodal form. For such a $p \in \mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\text{antip}}$, the antiholomorphic involutions $\kappa_p\colon X_p \to X_p$ and $\alpha_p\colon X_p \to X_p$ will induce distinct $\mathbb{P}\,\mathrm{Isom}(\Lambda)$-conjugacy classes of involutive anti-isometries of Λ; see Lemma 3.25. This observation will allow us to single out the antipodal periods and discard them. We mentioned this briefly in the opening paragraphs of the present section (Section 3), and this is more precisely elaborated in Remark 3.29.

3.4 Integral Copies of $\mathbb{RH}^5$ in $\mathbb{CH}^5$

It can be readily checked that, for each $\chi \in \mathrm{IAI}(\Lambda)$, the metric on Λ restricts to a metric on the $\mathbb{Z}$-module $\mathrm{Fix}(\chi) \cong \mathbb{Z}^6$ of signature $(1+,5-)$. Thus $\mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{1+,5-}$, and

$$\begin{array}{ccc} \mathbb{RH}\big(\mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big) & \cong & \mathbb{RH}^5 \\ \cap & & \cap \\ \mathbb{CH}(\Lambda \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C}) & \cong & \mathbb{CH}^5 \end{array}$$

Hence, we may make the following definition.

Definition 3.22 A copy of $\mathbb{RH}^5 \subset \mathbb{CH}^5$ is said to be *integral* if it is of the form $\mathbb{RH}\big(\mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big)$ for some $\chi \in \mathrm{IAI}(\Lambda)$.

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3.5 "Real" Octics Have "Real" Periods; "Antipodal" Octics Have "Antipodal" Periods

Recall that, for any smooth $p \in \mathcal{P}_0$,

$$\Lambda(X_p) \otimes_{\mathbb{Z}[\mathbf{i}]} \mathbb{C} \cong \underbrace{H^1_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})}_{\mathbb{C}^{1,5}=\mathbb{C}^{1+,5-}} = \underbrace{H^{1,0}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})}_{(+)} \oplus \underbrace{H^{0,1}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})}_{(-----)}.$$

On the other hand, consider an ordered pair (p,ν_p), where either $p \in \mathcal{P}_0^{\mathbb{R}}$ and $\nu_p = \kappa_p$, or $p \in \mathcal{P}_0^{\text{antip}}$ and $\nu_p = \alpha_p$. Recall that $\nu'_p \colon H^1(X_p,\mathbb{C}) \to H^1(X_p,\mathbb{C})$ preserves both the Hodge decomposition and the σ-eigenspace decomposition. Since $H^{1,0}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})$ is complex one-dimensional, $H^{1,0}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C}) \in \mathbb{CH}(\Lambda(X_p)\otimes\mathbb{C})$ is fixed by $[\nu'_p|_{\Lambda(X_p)}] = [\nu^*_p] \in \mathbb{P}\,\mathrm{IAI}(\Lambda(X_p))$. Hence, for a given framed smooth form $[\Lambda(X_p) \xrightarrow{i} \Lambda]$ over $p \in \mathcal{P}_0^{\mathbb{R}} \sqcup \mathcal{P}_0^{\text{antip}}$, and a fixed choice of ν_p $(=\kappa_p$ or $\alpha_p)$, the complex period $i\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p,\mathbb{C})\big) \in \mathbb{CH}^5 = \mathbb{CH}(\Lambda\otimes\mathbb{C})$ is fixed by the projective class $[\chi_{\nu_p}] = [i\circ\nu^*_p\circ i^{-1}] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$. It now makes sense to introduce the following two definitions:

Definition 3.23 For each $[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$, define $\mathbb{RH}^5_{[\chi]}$ to be the fixed point set of $[\chi]$ in $\mathbb{CH}(\Lambda\otimes_{\mathbb{Z}[\mathbf{i}]}\mathbb{C}) \cong \mathbb{CH}^5$, *i.e.*, $\mathbb{RH}^5_{[\chi]} := \{[v] \in \mathbb{CH}^5 \mid [\chi]([v]) = [v]\}$.

Definition 3.24 An element $x \in \mathbb{CH}^5$ is called a *real period* if $x \in \mathbb{RH}^5_{[\chi_{\kappa_p}]}$, for some $p \in \mathcal{P}_0^{\mathbb{R}}$. An element $x \in \mathbb{CH}^5$ is called an *antipodal period* if $x \in \mathbb{RH}^5_{[\chi_{\alpha_p}]}$, for some $p \in \mathcal{P}_0^{\text{antip}}$.

Let a representative $\chi \in [\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$ be fixed. It is straightforward to see that we have the equality

$$\mathbb{RH}^5_{[\chi]} = \{[v] \in \mathbb{CH}^5 \mid \exists v \in [v] \text{ with } \chi(v) = v\}.$$

It is also easy to see that given any $[v] \in \mathbb{RH}^5_{[\chi]}$, the representative $v \in [v]$ that is fixed by the given χ is unique up to real scalar multiples. This gives an identification between $\mathbb{RH}^5_{[\chi]}$ and $\mathbb{RH}\big(\mathrm{Fix}(\chi)\otimes_{\mathbb{Z}}\mathbb{R}\big) \cong \mathbb{RH}^5$. The fixed point set $\mathbb{RH}^5_{[\chi]}$ is therefore an integral copy of $\mathbb{RH}^5$ (hence its notation) and $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$ is isomorphic to a subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$. We see at once that the real and antipodal periods lie on integral copies of $\mathbb{RH}^5$ within $\mathbb{CH}^5$.

Lemma 3.25 *The images of* $\pi_0(\mathcal{P}_0^{\mathbb{R}})$ *and* $\pi_0(\mathcal{P}_0^{\text{antip}})$ *under the map in Definition* 3.19 *are disjoint in* $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$.

Proof This follows from the observation that every octic form in $\mathcal{P}_0^{\mathbb{R}}$ can deform within $\mathcal{P}_0^{\mathbb{R}}$ to a nodal octic (*i.e.*, a singular octic with one double root and no other singularities), whereas an octic in $\mathcal{P}_0^{\text{antip}}$ can only deform within $\mathcal{P}_0^{\text{antip}}$ to singular octics with at least two double points.

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Let $p_1 \in \mathcal{P}_0^{\mathbb{R}}$, and $[\chi_1] := [i_1 \circ \kappa_{p_1}^* \circ i_1^{-1}]$ be its image in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$, where i_1 is any smooth frame over p_1. Note that $\mathbb{RH}^5_{\chi_1} := \mathbb{RH}\big(\mathrm{Fix}(\chi_1) \otimes_{\mathbb{Z}} \mathbb{R}\big) \subset \mathbb{CH}^5$ contains periods of real octics of the same topological type as p_1. Now recall that periods of nodal octics lie on the collection $\mathcal{H} \subset \mathbb{CH}^5$ of hyperplanes which are orthogonal complements of vectors in Λ of squared norm -2. (See Section 2.3.) By the preceding observation, we see that the intersection of $\mathcal{H}$ and $\mathbb{RH}^5_{\chi_1}$ must contain some smooth points of $\mathcal{H}$.

On the other hand, let $p_2 \in \mathcal{P}_0^{\mathrm{antip}}$, and $[\chi_2] := [i_2 \circ \alpha_{p_2}^* \circ i_2^{-1}]$ be its image in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$, where i_2 is any smooth frame over p_2. Then

$$\mathbb{RH}^5_{\chi_2} := \mathbb{RH}\big(\mathrm{Fix}(\chi_2) \otimes_{\mathbb{Z}} \mathbb{R}\big) \subset \mathbb{CH}^5$$

contains periods of antipodal octics. By the preceding observation again, we see that the intersection of $\mathcal{H}$ and $\mathbb{RH}^5_{\chi_2}$ cannot contain any smooth point of $\mathcal{H}$.

Since the two intersection patterns described above are $\mathbb{P}\,\mathrm{Isom}(\Lambda)$-invariant, we must have $[\chi_1] \neq [\chi_2]$. ■

By Lemma 3.25, it makes sense to introduce the following:

Definition 3.26 Let $\mathbb{P}\,\mathrm{IAI}(\Lambda)^{\mathbb{R}}/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ and $\mathbb{P}\,\mathrm{IAI}(\Lambda)^{\mathrm{antip}}/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ be the images in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ of $\pi_0(\mathcal{P}_0^{\mathbb{R}})$ and $\pi_0(\mathcal{P}_0^{\mathrm{antip}})$, respectively, of the map

$$\pi_0(\mathcal{P}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{P}_0^{\mathrm{antip}}) \to \mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda),$$

as in Corollary 3.33.

3.6 The Real Period Map and the Allcock–Carlson–Toledo Construction of $\mathcal{M}_0^{\mathbb{R}}$

The G-invariant complex period map $\mathfrak{p}\colon \mathcal{F}_s \to \mathbb{CH}^5$ was an important ingredient towards constructing the $\mathbb{CH}^5$ quotient structure for the moduli space $\mathcal{M}_s$ of stable complex binary octics. We make use of it again to study the moduli space $\mathcal{M}_0^{\mathbb{R}}$ of real binary octics.

Definition 3.27 The *real period map* is the map

$$\mathfrak{p}^{\mathbb{R}}\colon \mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\mathrm{antip}} \to \mathbb{CH}^5 \times \mathbb{P}\,\mathrm{IAI}(\Lambda)$$

defined by

$$\mathfrak{p}^{\mathbb{R}}(p,[i]) := \begin{cases} \big(\mathfrak{p}(p,[i]), [i \circ \kappa_p^* \circ i^{-1}]\big), & \text{if } (p,[i]) \in \mathcal{F}_0^{\mathbb{R}}, \\ \big(\mathfrak{p}(p,[i]), [i \circ \alpha_p^* \circ i^{-1}]\big), & \text{if } (p,[i]) \in \mathcal{F}_0^{\mathrm{antip}}. \end{cases}$$

Remark 3.28 The codomain of the real period map $\mathfrak{p}^{\mathbb{R}}$ can be regarded (see remarks following [3, Lemma 2.1] for the analogous result in the case real cubic surfaces) as:

$$\mathcal{D}_0 := \bigsqcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{RH}^5_{[\chi]} - \mathcal{H}),$$

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recalling that $\mathcal{H} \subset \mathbb{CH}^5$ is the collection of hyperplanes orthogonal to vectors in Λ of squared norm -2. Recall also that $\mathcal{H}$ is precisely the set of periods of singular octics (see Section 2.3). Hereinafter, we regard $\mathcal{D}_0$ as the codomain of $\mathfrak{p}^{\mathbb{R}}$. We also define

$$\mathcal{D}_0^{\mathbb{R}} := \bigsqcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)^{\mathbb{R}}} (\mathbb{RH}^5_{[\chi]} - \mathcal{H})$$

Remark 3.29 We have observed in previous sections that real and antipodal periods lie on integral copies of $\mathbb{RH}^5$ within the period domain $\mathbb{CH}^5$. So, we restrict the domain of the complex period map $\mathfrak{p}\colon \mathcal{F}_s \to \mathbb{CH}^5$ to the collection $\mathcal{F}_0^{\mathbb{R}} \cup \mathcal{F}_0^{\mathrm{antip}}$ of framed smooth forms over smooth real and antipodal octic forms, and we restrict the codomain to the collection $\bigcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{RH}^5_{[\chi]} - \mathcal{H})$ of integral copies of $\mathbb{RH}^5$ in $\mathbb{CH}^5$. This extracts from the domain and codomain of the complex period map points that correspond to smooth real and antipodal octics. Furthermore, we keep track of the "topological type" of a real or antipodal octic form p by working with the disjoint unions $\mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\mathrm{antip}}$ and $\mathcal{D}_0$ instead. We stress again that, for a smooth octic $p \in \mathcal{P}_0^{\mathbb{R}} \cap \mathcal{P}_0^{\mathrm{antip}}$, this will differentiate p considered as a real octic from p considered as an antipodal octic. The reason we want to keep track of this distinction is that we will next discard $\mathcal{F}_0^{\mathrm{antip}}$ from $\mathrm{Domain}(\mathfrak{p}^{\mathbb{R}}) = \mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\mathrm{antip}}$; we will also discard all the integral copies of $\mathbb{RH}^5$ in $\mathcal{D}_0 = \mathrm{codomain}(\mathfrak{p}^{\mathbb{R}})$ corresponding to antipodal octics, obtaining $\mathcal{D}_0^{\mathbb{R}}$, the disjoint union of all integral copies of $\mathbb{RH}^5$ corresponding to real octics.

We now continue with the key properties of the real period map.

Definition 3.30 We let $\mathbb{P}\Gamma = \mathbb{P}\,\mathrm{Isom}(\Lambda)$ act on $\mathbb{CH}^5 \times \mathbb{P}\,\mathrm{IAI}(\Lambda)$ as follows: for $[\gamma] \in \mathbb{P}\Gamma$, and $(x, [\chi]) \in \mathbb{CH}^5 \times \mathbb{P}\,\mathrm{IAI}(\Lambda)$,

$$[\gamma] \cdot (x, [\chi]) := \big(\gamma(x), [\gamma \circ \chi \circ \gamma^{-1}]\big).$$

This induces an action of $\mathbb{P}\Gamma$ on

$$\mathrm{codomain}(\mathfrak{p}^{\mathbb{R}}) = \mathcal{D}_0 = \bigsqcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{RH}^5_{[\chi]} - \mathcal{H}).$$

Lemma 3.31 *The real period map is $\mathbb{P}\Gamma$-equivariant.*

Proof Let $[\gamma] \in \mathbb{P}\,\mathrm{Isom}(\Lambda)$. Let $(p, [i]) \in \mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\mathrm{antip}}$. Let ν_p be κ_p if $(p, [i]) \in \mathcal{F}_0^{\mathbb{R}}$ and let it be α_p if $(p, [i]) \in \mathcal{F}_0^{\mathrm{antip}}$. Then note that

$$\begin{aligned}\mathfrak{p}^{\mathbb{R}}\big(\gamma \cdot (p, [i])\big) &= \mathfrak{p}^{\mathbb{R}}\big((p, [\gamma \circ i])\big) = \big(\mathfrak{p}(p, [\gamma \circ i]), [(\gamma \circ i) \circ \nu_p^* \circ (\gamma \circ i)^{-1}]\big)\\ &= \Big((\gamma \circ i)\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p)\big), [\gamma \circ (i \circ \nu_p^* \circ i^{-1}) \circ \gamma^{-1}]\Big).\end{aligned}$$

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On the other hand,

$$\begin{aligned}\gamma \cdot \mathfrak{p}^{\mathbb{R}}(p,[i]) &= \gamma \cdot \big(\mathfrak{p}(p,[i]), [i \circ \nu_p^* \circ i^{-1}]\big)\\ &= \Big(\gamma\big(\mathfrak{p}(p,[i])\big), [\gamma \circ (i \circ \nu_p^* \circ i^{-1}) \circ \gamma^{-1}]\Big)\\ &= \Big(\gamma\Big(i\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p)\big)\Big), [\gamma \circ (i \circ \nu_p^* \circ i^{-1}) \circ \gamma^{-1}]\Big)\\ &= \Big((\gamma \circ i)\big(H^{1,0}_{\sigma=-\mathbf{i}}(X_p)\big), [\gamma \circ (i \circ \nu_p^* \circ i^{-1}) \circ \gamma^{-1}]\Big).\end{aligned}$$

The two calculations above show $\mathfrak{p}^{\mathbb{R}}\big(\gamma \cdot (p,[i])\big) = \gamma \cdot \mathfrak{p}^{\mathbb{R}}(p,[i])$, for arbitrary $[\gamma] \in \mathbb{P}\,\mathrm{Isom}(\Lambda)$ and $(p,[i]) \in \mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\mathrm{antip}}$. Thus, $\mathfrak{p}^{\mathbb{R}}$ is indeed $\mathbb{P}\Gamma$-equivariance. ∎

Lemma 3.32 *The real period map is $G^{\mathbb{R}}$-invariant with respect to the action of $G^{\mathbb{R}}$ on $\mathcal{F}_0^{\mathbb{R}}$ and it is G^{antip}-invariant with respect to the action on $\mathcal{F}_0^{\mathrm{antip}}$. In other words, it descends to a map, also denoted by $\mathfrak{p}^{\mathbb{R}}$,*

$$\mathfrak{p}^{\mathbb{R}} \colon (\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}}) \sqcup (\mathcal{F}_0^{\mathrm{antip}}/G^{\mathrm{antip}}) \to \bigsqcup_{[\chi]\in\mathbb{P}\,\mathrm{IAI}(\Lambda)} \mathbb{R}\mathbb{H}^5_{[\chi]}.$$

Furthermore, the real period map $\mathfrak{p}^{\mathbb{R}}$ restricts to a $\mathbb{P}\Gamma$-equivariant real-analytic diffeomorphism as follows:

$$\mathfrak{p}^{\mathbb{R}} \colon (\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}}) \sqcup (\mathcal{F}_0^{\mathrm{antip}}/G^{\mathrm{antip}}) \to \mathcal{D}_0 := \bigsqcup_{[\chi]\in\mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{R}\mathbb{H}^5_{[\chi]} - \mathcal{H}).$$

Proof Consider first $(p,[i]) \in \mathcal{F}_0^{\mathbb{R}}$ and $g \in G^{\mathbb{R}}$. Then

$$\begin{aligned}\mathfrak{p}^{\mathbb{R}}\big((p,[i])\cdot g\big) &= \mathfrak{p}^{\mathbb{R}}\big(p \circ g, [i \circ (g^*)^{-1}]\big)\\ &= \Big(\mathfrak{p}\big(p \circ g, [i \circ (g^*)^{-1}]\big), [i \circ (g^*)^{-1} \circ \kappa_p^* \circ g^* \circ i^{-1}]\Big)\\ &= \Big(\mathfrak{p}\big((p,[i])\cdot g\big), [i \circ \kappa_p^* \circ i^{-1}]\Big)\\ &= \Big(\mathfrak{p}\big((p,[i])\big), [i \circ \kappa_p^* \circ i^{-1}]\Big)\\ &= \mathfrak{p}^{\mathbb{R}}\big((p,[i])\big),\end{aligned}$$

where the third equality uses the fact that $g \in G^{\mathbb{R}}$ and κ_p^* commutes, and the fourth equality uses the G-invariance of the complex period map. This shows that the real period map $\mathfrak{p}^{\mathbb{R}}$ is indeed $G^{\mathbb{R}}$-invariant with respect to the $G^{\mathbb{R}}$-action on $\mathcal{F}_0^{\mathbb{R}}$. As for its G^{antip}-invariance with respect to the G^{antip}-action on $\mathcal{F}_0^{\mathrm{antip}}$, simply replace $\mathcal{F}_0^{\mathbb{R}}$, $G^{\mathbb{R}}$, κ_p with $\mathcal{F}_0^{\mathrm{antip}}$, G^{antip}, α_p, respectively, in the above calculations, noting that each element in G^{antip} commutes with α_p^*.

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The $\mathbb{P}\Gamma$-equivariance of $\mathfrak{p}^{\mathbb{R}}$ follows immediately from Lemma 3.31. Next, observe that the map $\mathfrak{p}^{\mathbb{R}}\colon (\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}}) \sqcup (\mathcal{F}_0^{\text{antip}}/G^{\text{antip}}) \to \mathcal{D}_0$ locally has rank 5 everywhere since the complex period map $\mathfrak{p}$ does. It therefore remains only to prove bijectivity.

We now prove the surjectivity of $\mathfrak{p}^{\mathbb{R}}$. Let $[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$, and $x \in \mathbb{RH}^5_{[\chi]} - \mathcal{H}$. The surjectivity of the complex period map implies that there exists $(p, [i]) \in \mathcal{F}_0$ such that $\mathfrak{p}(p, [i]) = x$.

Claim 1 There exists an antilinear involution or antilinear anti-involution ν on $\mathbb{C}^2$ such that

- $\nu \cdot p = p$; in other words, $\overline{p\big(\nu(x_0, x_1)\big)} = p(x_0, x_1)$,
- ν induces an antiholomorphic map on $\mathbb{P}(1,1,2)$ which preserves $X_p \subset \mathbb{P}(1,1,2)$, and
- $\nu^* \in \mathrm{IAI}(\Lambda(X_p))$ coincides with $i^{-1} \circ \chi \circ i$, up to $\mathbb{Z}[\mathbf{i}]$-unit scalars.

Recall that the action of ν on $\mathbb{P}(1,1,2)$ is induced by the following extended action of ν on $\mathbb{C}^3$:

$$\nu(x_0, x_1, y) := \big(\nu(x_0, x_1), \overline{y}\big)\,.$$

A simple calculation shows that, under this action, the preservation of $X_p \subset \mathbb{P}(1,1,2)$ by $\mathbb{P}(1,1,2) \xrightarrow{\nu} \mathbb{P}(1,1,2)$ is an immediate consequence of the property that $p \cdot \nu = p$.

Granting Claim 1, we see, by Remark 3.4, that for such an antilinear $\mathbb{C}^2 \xrightarrow{\nu} \mathbb{C}^2$, there exists $g \in \mathrm{GL}(2, \mathbb{C})$ such that either $g^{-1} \circ \nu \circ g = \kappa$, or $g^{-1} \circ \nu \circ g = \alpha$. In the case $g^{-1} \circ \nu \circ g = \kappa$, it follows that

$$\begin{aligned}\big(\kappa \cdot (p \circ g)\big)(x_0, x_1) &= \overline{(p \circ g)\big(\kappa(x_0, x_1)\big)} = \overline{p\big(\nu \circ g(x_0, x_1)\big)} \\ &= (\nu \cdot p)\big(g(x_0, x_1)\big) = (p \circ g)(x_0, x_1).\end{aligned}$$

Thus, $\kappa \cdot (p \circ g) = p \circ g$; hence $p \circ g \in \mathcal{P}_0^{\mathbb{R}}$, and

$$\begin{aligned}\mathfrak{p}^{\mathbb{R}}\big((F, [i]) \cdot g\big) &= \Big(\mathfrak{p}\big((F, [i]) \cdot g\big), [i \circ (g^{-1})^* \circ \kappa^* \circ g^* \circ i^{-1}]\Big) \\ &= \big(\mathfrak{p}(F, [i]), [i \circ (g \circ \kappa \circ g^{-1})^* \circ i^{-1}]\big) \\ &= (x, [i \circ \nu^* \circ i^{-1}]) \\ &= (x, [\chi]).\end{aligned}$$

In the other case, *i.e.*, $g^{-1} \circ \nu \circ g = \alpha$, we similarly have $p \circ g \in \mathcal{P}_0^{\text{antip}}$, and

$$\begin{aligned}\mathfrak{p}^{\mathbb{R}}\big((F, [i]) \cdot g\big) &= \Big(\mathfrak{p}\big((F, [i]) \cdot g\big), [i \circ (g^{-1})^* \circ \alpha^* \circ g^* \circ i^{-1}]\Big) \\ &= \big(\mathfrak{p}(F, [i]), [i \circ (g \circ \alpha \circ g^{-1})^* \circ i^{-1}]\big) \\ &= (x, [i \circ \nu^* \circ i^{-1}]) \\ &= (x, [\chi]).\end{aligned}$$

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The above argument therefore shows that Claim 1 implies the surjectivity of $\mathfrak{p}^{\mathbb{R}}$.

We now prove Claim 1. Apply Lemma A.2 with $Y = \mathcal{F}'_0$, $y = (p, [i])$, $R = G'$, and $L = \mathbb{P}\Gamma'$. Then $l = x = \mathfrak{p}(y) = \mathfrak{p}(p, [i]) \in \mathbb{CH}^5 - \mathcal{H} = Y/R$, and $r = p \in \mathcal{P}_0 = \mathbb{P}\Gamma' \backslash \mathcal{F}'_0 = L\backslash Y$. Taking ϕ to be $[\chi] \in (\mathbb{P}\Gamma')_x = L_l$, we therefore get, by Lemma A.2, $\hat{\phi} = R_r = (G')_p$ such that $\phi \cdot (p, [i]) \cdot \hat{\phi} = (p, [i])$. Since $[\chi]$ is antilinear, it interchanges the two connected components of $\mathcal{F}'_0$ (each being a copy of $\mathcal{F}_0$). The preceding equality therefore implies that $\hat{\phi}$ likewise must interchange the components of $\mathcal{F}'_0$. Hence, $\hat{\phi} \in G'$ must be antilinear. Secondly, since $[\chi]$ has order two, so does $\hat{\phi}$, since they are related by the anti-isomorphism $\mathbb{P}\Gamma'_x = L_l \to R_r = G'_p$ mentioned in Lemma A.2. Thus, $\hat{\phi} \in G' = \mathrm{GL}(2, \mathbb{C})'/\langle\text{eighth roots of unity}\rangle$ is an antilinear involution. Let $\nu \in \mathrm{GL}(2, \mathbb{C})'$ be any lifting of $\hat{\phi} \in G'$. Then ν is antilinear and $\nu^2 \colon \mathbb{C}^2 \to \mathbb{C}^2$ acts on $\mathbb{C}^2$ as scalar multiplication by an eighth root of unity. Lemma A.1 implies ν^2 must be a real scalar. Thus $\nu^2 = \pm\,\mathrm{id}_{\mathbb{C}^2}$, *i.e.*, ν is either an antilinear involution, or it is an antilinear anti-involution.

Lastly,

$$
\begin{aligned}
(p, [i]) &= \phi \cdot (p, [i]) \cdot \hat{\phi} = [\chi] \cdot (p, [i]) \cdot \hat{\phi} \\
&= (p, [\chi \circ i]) \cdot \hat{\phi} = \big(\overline{p \circ \hat{\phi}}, [\chi \circ i \circ (\hat{\phi}^*)^{-1}]\big) \\
&\iff p = p \cdot \hat{\phi} = p \cdot \nu,
\end{aligned}
$$

and $[i] = [\chi \circ i \circ (\hat{\phi}^*)^{-1}] \iff [i^{-1} \circ \chi \circ i] = [\hat{\phi}^*] = [\nu^*]$. Thus $\nu \in \mathrm{GL}(2, \mathbb{C})'$ is an antilinear involution or antilinear anti-involution with the desired properties as in Claim 1. This completes the proof of Claim 1 and the surjectivity of $\mathfrak{p}^{\mathbb{R}}$.

We next prove injectivity of $\mathfrak{p}^{\mathbb{R}}$. First, for each $[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$, define

$$
\begin{aligned}
\mathcal{F}^{\mathbb{R}}_{0,[\chi]} &:= \{(p, [i]) \in \mathcal{F}_0 \mid p \in \mathcal{P}^{\mathbb{R}}_0, [i \circ \kappa^*_p \circ i^{-1}] = [\chi]\}, \\
\mathcal{F}^{\mathrm{antip}}_{0,[\chi]} &:= \{(p, [i]) \in \mathcal{F}_0 \mid p \in \mathcal{P}^{\mathrm{antip}}_0, [i \circ \alpha^*_p \circ i^{-1}] = [\chi]\}.
\end{aligned}
$$

Recall that $\mathcal{F}^{\mathbb{R}}_0$ and $\mathcal{F}^{\mathrm{antip}}_0$ are the preimages of $\mathcal{P}^{\mathbb{R}}_0$ and $\mathcal{P}^{\mathrm{antip}}_0$ respectively under the covering $\mathcal{F}_0 \to \mathcal{P}_0$. Note the following equalities

$$
\mathcal{F}^{\mathbb{R}}_0 = \bigcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} \mathcal{F}^{\mathbb{R}}_{0,[\chi]}, \quad \text{and} \quad \mathcal{F}^{\mathrm{antip}}_0 = \bigcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} \mathcal{F}^{\mathrm{antip}}_{0,[\chi]}.
$$

Suppose $\mathfrak{p}^{\mathbb{R}}(p_1, [i_1]) = \mathfrak{p}^{\mathbb{R}}(p_2, [i_2])$. Recall that the real period map

$$
\mathfrak{p}^{\mathbb{R}} \colon (\mathcal{F}^{\mathbb{R}}_0/G^{\mathbb{R}}) \sqcup (\mathcal{F}^{\mathrm{antip}}_0/G^{\mathrm{antip}}) \to \mathcal{D}_0 := \bigsqcup_{[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{RH}^5_{[\chi]} - \mathcal{H})
$$

is induced from the following $G^{\mathbb{R}}$-invariant and G^{antip}-invariant map

$$
\mathfrak{p}^{\mathbb{R}}(p, [i]) := \begin{cases} \big(\mathfrak{p}(p, [i]), [i \circ \kappa^*_p \circ i^{-1}]\big), & \text{if } (p, [i]) \in \mathcal{F}^{\mathbb{R}}_0, \\ \big(\mathfrak{p}(p, [i]), [i \circ \alpha^*_p \circ i^{-1}]\big), & \text{if } (p, [i]) \in \mathcal{F}^{\mathrm{antip}}_0. \end{cases}
$$

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Next, observe that, for $p_1, p_2 \in \mathcal{P}_0^{\mathbb{R}}$, the inequality $[\chi_1] := [i_1 \circ \kappa_{p_1}^* \circ i_1^{-1}] \neq [i_2 \circ \kappa_{p_2}^* \circ i_2^{-1}] =: [\chi_2]$ would force $\mathfrak{p}^{\mathbb{R}}(p_1, [i_1])$ and $\mathfrak{p}^{\mathbb{R}}(p_2, [i_2])$ to be unequal. Similarly, for $p_1, p_2 \in \mathcal{P}_0^{\text{antip}}$, the inequality $[\chi_1] := [i_1 \circ \alpha_{p_1}^* \circ i_1^{-1}] \neq [i_2 \circ \alpha_{p_2}^* \circ i_2^{-1}] =: [\chi_2]$ would force $\mathfrak{p}^{\mathbb{R}}(p_1, [i_1])$ and $\mathfrak{p}^{\mathbb{R}}(p_2, [i_2])$ to be unequal. Lemma 3.25 implies that for $p_1 \in \mathcal{P}_0^{\mathbb{R}}$ and $p_2 \in \mathcal{P}_0^{\text{antip}}$, $[\chi_1] := [i_1 \circ \kappa_{p_1}^* \circ i_1^{-1}]$ and $[i_2 \circ \alpha_{p_2}^* \circ i_2^{-1}] =: [\chi_2]$ will be distinct. Thus, to prove injectivity of $\mathfrak{p}^{\mathbb{R}}$, it suffices to prove, for each fixed $[\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$, the injectivity of the restriction of $\mathfrak{p}^{\mathbb{R}}$ to $\mathcal{F}_{0,[\chi]}^{\mathbb{R}}/G^{\mathbb{R}}$ when $\mathcal{F}_{0,[\chi]}^{\mathbb{R}} \subset \mathcal{F}_0^{\mathbb{R}}$, or to $\mathcal{F}_{0,[\chi]}^{\text{antip}}/G^{\text{antip}}$ when $\mathcal{F}_{0,[\chi]}^{\text{antip}} \subset \mathcal{F}_0^{\text{antip}}$.

For this, we appeal to Lemma A.3 as follows: Identify $\mathcal{F}_0 = \mathcal{F}_0^{\mathbb{R}} \sqcup \mathcal{F}_0^{\text{antip}}$ with $\mathcal{F}_0'^{\mathbb{R}}/\langle\kappa\rangle \sqcup \mathcal{F}_0'^{\text{antip}}/\langle\alpha\rangle$. Let $\mathbb{P}\Gamma'$ act on $Y = \mathcal{F}_0'^{\mathbb{R}}/\langle\kappa\rangle \sqcup \mathcal{F}_0'^{\text{antip}}/\langle\alpha\rangle$ as follows:

$$\gamma \cdot (p, [i]) := \begin{cases} (p, [\gamma \circ i]), & \text{if } p \in \mathcal{P}_0^{\mathbb{R}} \sqcup \mathcal{P}_0^{\text{antip}}, \text{ and } \gamma \text{ is linear}, \\ (p \cdot \kappa, [\gamma \circ i \circ \kappa^*]), & \text{if } p \in \mathcal{P}_0^{\mathbb{R}}, \text{ and } \gamma \text{ is antilinear}, \\ (p \cdot \alpha, [\gamma \circ i \circ \alpha^*]), & \text{if } p \in \mathcal{P}_0^{\text{antip}}, \text{ and } \gamma \text{ is antilinear}. \end{cases}$$

Let $Y = \mathcal{F}_0$ with $\mathcal{F}_0$ regarded as above. Let the group H in the statement of Lemma A.3 be $G := \mathrm{GL}(2,\mathbb{C})/\langle$ all eighth roots of unity $\rangle$. Let $\phi = [\chi] \in \mathbb{P}\,\mathrm{IAI}(\Lambda)$. It is straightforward to show that $Y^{\phi} = \mathcal{F}_0 X$. Consider first the case where $[\chi]$ is induced by κ. Then $Z = G^{\mathbb{R}}$ and it follows from Lemma A.3 that $\mathcal{F}_0^{[\chi]}/G^{\mathbb{R}} \to \mathcal{F}_0/G = (\mathbb{CH}^5 - \mathcal{H})$ is injective. For the remaining case where $[\chi]$ is induced by α, we then have $Z = G^{\text{antip}}$ and it follows from Lemma A.3 again that $\mathcal{F}_0^{[\chi]}/G^{\text{antip}} \to \mathcal{F}_0/G = (\mathbb{CH}^5 - \mathcal{H})$ is injective. ■

Corollary 3.33 *The map defined in Definition* 3.19

$$\pi_0(\mathcal{P}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{P}_0^{\text{antip}}) \to \mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$$

is surjective. Consequently, the cardinality of $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ *is at most seven.*

Proof The surjectivity statement follows immediately from the proof of the last statement of the preceding lemma. The cardinality bound then trivially follows Lemma 3.14:

$$|\pi_0(\mathcal{P}_0^{\mathbb{R}})| = 6, \quad \text{and} \quad |\pi_0(\mathcal{P}_0^{\text{antip}})| = 1. \qquad ■$$

Lemma 3.34 *The images of* $\{\mathcal{P}_0^{\mathbb{R},0}\}$, $\{\mathcal{P}_0^{\mathbb{R},1}\}$, $\{\mathcal{P}_0^{\mathbb{R},2}\}$, $\{\mathcal{P}_0^{\mathbb{R},3}\}$, *and* $\{\mathcal{P}_0^{\mathbb{R},4+}, \mathcal{P}_0^{\mathbb{R},4-}\}$ *(considered as subsets of* $\pi_0(\mathcal{P}_0^{\mathbb{R}})$*) under the map*

$$\pi_0(\mathcal{P}_0^{\mathbb{R}}) \sqcup \pi_0(\mathcal{P}_0^{\text{antip}}) \to \mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$$

as in Definition 3.19 *are pairwise distinct.*

Proof Appendix B.1 exhibits five involutive anti-isometries of Λ. In Appendix B.4, it is shown that their fixed $\mathbb{Z}$-lattices have pairwise distinct Vinberg diagrams. Hence, they represent five distinct $\mathbb{P}\,\mathrm{Isom}(\Lambda)$-conjugacy classes in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$. Appendices B.5 and B.6 show that all five involutive anti-isometries are induced by real octics and identify their deformation types. ■

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Remark 3.35 We stress that Lemma 3.34 does not assert that $\mathcal{P}_0^{\mathbb{R},4+}$ and $\mathcal{P}_0^{\mathbb{R},4-}$ induce the same conjugacy class in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$; they may or may not. However, this ambiguity does not pose a problem since our goal is just to describe the five connected components of $\mathcal{M}_0^{\mathbb{R}}$ as abstract real hyperbolic quotients: The complex linear change of variables $(x_0, x_1) \longmapsto \big(\exp(\mathbf{i}\pi/8)x_0, \exp(\mathbf{i}\pi/8)x_1\big)$ maps every $p(x_0, x_1) \in \mathcal{P}_0$ to $-p(x_0, x_1)$. Consequently, even if the induced conjugacy classes in $\mathbb{P}\,\mathrm{IAI}(\Lambda)/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ of $\mathcal{P}_0^{\mathbb{R},4+}$ and $\mathcal{P}_0^{\mathbb{R},4-}$ are different, the respective real hyperbolic quotients will still be isomorphic.

Proposition 3.36 *By further restricting the domain and codomain, and taking the quotient by $\mathbb{P}\Gamma$, the ($\mathbb{P}\Gamma$-equivariant) real period map*

$$\mathfrak{p}^{\mathbb{R}}\colon (\mathcal{F}_0^{\mathbb{R}}/G^{\mathbb{R}}) \sqcup (\mathcal{F}_0^{\mathrm{antip}}/G^{\mathrm{antip}}) \to \mathcal{D}_0 := \bigsqcup_{[\chi]\in\mathbb{P}\,\mathrm{IAI}(\Lambda)} (\mathbb{RH}^5_{[\chi]} - \mathcal{H})$$

descends to the following real-analytic manifold isomorphism:

$$\mathcal{M}_0^{\mathbb{R}} \sqcup \mathcal{M}_0^{\mathrm{antip}} \cong \mathbb{P}\Gamma\backslash\big((\mathcal{F}_0^{\mathbb{R},0} \sqcup \mathcal{F}_0^{\mathbb{R},1} \sqcup \mathcal{F}_0^{\mathbb{R},2} \sqcup \mathcal{F}_0^{\mathbb{R},3} \sqcup \mathcal{F}_0^{\mathbb{R},4+}/G^{\mathbb{R}}) \sqcup (\mathcal{F}_0^{\mathrm{antip}}/G^{\mathrm{antip}})\big).$$

In particular,

$$\mathcal{M}_0^{\mathbb{R}} \cong \mathbb{P}\Gamma\backslash\big(\mathcal{F}_0^{\mathbb{R},0} \sqcup \mathcal{F}_0^{\mathbb{R},1} \sqcup \mathcal{F}_0^{\mathbb{R},2} \sqcup \mathcal{F}_0^{\mathbb{R},3} \sqcup \mathcal{F}_0^{\mathbb{R},4+}/G^{\mathbb{R}}\big).$$

Combining Lemmas 3.32, 3.25, 3.34, and Proposition 3.36, we get the following.

Corollary 3.37 *Let $\chi_0, \chi_1, \chi_2, \chi_3$ be any representatives of the conjugacy classes in $\mathbb{P}\,\mathrm{IAI}(\Lambda)^{\mathbb{R}}/\mathbb{P}\,\mathrm{Isom}(\Lambda)$ induced by $\mathcal{P}_0^{\mathbb{R},0}$, $\mathcal{P}_0^{\mathbb{R},1}$, $\mathcal{P}_0^{\mathbb{R},2}$, $\mathcal{P}_0^{\mathbb{R},3}$, respectively. Let χ_4 be any representative from either the conjugacy class induced by $\mathcal{P}_0^{\mathbb{R},4+}$ or that induced by $\mathcal{P}_0^{\mathbb{R},4-}$. Then*

$$\mathcal{M}_0^{\mathbb{R},i} \cong \mathbb{P}\Gamma_i^{\mathbb{R}}\backslash(\mathbb{RH}^5_{[\chi_i]} - \mathcal{H}), \quad \textit{where } \mathbb{P}\Gamma_i^{\mathbb{R}} := \mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}(\Lambda)}(\mathbb{RH}^5_{[\chi_i]}).$$

Consequently,

$$\mathcal{M}_0^{\mathbb{R}} = \bigsqcup_{i=0}^{4} \mathcal{M}_0^{\mathbb{R},i} \cong \bigsqcup_{i=0}^{4} \mathbb{P}\Gamma_i^{\mathbb{R}}\backslash(\mathbb{RH}^5_{[\chi_i]} - \mathcal{H}).$$

4 Relationship between $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$ and $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi)$

In this section, we need to work simultaneously with projective equivalence classes of vectors, isometries and anti-isometries in various $\mathbb{Z}$-lattices and $\mathbb{Z}[\mathbf{i}]$-lattices. For the sake of clarity, we will use slightly more cumbersome notation such as $[v]_{\mathbb{C}} \in \mathbb{CH}^5$, $[A]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda$ or $[A]_{\mathbb{Z}} \in \mathbb{P}_{\mathbb{Z}}\,\mathrm{Isom}(\mathrm{Fix}\,\chi)$ to indicate that the projectivization is done over $\mathbb{C}$, $\mathcal{G} = \mathbb{Z}[\mathbf{i}]$ and $\mathbb{Z}$, respectively.

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4.1 Characterization of $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$

Let $[\chi]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}}\,\mathrm{IAI}(\Lambda)$ be fixed. Then

$$\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]}) := \{[A]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda \mid [A]_{\mathcal{G}}(\mathbb{RH}^5_{[\chi]}) \subseteq \mathbb{RH}^5_{[\chi]}\}.$$

Furthermore, let a representative $\chi \in [\chi]_{\mathcal{G}}$ be fixed. Then for $[A]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}(\Lambda)$,

$$[A]_{\mathcal{G}} \in \mathrm{Stab}_{\mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]}) \iff \begin{cases} \text{For each } A \in [A]_{\mathcal{G}}, \text{ the following holds:} \\ \text{for each } [v]_{\mathbb{C}} \in \mathbb{RH}^5_{[\chi]} \text{ and } v \in [v]_{\mathbb{C}} \text{ with} \\ \chi(v) = v, \exists \text{ unique } \beta \in \mathbb{C}^* \text{ with } |\beta| = 1 \\ \text{and } \chi\big(A(v)\big) = \beta A(v). \end{cases}$$

Remark 4.1 The uniqueness (once the representatives $A \in [A]_{\mathcal{G}}$ and $\chi \in [\chi]_{\mathcal{G}}$ are fixed) and unimodularity of β above are clear. Since both A and χ preserve primitiveness of lattice vectors, we see that β is in fact a unit Gaussian integer whenever $v \in \mathrm{Fix}(\chi)$ is primitive in Λ. If $v \in \mathrm{Fix}(\chi)$ is only primitive in the $\mathbb{Z}$-lattice $\mathrm{Fix}(\chi)$, but not in Λ, then $v = (1 + \mathbf{i})w$, for some w primitive in Λ. It can be readily shown that $\chi\big(A(v)\big) = \beta A(v)$ implies $\chi\big(A(w)\big) = \mathbf{i}\beta A(w)$. Λ-primitiveness of w then again shows that β must be a unit Gaussian integer.

Lemma 4.2 *Let $\chi \in \mathrm{IAI}(\Lambda)$ be given. Let $A \in \mathrm{Isom}(\Lambda)$ be such that $[A]_{\mathcal{G}} \in \mathrm{Stab}_{\mathbb{P}_{\mathcal{G}}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$. Then there exists a unique $\beta \in \mathbb{C}^*$ such that $\chi\big(A(v)\big) = \beta A(v)$, for all $v \in \mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, β is in fact a unit Gaussian integer.*

Proof From the preceding remark, we know that for each given $v \in \mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$, there exists a unique unimodular $\beta \in \mathbb{C}^*$ such that $\chi\big(A(v)\big) = \beta A(v)$. Furthermore, β is a unit Gaussian integer whenever v is primitive in the $\mathbb{Z}$-lattice $\mathrm{Fix}(\chi)$. So, it remains to show only that β is in fact the same for all $v \in \mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$. For this, let $b_1, \dots, b_6$ be a $\mathbb{Z}$-basis for $\mathrm{Fix}(\chi)$, and let $r_1, \dots, r_6 \in \mathbb{R}$ be six arbitrary real numbers. Set $v = r_1 b_1 + \cdots + r_6 b_6$. Then $v \in \mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$, and v is fixed by the extension of χ to $\mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$, which we also denote by χ. Note that there exist unit Gaussian integers $\beta_1, \dots, \beta_6 \in \mathbb{Z}[\mathbf{i}]$ such that $\chi\big(A(b_k)\big) = \beta_k A(b_k)$, unique for each $k = 1, \dots, 6$. Also, there exists unique unimodular $\beta \in \mathbb{C}^*$ such that $\chi\big(A(v)\big) = \beta A(v)$. Now, recall that $\mathrm{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}$ is a maximal totally real subspace of $\mathbb{C}^{1,5}$. In particular, $b_1, \dots, b_6$ are linearly independent over $\mathbb{C}$. Now, on the one hand,

$$\chi(Av) = \beta Av = \beta A\Big(\sum_{k=1}^{6} r_k b_k\Big) = \sum_{k=1}^{6} r_k \beta A(b_k).$$

On the other hand,

$$\chi(Av) = \chi\bigg(A\Big(\sum_{k=1}^{6} r_k b_k\Big)\bigg) = \sum_{k=1}^{6} r_k \chi\big(A(b_k)\big) = \sum_{k=1}^{6} r_k \beta_k A(b_k).$$

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$A(\operatorname{Fix}\chi)$	$-A(\operatorname{Fix}\chi)$	$\mathbf{i}A(\operatorname{Fix}\chi)$	$-\mathbf{i}A(\operatorname{Fix}\chi)$
$\operatorname{Fix}(\chi)$	$\operatorname{Fix}(\chi)$	$\operatorname{Fix}(-\chi)$	$\operatorname{Fix}(-\chi)$
$\operatorname{Fix}(-\chi)$	$\operatorname{Fix}(-\chi)$	$\operatorname{Fix}(\chi)$	$\operatorname{Fix}(\chi)$
$\operatorname{Fix}(\mathbf{i}\chi)$	$\operatorname{Fix}(\mathbf{i}\chi)$	$\operatorname{Fix}(-\mathbf{i}\chi)$	$\operatorname{Fix}(-\mathbf{i}\chi)$
$\operatorname{Fix}(-\mathbf{i}\chi)$	$\operatorname{Fix}(-\mathbf{i}\chi)$	$\operatorname{Fix}(\mathbf{i}\chi)$	$\operatorname{Fix}(\mathbf{i}\chi)$

Table 4.1: Recall that $A(\operatorname{Fix}\chi) = \operatorname{Fix}(\beta\chi)$, where β is one the following four unit Gaussian integers. This table summarizes how β determines $-A(\operatorname{Fix}\chi)$, $\mathbf{i}A(\operatorname{Fix}\chi)$, and $-\mathbf{i}A(\operatorname{Fix}\chi)$. For example, the entry $\operatorname{Fix}(-\chi)$ in the third row and second column says that if $\beta = -1$ (equivalently, $A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$), then $-A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$.

Hence,

$$\sum_{k=1}^{6} r_k\beta A(b_k) = \chi(Av) = \sum_{k=1}^{6} r_k\beta_k A(b_k),$$

which implies

$$r_k(\beta - \beta_k) = 0, \quad \text{for } k = 1, \dots, 6, \text{ and for arbitrary } r_1, \dots, r_6 \in \mathbb{R}.$$

This implies that $\beta_1 = \cdots = \beta_6 = \beta$ and completes the proof. ■

Proposition 4.3 *Suppose $\chi \in \operatorname{IAI}(\Lambda)$ is an involutive anti-isometry, and let $[\chi]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}}\operatorname{IAI}(\Lambda)$ be a representative of $\chi \in [\chi]_{\mathcal{G}}$. Then, for each $[A]_{\mathcal{G}} \in \operatorname{Stab}_{\mathbb{P}_{\mathcal{G}}\operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$, exactly one of the following holds:*

$$\text{either} \quad \exists A \in [A]_{\mathcal{G}}, \text{ unique up to sign, such that } A\big(\operatorname{Fix}(\chi)\big) \subseteq \operatorname{Fix}(\chi),$$
$$\text{or} \quad \exists A \in [A]_{\mathcal{G}}, \text{ unique up to sign, such that } A\big(\operatorname{Fix}(\chi)\big) \subseteq \operatorname{Fix}(\mathbf{i}\chi).$$

Proof We already know that, for an arbitrary representative $A \in [A]_{\mathcal{G}}$, we have $A(\operatorname{Fix}\chi) = \operatorname{Fix}(\beta\chi)$, where β is one of the four unit Gaussian integers. The proposition thus trivially follows from observing what the other associates of A are doing to $\operatorname{Fix}(\chi)$, as shown in Table 4.1.

For explicitness, we verify the entry in the third row and the second column of Table 4.1. So, suppose $A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$. We verify that we then indeed have $-A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$. Let $v \in \operatorname{Fix}\chi$, *i.e.*, $\chi(v) = v$. Hence, $-\chi\big(-A(v)\big) = (-1)(-\chi)(Av) = -A(v)$, where the second equality follows immediately from the assumption that $A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$. In other words, $A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$ implies that $-A(\operatorname{Fix}\chi) \subseteq \operatorname{Fix}(-\chi)$, hence $-A(\operatorname{Fix}\chi) = \operatorname{Fix}(-\chi)$, as required. ■

Remark 4.4 Note that if $A(\operatorname{Fix}\chi) = \operatorname{Fix}(\chi)$, then A also preserves each of $\operatorname{Fix}(\mathbf{i}\chi)$, $\operatorname{Fix}(-\chi)$, $\operatorname{Fix}(-\mathbf{i}\chi)$. On the other hand, if $A(\operatorname{Fix}\chi) = \operatorname{Fix}(\mathbf{i}\chi)$, then A maps $\operatorname{Fix}(\mathbf{i}\chi)$ to $\operatorname{Fix}(-\chi)$, $\operatorname{Fix}(-\chi)$ to $\operatorname{Fix}(-\mathbf{i}\chi)$, and $\operatorname{Fix}(-\mathbf{i}\chi)$ to $\operatorname{Fix}(\chi)$. Hence, we have the following.

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Proposition 4.5 *Let $\chi \in [\chi]_{\mathcal{G}}$ be fixed. Then the stabilizer* $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *can be characterized as follows:*

$$\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]}) = \mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}\left(\mathrm{Fix}(\chi) \cup \mathrm{Fix}(\mathbf{i}\chi) \cup \mathrm{Fix}(-\chi) \cup \mathrm{Fix}(-\mathbf{i}\chi)\right).$$

We seek an even more algebraically transparent expression for $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ in terms of $\mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}(\mathrm{Fix}\,\chi)$.

Definition 4.6 $[A]_{\mathcal{G}} \in \mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ is said to be of *type I* if there exists $A \in [A]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda$ such that $A(\mathrm{Fix}\,\chi) \subset \mathrm{Fix}(\chi)$, and it is said to be of *type II* if there exists $A \in [A]_{\mathcal{G}} \in \mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda$ such that $A(\mathrm{Fix}\,\chi) \subset \mathrm{Fix}(\mathbf{i}\chi)$.

Remark 4.7 Note that $[A]_{\mathcal{G}} \in \mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ is either of type I or type II by Proposition 4.3. Some simple calculations will furthermore show the following:

Lemma 4.8

(i) *If two elements in* $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *are of the same type, then their composition is an element of type I.*

(ii) *If two elements in* $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *are of different types, then their composition in either order is of type II.*

(iii) *Taking inverses in* $\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *preserves types.*

It is already clear that either

$$\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]}) = \mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}(\mathrm{Fix}\,\chi), \quad \text{or} \quad \frac{\mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})}{\mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}(\mathrm{Fix}\,\chi)} \cong \mathbb{Z}/2\mathbb{Z}.$$

We will next show, under the further assumption that $\mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}(\mathrm{Fix}\,\chi)$ is a reflection group, that the following short exact sequence

$$1 \to \mathbb{P}_{\mathbb{Z}} \mathrm{Stab}_{\operatorname{Isom}\Lambda}(\mathrm{Fix}\,\chi) \to \mathrm{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]}) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

is split (Proposition 4.11). We start with the following general result.

Proposition 4.9 *Let G be a discrete subgroup of* $\operatorname{Isom}(\mathbb{RH}^n)$*. Suppose that H is a normal subgroup of G which is generated by reflections. (H need not be the full reflection subgroup of G.) Fix a fundamental domain P of H, and let $K := \{g \in G \mid g \cdot P = P\}$. Then $G = H \rtimes K$, where the action of K on H is, as usual, by conjugation.*

Proof Since every non-identity element of H maps P to a translate of P, we see that $H \cap K = \{1\}$. So, $H \rtimes K \subseteq G$. It remains to prove the reverse inclusion, which is equivalent to the set equality $G = HK$. We now make the following statement.

Claim Every element of G preserves the union of the mirrors of the reflections in H. Consequently, every element of G preserves the complement of this union of mirrors; in particular, it maps every fundamental domain of H to a fundamental domain of H.

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

Proof of Claim Let $g \in G$ and $R_x \in H$ be the reflection in the space-like vector $x \in \mathbb{R}^{1,n}$. Then $g \circ R_x \circ g^{-1} = R_{g(x)}$ is the reflection in $g(x) \in \mathbb{R}^{1,n}$, and g maps the mirror of R_x to that of $R_{g(x)}$. It therefore remains only to prove that $R_{g(x)}$ is an element of H, but this is immediate by the hypothesis that H is normal in G. The claim is proved.

Now, let $g \in G$. By the preceding claim, g maps P to a translate of P by some element $h \in H$, *i.e.*, $g \cdot P = h \cdot P$; hence $(h^{-1}g) \cdot P = P$. Thus, $g = hk$, where $k := h^{-1}g \in K \subset G$. This completes the proof of this Proposition. ■

Remark 4.10 K in Proposition 4.9 is a subgroup of the symmetry group of the fundamental domain P. K may be trivial even if the symmetry group of P is not. Obviously, if K is trivial, then $G = H$.

Recall that $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$ is a normal subgroup of $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ of index two or one, depending on whether or not there are elements of type II. Using Proposition 4.9, we now obtain the following expression for $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ in terms of $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$:

Proposition 4.11 *Suppose* $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$ *is generated by reflections. Then exactly one of the following holds:*

- $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *has no elements of type II, in which case,*
$$\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]}) = \mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi),$$
- $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *contains elements of type II, in which case, the fundamental domain of the group action* $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi) \curvearrowright \mathbb{RH}^5$ *admits a* $(\mathbb{Z}/2\mathbb{Z})$*-symmetry, and via its norm-preserving action on the roots of* $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$*, this* $(\mathbb{Z}/2\mathbb{Z})$*-symmetry induces an order-two element* $[T] \in \operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ *of type II such that*
$$\begin{aligned}\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]}) &= \mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi) \rtimes \langle [T] \rangle \\ &\cong \mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi) \rtimes (\mathbb{Z}/2\mathbb{Z}).\end{aligned}$$

Remark 4.12 Any representative $T \in \operatorname{Isom}(\Lambda)$ of the type II and order-two element $[T] \in \operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{RH}^5_{[\chi]})$ maps $\operatorname{Fix}(\chi)$ to $\operatorname{Fix}(\mathbf{i}\chi)$, rather than back to $\operatorname{Fix}(\chi)$ itself. T induces an action on $\mathbb{RH}^5_{[\chi]} \cong \mathbb{RH}\big(\operatorname{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big)$ by identifying $\mathbb{RH}\big(\operatorname{Fix}(\mathbf{i}\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big)$ with $\mathbb{RH}\big(\operatorname{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big)$ via scalar multiplication by $(1-\mathbf{i})/\sqrt{2}$; more explicitly,

$$\begin{array}{ccc}\mathbb{RH}\big(\operatorname{Fix}(\mathbf{i}\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big) & \to & \mathbb{RH}\big(\operatorname{Fix}(\chi) \otimes_{\mathbb{Z}} \mathbb{R}\big) \\ [w] & \longmapsto & \Big[\frac{1-\mathbf{i}}{\sqrt{2}} w\Big].\end{array}$$

This identification is canonical due to the following observation:

$$\mathbf{i}\chi(w) = w \Longleftrightarrow \Big(\frac{1+\mathbf{i}}{\sqrt{2}}\Big)\Big(\frac{1+\mathbf{i}}{\sqrt{2}}\Big)\chi(w) = w \Longleftrightarrow \chi\Big(\frac{1-\mathbf{i}}{\sqrt{2}} w\Big) = \Big(\frac{1-\mathbf{i}}{\sqrt{2}}\Big) w.$$

We emphasize that while T preserves the $\mathbb{R}$-span of $\operatorname{Fix}(\chi)$ via the above canonical induced action, it fails to preserve the $\mathbb{Z}$-lattice $\operatorname{Fix}(\chi)$ itself due to the occurrence of the $1/\sqrt{2}$ factor above.

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

4.2 A Sufficient Condition for the Nonexistence of Isometries of Type II

Note that $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$ is merely the subgroup of the isometry group $\mathbb{P}_{\mathbb{Z}} \operatorname{Isom}(\operatorname{Fix}\chi)$ of the abstract $\mathbb{Z}$-lattice $\operatorname{Fix}(\chi)$ consisting of elements that extend to an action on the whole $\mathbb{Z}[\mathbf{i}]$-lattice Λ. In the case where $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$ is a reflection group and $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]})$ contains type II elements, we see that we have the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi) & \hookrightarrow & \mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi) \rtimes \langle [T] \rangle = \operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]}) \\
\big\downarrow & & \big\downarrow \\
\mathbb{P}_{\mathbb{Z}} \operatorname{Isom}(\operatorname{Fix}\chi) & \hookrightarrow & \mathbb{P}_{\mathbb{Z}} \operatorname{Isom}(\operatorname{Fix}\chi) \rtimes \langle [T] \rangle
\end{array}$$

where $[T] \in \operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]})$ is an element of type II and order two. Proposition 4.9 therefore implies the following:

Corollary 4.13 *Suppose $\mathbb{P}_{\mathbb{Z}} \operatorname{Isom}(\operatorname{Fix}\chi)$ is generated by reflections, and suppose one of the following conditions holds:*

- *The fundamental domain of $\mathbb{P}_{\mathbb{Z}} \operatorname{Isom}(\operatorname{Fix}\chi)$ admits no $(\mathbb{Z}/2\mathbb{Z})$-symmetries.*
- *It admits $(\mathbb{Z}/2\mathbb{Z})$-symmetries, but none of them induces an order-two element of $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]})$ of type II.*

Then $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]})$ in fact has no elements of type II, and

$$\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]}) = \mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi).$$

Remark 4.14 In the author's thesis [4], $\operatorname{Stab}_{\mathbb{P}_{\mathcal{G}} \operatorname{Isom}\Lambda}(\mathbb{R}\mathbb{H}^5_{[\chi]})$ was mistakenly identified with $\mathbb{P}_{\mathbb{Z}} \operatorname{Stab}_{\operatorname{Isom}\Lambda}(\operatorname{Fix}\chi)$, which need not be the case in general, as we saw in this section. The main results stated there are nonetheless correct, since for the specific cases therein (*i.e.*, $\chi = \chi_0, \chi_1, \chi_2, \chi_4$), the above equality indeed holds.

5 Distinguishing the Deformation Types

In this section, we describe a strategy to identify the deformation types of the real octics that give rise to the involutive anti-isometries of Λ.

5.1 The Isomorphism $\mathrm{O}\big(\Lambda/(1+\mathbf{i})\Lambda, q\big) \cong S_8$

Let h be the $\mathbb{Z}[\mathbf{i}]$-valued inner product of Λ and Q be the associated $\mathbb{Z}$-valued quadratic form. Q is "even-valued," and $\frac{1}{2}Q$ is thus a well-defined $\mathbb{Z}$-valued function on Λ. On the other hand, $\mathbb{Z}[\mathbf{i}]/(1+\mathbf{i})\mathbb{Z}[\mathbf{i}] \cong \mathbb{F}_2$, as rings (hence as fields), where $\mathbb{F}_2$ denotes the field with two elements. $V := \Lambda/(1+\mathbf{i})\Lambda$ is a six-dimensional $\mathbb{F}_2$-vector space. The $\mathbb{F}_2$-valued function q on V defined by $x \overset{q}{\longmapsto} \frac{1}{2}Q(x) \bmod (1+\mathbf{i})$ is an $\mathbb{F}_2$-valued quadratic form on V. It turns out that the orthogonal group $\mathrm{O}(V, q)$ is isomorphic to S_8, the symmetric group on eight objects.

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We will not give complete proofs of the above assertions but refer the reader to [4] and [6]. However, we give an intuitive description of the isomorphism between $O(V,q)$ and S_8.

Since $\dim_{\mathbb{F}_2}(V) = 6$, we immediately see that the cardinality of V is $2^6 = 64$. Let $P_8 := \{1, \ldots, 8\}$. It turns out that, as a set, V is in one-to-one correspondence with

$$W := \{\text{even-cardinality subsets of } P_8\}/\{B \sim \text{complement of } B \text{ in } P_8\}.$$

Each element of W can be considered as a pair of even-cardinality subsets of P_8, where the two subsets in each such pair are complements of each other. The cardinality of W is also 64. The $\mathbb{F}_2$-valued quadratic form on V corresponds to the $\mathbb{F}_2$-valued function on W given by:

$$W \to \mathbb{F}_2$$
$$s \longmapsto \frac{1}{2}(\text{cardinality of } s) \bmod 2.$$

Furthermore, elements of $O(V,q)$ correspond to maps from P_8 to itself which preserve the cardinality of every even-cardinality subset of P_8. Such a map is just a permutation of P_8, namely, an element of the symmetric group S_8. It turns out (see [6]) that this map $O(V,q) \to S_8$ is an isomorphism of groups. We denote its inverse by $\Phi\colon S_8 \to O(V,q)$.

5.2 Two Invariants of Involutions in S_8 Which Can Distinguish Cycle Structures

In this section, we introduce two integer invariants for the conjugacy classes in S_8 which can be used to distinguish these classes by their cycle structures. These invariants are explained in the following paragraphs and their values are shown in the third and fourth columns in Table 5.2. We then explain how these invariants will be used to distinguish the deformation types of real binary octics by their induced involutive anti-isometries.

Recall that the eight distinct roots of a smooth real binary octic are preserved as a set by complex conjugation κ on $\mathbb{CP}^1$. The collection P_8 of roots comprises a number $2n \in \{0, 2, 4, 6, 8\}$ of real points (lying on $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\} \subseteq \mathbb{CP}^1$) together with a number $(8-2n)/2$ of complex conjugate pairs. The number $2n$ determines the deformation type of a real binary octic.

On the other hand, note that when κ is restricted to the collection P_8 of the eight distinct roots of a real binary octic, it becomes an order-two permutation on P_8. Table 5.1 shows the one-to-one correspondence between the deformation types of octics and the cycle structures of $\kappa|_{P_8}$.

Of course the cycle structure of $\kappa|_{P_8}$ determines a conjugacy class in S_8. Now we make the following observations:

- κ induces an involutive antiholomorphic diffeomorphism $\kappa\colon X_p \to X_p$ on the 4-sheeted cyclic cover $X_p \to \mathbb{CP}^1$ branched over the roots of a smooth real binary octic form p. κ in turn induces an involutive anti-isometry on the $\mathbb{Z}[\mathbf{i}]$-lattice $\Lambda(X_p) \cong \Lambda$.

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Type of octic	$2n$	cycle structure of $\kappa\|_{P_8}$
0	8	(1)(2)(3)(4)(5)(6)(7)(8)
1	6	(1)(2)(3)(4)(5)(6)(78)
2	4	(1)(2)(3)(4)(56)(78)
3	2	(1)(2)(34)(56)(78)
4	0	(12)(34)(56)(78)

Table 5.1

- $(V,q) := \big(\Lambda/(1+\mathbf{i})\Lambda, q\big)$ is an orthogonal space over $\mathbb{Z}[\mathbf{i}]/(1+\mathbf{i}) \cong \mathbb{F}_2$ such that $\mathrm{O}(V,q) \cong S_8$.
- The above "abstract" isomorphism $\mathrm{O}(V,q) \cong S_8$ is geometrically realized by the permutation of the eight ramification points of the branched cover of $X \to \mathbb{CP}^1$. This fact is an immediate consequence of the fact that the monodromy group $\mathbb{P}\Gamma = \mathbb{P}\,\mathrm{Isom}(\Lambda)$ is generated by transposing pairs of roots by "continuous half turns". See [12].
- An involutive anti-isometry of Λ descends to an involutive isometry of (V,q) (because complex conjugation on $\mathbb{Z}[\mathbf{i}]$ descends to the identity on $\mathbb{Z}[\mathbf{i}]/(1+\mathbf{i})\mathbb{Z}[\mathbf{i}] \cong \mathbb{F}_2$).

The above observations show the following: Given $\chi \in \mathrm{IAI}(\Lambda)$, we can determine the deformation type of the real binary octic that gives rise to χ in the first place by determining the element (or conjugacy class) in $S_8 \cong \mathrm{O}(V,q)$ that χ descends to. In order to do this, it is sufficient to examine two invariants:

Lemma 5.1 *Let $\Phi\colon S_8 \to \mathrm{O}(V,q)$ be the isomorphism (unique up to conjugacy) constructed earlier. Then the invariants $\dim_{\mathbb{F}_2} \mathrm{Fix}\big(\Phi(\tau_i)\big)$ and the number of norm-one vectors in $\mathrm{Fix}\big(\Phi(\tau_i)\big)$ of the various cycle structures are as shown in Table* 5.2.

Outline of Proof Let $P_8 = \{1,2,\dots,8\}$. Recall that norm-one vectors in V correspond to cardinality-two subsets of P_8. The computations for all the cases are similar; we show only those for τ_6: The number of even-cardinality subsets of P_8 fixed by

Type	cycle structure of $\kappa\|_{P_8}$	$\dim_{\mathbb{F}_2} \mathrm{Fix}\big(\Phi(\tau_i)\big)$	*number of norm-one vectors in* $\mathrm{Fix}\big(\Phi(\tau_i)\big)$
0	$\tau_8 = (1)(2)(3)(4)(5)(6)(7)(8)$	6	28
1	$\tau_6 = (1)(2)(3)(4)(5)(6)(78)$	5	16
2	$\tau_4 = (1)(2)(3)(4)(56)(78)$	4	8
3	$\tau_2 = (1)(2)(34)(56)(78)$	3	4
4	$\tau_0 = (12)(34)(56)(78)$	4	4

Table 5.2

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$\tau_6 = (1)(2)(3)(4)(5)(6)(78)$ is given by

$$2 \times \binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 2 \times (1 + 15 + 15 + 1) = 2 \times 32.$$

Hence, $\dim_{\mathbb{F}_2} \operatorname{Fix}\big(\Phi(\tau_6)\big) = \log_2\left(\frac{2 \times 32}{2}\right) = \log_2(2^5) = 5$, and

$$\begin{pmatrix}\text{the number of}\\ \text{norm-one}\\ \text{vectors}\\ \text{in Fix}\big(\Phi(\tau_6)\big)\end{pmatrix} = \begin{pmatrix}\text{the number of}\\ \text{cardinality-two}\\ \text{subsets}\\ \text{preserved by } \tau_6\end{pmatrix} = 1 + \begin{pmatrix}\text{the number of all}\\ \text{cardinality-two}\\ \text{subsets of } \{1,\dots,6\}\end{pmatrix}$$
$$= 1 + \binom{6}{2} = 16.$$

■

Remark 5.2 The antipodal map $\mathbb{CP}^1 \xrightarrow{\alpha} \mathbb{CP}^1$ permutes the roots of a smooth antipodal octic in the same way as complex conjugation $\mathbb{CP}^1 \xrightarrow{\kappa} \mathbb{CP}^1$ does the roots of a smooth real octic of type 4 (*i.e.*, the roots are four complex conjugate pairs). The cycle structure for both is $(12)(34)(56)(78)$. Hence, Lemma 5.1 is insufficient to distinguish an antipodal octic from a real octic of type 4. To achieve this, we will need the idea of the proof of Lemma 3.25 instead. See section B.6.

6 The Main Theorem

Theorem 6.1

(i) *Under the real period map as in Proposition* 3.36*, the moduli space* $\mathcal{M}_0^{\mathbb{R},i}$ *of smooth real binary octics of type* $i = 0,\dots,4$ *is isomorphic as a real-analytic manifold to*

$$\mathbb{P}\Gamma_i^{\mathbb{R}} \backslash (\mathbb{RH}^5_{[\chi_i]} - \mathcal{H}),$$

where $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4 \in \mathrm{IAI}(\Lambda)$ *are defined as in equation* (B.1)*, and* $\mathbb{P}\Gamma_i^{\mathbb{R}} := \mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]})$*, where*

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]}) \cong \begin{cases} \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i), & i = 0,1,2,4 \\ \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i) \rtimes (\mathbb{Z}/2\mathbb{Z}), & i = 3 \end{cases} \tag{6.1}$$

(ii) *For each* $i = 0,\dots,4$*,* $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ *is isomorphic to the following subgroup of* $\mathbb{P}\,\mathrm{Isom}(L_i)$*:*

$$\mathbb{P}\big(\{M \in \mathrm{Isom}(L_i) \mid \mathfrak{B}_i \cdot M \cdot \mathfrak{B}_i^{-1} \in \mathbb{Z}[\mathbf{i}]^{6\times 6}\}\big),$$

where L_i *is a* $\mathbb{Z}$*-lattice given in Appendix* B.3*, and* $\mathfrak{B}_i \in \mathbb{Z}[\mathbf{i}]^{6\times 6}$ *is given in Appendix* B.2*. Each* $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$*,* $i = 0,\dots,4$*, is thus an arithmetic subgroup of* $\mathrm{Isom}(\mathbb{RH}^5)$*. Hence, each has finite co-volume and is isomorphic to a finite-index subgroup of* $\mathbb{P}\,\mathrm{Isom}(L_i)$*. In particular, each* $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ *is commensurable with* $\mathbb{P}\,\mathrm{Isom}(L_i)$*.*

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(iii) *Each* $\mathbb{P}\,\mathrm{Isom}(L_i)$, $i = 0, \dots, 4$, *is a discrete reflection subgroup of* $\mathrm{Isom}(\mathbb{RH}^5)$ *with Vinberg diagram given as in Appendix* B.4.

Proof First, we prove statement (iii). Each $\mathbb{P}\,\mathrm{Isom}(L_i)$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$ since each L_i is a $\mathbb{Z}$-lattice of signature $(+,-,-,-,-)$. Standard computations (see [17]) show that the Vinberg algorithm terminates after finitely many iterations for each of $\mathbb{P}\,\mathrm{Isom}(L_i)$. See Appendix B.4 for the results of these computations and the Vinberg diagrams of the fundamental domains in $\mathbb{RH}^5$ of $\mathbb{P}\,\mathrm{Isom}(L_i)$, $i = 0, \dots, 4$. The termination after finitely many iterations of the Vinberg algorithm implies that the reflection subgroup of each of $\mathbb{P}\,\mathrm{Isom}(L_i)$ has finite index in $\mathbb{P}\,\mathrm{Isom}(L_i)$, and that $\mathbb{P}\,\mathrm{Isom}(L_i)$ is a semidirect product of its reflection subgroup with a subgroup of the symmetry group of its fundamental domain; see [17]. The fact that each of the Vinberg diagrams has no symmetries (taking norms of roots into account) implies that each $\mathbb{P}\,\mathrm{Isom}(L_i)$ in fact equals its own reflection subgroup; hence, each $\mathbb{P}\,\mathrm{Isom}(L_i)$ is a discrete reflection subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$. This proves statement (iii).

Next, we establish statement (ii). It is straightforward to verify that χ_0, χ_1, χ_2, χ_3, and χ_4 are involutive anti-isometries of Λ. Obviously, for $\chi = \chi_0, \dots, \chi_4$, $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi)$ is the subgroup of $\mathbb{P}\,\mathrm{Isom}(\mathrm{Fix}\,\chi)$ consisting of elements that extend back to isometries of the full $\mathbb{Z}[\mathbf{i}]$-lattice Λ. Since $\mathrm{Fix}(\chi_i) \cong L_i$ as $\mathbb{Z}$-lattices, we see that, for $i = 0, \dots, 4$,

$$\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i) \cong \mathbb{P}\Big(\big\{M \in \mathrm{Isom}(L_i) \mid \mathfrak{B}_i \cdot M \cdot \mathfrak{B}_i^{-1} \in \mathbb{Z}[\mathbf{i}]^{6\times 6}\big\}\Big),$$

where each $\mathfrak{B}_i$ is given in Appendix B.2. Since $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ is defined by algebraic equations with coefficients in $\mathbb{Z}[\mathbf{i}]$, it is an arithmetic subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$. Each therefore has finite co-volume and is isomorphic to a finite-index subgroup of $\mathbb{P}\,\mathrm{Isom}(L_i)$. In particularly, each $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ is commensurable with $\mathbb{P}\,\mathrm{Isom}(L_i)$. This proves statement (ii).

Lastly, we prove statement (i). Let $\phi_0, \dots, \phi_4$ be the induced maps on $V = \Lambda/(1+\mathbf{i})\Lambda$ of $\chi_0, \dots, \chi_4$, respectively. Then $\phi_0, \dots, \phi_4 \in \mathrm{O}(V, q) \cong S_8$. By the surjectivity statement in Corollary 3.33, we know that each of $\chi_0, \dots, \chi_4$ must be induced by either real binary octics or antipodal ones. The values of $\dim_{\mathbb{F}_2}\big(\mathrm{Fix}(\phi_i)\big)$ and the numbers of norm-one vectors in $\mathrm{Fix}(\phi_i)$, $i = 0, \dots, 4$, are tabulated in Table B.1. Comparison between Table 5.1 and Table B.1 now shows that χ_0, χ_1, χ_2, χ_3 are induced by smooth real octics of types 0, 1, 2, and 3, respectively. The same comparison also shows that χ_4 is induced either by real binary octics of type 4 or by antipodal octics. However, one of the nodes in the Vinberg diagram of the $\mathbb{Z}$-lattice $\mathrm{Fix}(\chi_4)$ is of the form $(1+\mathbf{i})w$, where w is a primitive vector in Λ of squared norm -2. The orthogonal complement of w in $\mathbb{CH}^5$ is thus one of the constituent hyperplanes in the hyperplane arrangement $\mathcal{H}$ (see Section 2.3). This shows that octics parametrized by $\mathbb{RH}^5_{[\chi_4]}$ can deform to singular octics with exactly one node. This in turn shows that χ_4 is induced by real binary octics since smooth antipodal octics cannot deform to singular octics with just one node. Comparison of the last rows of Table 5.1 and Table B.1 now shows that χ_4 is induced by real binary octics of type 4. (See also Appendix B.6.) Corollary 3.37 now implies that the real period map gives

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the following isomorphism of real-analytic manifolds:

$$\mathcal{M}_0^{\mathbb{R},i} \cong \mathbb{P}\Gamma_i^{\mathbb{R}} \backslash (\mathbb{RH}^5_{[\chi_i]} - \mathcal{H}),$$

where $\mathbb{P}\Gamma_i^{\mathbb{R}} := \mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]})$, for $i = 0, \dots, 4$.

We now prove the formula (6.1). Since, for each $i = 0, \dots, 4$, $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ is a subgroup of the reflection group $\mathbb{P}\,\mathrm{Isom}(L_i)$, $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i)$ is itself a reflection group. Hence, Proposition 4.11 applies for each $i = 0, \dots, 4$, and we have

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]}) \cong \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i) \rtimes (\mathbb{Z}/2\mathbb{Z}) \quad \text{or} \quad \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i),$$

according to whether $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]})$ contains elements of type II. Since the Vinberg diagram of $\mathbb{P}\,\mathrm{Isom}(L_0)$, $\mathbb{P}\,\mathrm{Isom}(L_1)$ and $\mathbb{P}\,\mathrm{Isom}(L_4)$ have no $(\mathbb{Z}/2)$-symmetries, Corollary 4.13 immediately implies that

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]}) \cong \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i), \quad \text{for } i = 0, 1, 4.$$

The computations in Appendix B.7 show that $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]})$ contains no elements of type II, whereas the computations in Appendix B.8 show that $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_3]})$ does contain elements of type II. Consequently,

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_i]}) \cong \begin{cases} \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i), & i = 2, \\ \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_i) \rtimes (\mathbb{Z}/2\mathbb{Z}), & i = 3. \end{cases}$$

This completes the proof of (6.1) as well as that of the theorem. ∎

A Technical Lemmas

Lemma A.1 *Let $\phi\colon \mathbb{C}^n \to \mathbb{C}^n$ be an antilinear map such that ϕ^2 acts on $\mathbb{C}^n$ by multiplication of some scalar $\alpha \in \mathbb{C}$. Then α in fact must be real.* ∎

Proof Note that $\phi^2 = \alpha \cdot \mathrm{id}_{\mathbb{C}^n}$ implies that $\mathrm{trace}(\phi^2) = n \cdot \alpha$. So, it suffices to prove that $\mathrm{trace}(\phi^2)$ is real. Since ϕ is antilinear, there exists some $A \in \mathbb{C}^{n\times n}$ such that $\phi(v) = A \cdot \overline{v}$, for all $v \in \mathbb{C}^n$. Hence $\phi^2(v) = A \cdot \overline{A} \cdot v$ and $\mathrm{trace}(\phi^2) = \mathrm{trace}(A \cdot \overline{A})$. Now, recall that for any two complex square matrices C, D of the same dimensions, we have $\mathrm{trace}(C \cdot D) = \mathrm{trace}(D \cdot C)$. Thus,

$$\mathrm{trace}(A \cdot \overline{A}) = \mathrm{trace}(\overline{A} \cdot A) = \mathrm{trace}(\overline{A \cdot \overline{A}}) = \overline{\mathrm{trace}(A \cdot \overline{A})} \quad \Longrightarrow \quad \mathrm{trace}(A \cdot \overline{A}) \in \mathbb{R}.$$

This completes the proof of the lemma. ∎

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Lemma A.2 *Let Y be a set, L a group with a free left action on Y and R a group with a free right action on Y such that the two actions commute. Let $y \in Y$, and $l := y \cdot R \in Y/R$ and $r := L \cdot y \in L\backslash Y$. Then for every element $\phi \in L$ preserving l, there exists a unique element $\hat{\phi} \in R$ such that $\phi \cdot y \cdot \hat{\phi} = y$. Furthermore, the map $L_l \to R_r \colon \phi \longmapsto \hat{\phi}$ is an anti-isomorphism of the two stabilizer groups.*

Proof Immediate. ■

Lemma A.3 *Let H be a group acting freely on a set Y, ϕ be a permutation on Y normalizing H, and Z the centralizer of ϕ in H (considering both ϕ and H as contained in the symmetric group on Y). Denote the set of fixed points of ϕ by Y^ϕ. Then the natural map $Y^\phi/Z \xrightarrow{\Psi} Y/H$ is injective.*

Proof Let $y_1, y_2 \in Y^\phi$ be such that $\Psi(Z \cdot y_1) = \Psi(Z \cdot y_2)$. We want to show $Z \cdot y_1 = Z \cdot y_2$, *i.e.*, there exists some $z \in Z$ such that $z \cdot y_1 = y_2$. Now, since $\Psi(Z \cdot y_1) = H \cdot y_1$ and $\Psi(Z \cdot y_2) = H \cdot y_2$, the hypothesis $\Psi(Z \cdot y_1) = \Psi(Z \cdot y_2)$ is equivalent to $H \cdot y_1 = H \cdot y_2$, *i.e.*, there exists some $g \in H$ such that $g \cdot y_1 = y_2$. We claim that in fact $g \in Z$, which will complete the proof. For the claim, note that $y_1, y_2 \in Y^\phi$ implies the following:

$$(g^{-1} \circ \phi^{-1} \circ g \circ \phi)(y_1) = (g^{-1} \circ \phi^{-1} \circ g)(y_1) = (g^{-1} \circ \phi^{-1})(y_2) = g^{-1}(y_2) = y_1.$$

Since ϕ normalizes H, $\phi^{-1} \circ g \circ \phi$ is an element of H; hence, so is $g^{-1} \circ \phi^{-1} \circ g \circ \phi$. Freeness of the action of H on X now implies that $g^{-1} \circ \phi^{-1} \circ g \circ \phi = \mathrm{id}_Y$; equivalently, $g \circ \phi = \phi \circ g$, *i.e.*, $g \in Z$. ■

B Computational Results used in the Proof of Theorem 6.1

In this appendix, we collect and elaborate the computational arguments used in the proof of Theorem 6.1.

We define five involutive anti-isometries of Λ in B.1. We show that they are induced by the five deformation types of real binary octics by determining the cycle structures of their induced maps in $\mathrm{O}\big(\Lambda/(1+\mathbf{i})\Lambda, q\big) \cong S_8$ in B.5 and B.6. We give explicit descriptions of $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$, $i = 0, \ldots, 4$, in B.4, B.7, and B.8.

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B.1 The Five Involutive Anti-isometries

Define the map $\chi_2 \colon \Lambda \to \Lambda$ by

$$\chi_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} := \begin{pmatrix} \overline{z_2} \\ \overline{z_1} \\ \overline{z_4} \\ \overline{z_3} \\ -\mathbf{i}\overline{z_5} \\ \overline{z_6} \end{pmatrix}.$$

Then $\chi_2 \in \mathrm{IAI}(\Lambda)$. Next, define

$$A_0 := \begin{bmatrix} 0 & -\mathbf{i} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{i} & -1-\mathbf{i} & 1 & 1 \end{bmatrix},$$

$$A_4 := \begin{bmatrix} 0 & -\mathbf{i} & 0 & 0 & \mathbf{i} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{i} & \mathbf{i} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \mathbf{i} & -1-\mathbf{i} & \mathbf{i} & -1-\mathbf{i} & 2 & 1 \end{bmatrix}.$$

Consider $A_0, \dots, A_4$, as $\mathbb{Z}[\mathbf{i}]$-linear endomorphisms on Λ, via $v \mapsto A_i \cdot v$. Define

(B.1) $\chi_0 := A_0 \circ \chi_2, \chi_1 := A_1 \circ \chi_2, \chi_2 := A_2 \circ \chi_2 = \chi_2, \chi_3 := A_3 \circ \chi_2, \chi_4 := A_4 \circ \chi_2.$

It is straightforward to verify that $A_0, \dots, A_4 \in \mathrm{Isom}(\Lambda)$, and $\chi_0, \dots, \chi_4 \in \mathrm{IAI}(\Lambda)$. As the notation suggests, $\chi_0, \dots, \chi_4$ shall correspond to real binary octics of types

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$0, \ldots, 4$ respectively, as will be shown in this appendix. Appealing to the theory developed in the preceding sections, we now present a series of straightforward computational results which will establish this correspondence (see also Proof of Theorem 6.1). We will omit the details of these computations due to their routine but tedious nature.

B.2 $\mathbb{Z}$-bases for the Fixed Lattices of $\chi_0, \ldots, \chi_4$

The column vectors of the following matrices form respectively $\mathbb{Z}$-bases for the fixed $\mathbb{Z}$-lattices of the anti-involutions $\chi_0, \ldots, \chi_4$:

$$\mathfrak{B}_0 := \begin{bmatrix} 0 & 0 & 0 & 1-\mathbf{i} & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\mathbf{i} & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 \\ 1-\mathbf{i} & 0 & 0 & 0 & -1+\mathbf{i} & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\mathfrak{B}_1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1-\mathbf{i} \\ 0 & 1 & 0 & 0 & 0 & 1+\mathbf{i} \\ 0 & 0 & 1-\mathbf{i} & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 & 0 \\ 1-\mathbf{i} & 0 & 0 & 0 & -1+\mathbf{i} & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\mathfrak{B}_2 := \begin{bmatrix} 0 & 1 & 0 & 0 & 1-\mathbf{i} & 0 \\ 0 & 1 & 0 & 0 & 1+\mathbf{i} & 0 \\ -1 & 0 & 1 & 1 & 0 & 1-\mathbf{i} \\ -1 & 0 & 1 & 1 & 0 & 1+\mathbf{i} \\ 1-\mathbf{i} & 0 & -1+\mathbf{i} & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

$$\mathfrak{B}_3 := \begin{bmatrix} 0 & 0 & 1 & 1-\mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & 1+\mathbf{i} & 0 & 0 \\ \mathbf{i} & \mathbf{i} & 0 & 0 & 1-\mathbf{i} & 1-\mathbf{i} \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1-\mathbf{i} & 1-\mathbf{i} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1-\mathbf{i} \end{bmatrix},$$

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$$\mathfrak{B}_4 := \begin{bmatrix} 1-\mathbf{i} & 0 & 0 & \mathbf{i} & 0 & 0 \\ 2 & 2 & -1 & 0 & 0 & 2 \\ 0 & 1-\mathbf{i} & 0 & \mathbf{i} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\mathbf{i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mathbf{i} \end{bmatrix}.$$

B.3 The Induced Integral Quadratic Forms on $\mathrm{Fix}(\chi_0), \dots, \mathrm{Fix}(\chi_4)$

These are determined by inner product matrices of $\mathfrak{B}_0, \dots, \mathfrak{B}_4$, which are given respectively by

$$L_0 := \mathrm{diag}(+2,-2,-2,-2,-2,-2), \quad L_1 := \mathrm{diag}(+2,-2,-2,-2,-2,-4),$$
$$L_2 := \mathrm{diag}(+2,-2,-2,-2,-4,-4), \quad L_3 := \mathrm{diag}(+2,-2,-2,-4,-4,-4),$$
$$L_4 := \begin{bmatrix} -4 & -4 & 2 & 0 & 0 & -4 \\ -4 & -12 & 6 & 0 & 0 & -8 \\ 2 & 6 & -4 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 & 2 & -2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -4 & -8 & 4 & -2 & 0 & -8 \end{bmatrix}.$$

B.4 The Vinberg Diagrams

The Vinberg diagrams [17] of the reflection subgroups of the (integral) isometry groups $\mathbb{P}\,\mathrm{Isom}(L_0), \dots, \mathbb{P}\,\mathrm{Isom}(L_4)$ are shown in Figure B.1. In these diagrams, the following convention is used: No bond between two nodes means the two corresponding hyperplanes meet orthogonally; a single bond means they meet with interior angle $\pi/3$; a double bond means the interior angle is $\pi/4$; a bond marked with ∞ means the two hyperplanes are parallel; a dotted bond means they are ultraparallel.

The number of subdivisions within each node is minus one-half of the squared norm of the corresponding root. Equivalently, the squared norm of a root is equal to -2 times the number of subdivisions in its corresponding node in the Vinberg diagram. For example, consider the Vinberg diagram of L_2. The node r_4 has four subdivisions, which indicates that the corresponding root has norm $-8 = -2 \times 4$. Similarly, the roots corresponding to r_1, r_2, r_3, r_5, r_6, and r_7 have norms -4, -4, -2, -4, -4, and -4, respectively.

The labeling of the nodes of the diagrams for L_2 and L_3 will be used in Sections B.7 and B.8. The common labeling of these two sets of nodes is for economy of notation; the two sets otherwise have no relation to each other. Each of these five Vinberg diagrams has no symmetries, when norms of roots are taken into account. This implies that each of $\mathbb{P}\,\mathrm{Isom}(L_0), \dots, \mathbb{P}\,\mathrm{Isom}(L_4)$ is a discrete reflection subgroup of $\mathrm{Isom}(\mathbb{RH}^5)$. Hence, Corollary 4.13 applies to each of them.

Ignoring norms of roots, only the Vinberg diagrams of $\mathbb{P}\,\mathrm{Isom}(L_2)$ and $\mathbb{P}\,\mathrm{Isom}(L_3)$

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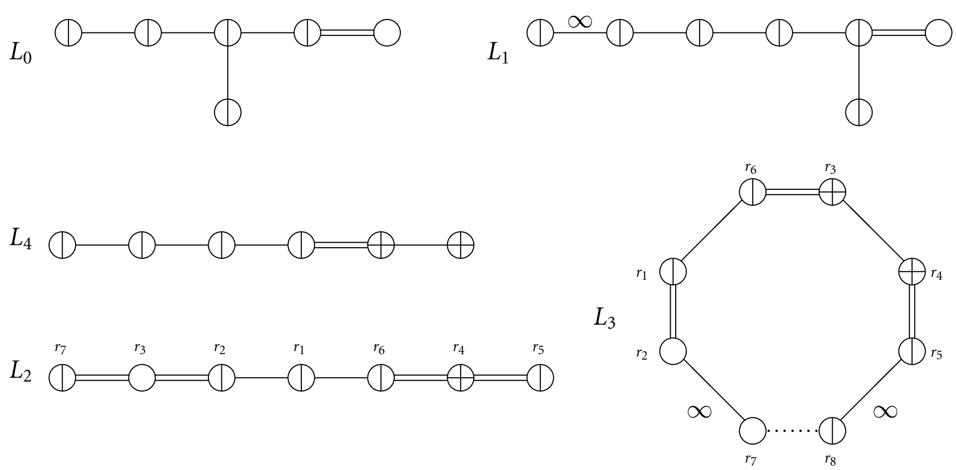

Figure B.1

	$\dim_{\mathbb{F}_2}\big(\mathrm{Fix}(\cdot)\big)$	number of norm-one vectors in $\mathrm{Fix}(\cdot)$
ϕ_0	6	28
ϕ_1	5	16
ϕ_2	4	8
ϕ_3	3	4
ϕ_4	4	4

Table B.1

have a $(\mathbb{Z}/2\mathbb{Z})$-symmetry, which implies (by Corollary 4.13) that

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]}) = \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi), \quad \text{for } \chi = \chi_0, \chi_1, \chi_4.$$

B.5 The Invariants of the Induced Isometries on $V = \Lambda/(1+\mathbf{i})\Lambda$

Let $\phi_0, \ldots, \phi_4 \in O(V, q)$ be the involutive isometries on $V = \Lambda/(1+\mathbf{i})\Lambda$ induced by $\chi_0, \ldots, \chi_4$ respectively. Then straightforward computations show that the two invariants mentioned in Lemma 5.1 of $\phi_0, \ldots, \phi_4$ are as tabulated in Table B.1.

We show the explicit verification for ϕ_2. Now,

$$\phi_2 \in \mathrm{O}(V, q) = \mathrm{O}\big(\Lambda/(1+\mathbf{i})\Lambda, q\big)$$

is induced by $\chi_2 \in \mathrm{IAI}(\Lambda)$, and χ_2 is defined by

$$\chi_2 \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} \overline{z_2} \\ \overline{z_1} \\ \overline{z_4} \\ \overline{z_3} \\ -\mathbf{i}\overline{z_5} \\ \overline{z_6} \end{pmatrix}.$$

Next, observe that the endomorphism $\xi \longmapsto \mathbf{i} \cdot \xi$ on $\mathbb{Z}[\mathbf{i}]$ descends to the identity map on $\mathbb{F}_2 = \mathbb{Z}[\mathbf{i}]/(1+\mathbf{i})\mathbb{Z}[\mathbf{i}]$, since $\mathbf{i} = 1 + \mathbf{i} \cdot (1+\mathbf{i}) \equiv 1 \bmod (1+\mathbf{i})$. Similarly, the endomorphism $\xi \longmapsto -\xi$ on $\mathbb{Z}[\mathbf{i}]$ also descends to the identity map on $\mathbb{F}_2$, since $-(m+\mathbf{i}n) = (m+\mathbf{i}n) - 2\cdot(m+\mathbf{i}n) = (m+\mathbf{i}n) - (1+\mathbf{i})(1-\mathbf{i})(m+\mathbf{i}n) \equiv m+\mathbf{i}n \bmod (1+\mathbf{i})$. As a result, complex conjugation on $\mathbb{Z}[\mathbf{i}]$ descends to the identity map on $\mathbb{F}_2$ as well, since $\overline{m+\mathbf{i}n} = m - \mathbf{i}n \equiv m + \mathbf{i}n \bmod (1+\mathbf{i})$. Hence, ϕ_2 is explicitly given by

$$\phi_2 \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \xi_1 \\ \xi_4 \\ \xi_3 \\ \xi_5 \\ \xi_6 \end{pmatrix}$$

Consequently, the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in V$$

form a basis for $\mathrm{Fix}(\phi_2)$, and we see that

$$\dim_{\mathbb{F}_2} \mathrm{Fix}(\phi_2) = 4.$$

Next, we count the norm-one vectors in $\mathrm{Fix}(\phi_2)$: Since v_1, v_2, v_3, v_4 form a basis for $\mathrm{Fix}(\phi_2)$, we see that $\mathrm{Fix}(\phi_2)$ contains exactly 16 vectors and the general form of a vector in $\mathrm{Fix}(\phi_2)$ is $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$, where $c_1, \ldots, c_4 \in \mathbb{F}_2$. The norm $q(v) \in \mathbb{F}_2$ of a vector $v \in V = \Lambda/(1+\mathbf{i})\Lambda$ is defined in Section 5.1 and can be readily calculated. The sixteen vectors in $\mathrm{Fix}(\phi_2)$ and their norms are tabulated in Table B.2, from which it is immediate that $\mathrm{Fix}(\phi_2)$ contains exactly eight vectors of norm one.

Remark B.1 Comparing Table 5.2 with Table B.1, we may conclude that χ_0, χ_1, χ_2, and χ_3 correspond to real binary octics of types 0, 1, 2, and 3 respectively. By Remark 5.2, it is furthermore clear that χ_4 is induced by either real binary octics of type 4 or antipodal binary octics. We show that χ_4 is in fact induced by real binary octics in Section B.6.

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c_1	c_2	c_3	c_4	$v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$	$q(v)$
0	0	0	0	$(000000)^T$	0
0	0	0	1	$(000001)^T$	0
0	0	1	0	$(000010)^T$	0
0	0	1	1	$(000011)^T$	1
0	1	0	0	$(001100)^T$	1
0	1	0	1	$(001101)^T$	1
0	1	1	0	$(001110)^T$	1
0	1	1	1	$(001111)^T$	0
1	0	0	0	$(110000)^T$	1
1	0	0	1	$(110001)^T$	1
1	0	1	0	$(110010)^T$	1
1	0	1	1	$(110011)^T$	0
1	1	0	0	$(111100)^T$	0
1	1	0	1	$(111101)^T$	0
1	1	1	0	$(111110)^T$	0
1	1	1	1	$(111111)^T$	1

Table B.2: All sixteen vectors in $\mathrm{Fix}(\phi_2)$ and their norms. The superscript T denotes transposition.

B.6 χ_4 is Induced by Real Binary Octics of Type 4

We can determine that χ_4 is induced by real binary octics of type 4 (rather than by antipodal binary octics) by the following observations:

- Recall that the collection $\mathcal{H}$ of discriminant hyperplanes in $\mathbb{CH}^5$ consists of orthogonal complements of vectors in Λ of squared norm -2, and that the smooth points of $\mathcal{H}$ correspond to nodal binary octics, *i.e.*, singular binary octics with one double point and no other singularities.
- One of the roots of L_4 is of the form $(1 + \mathbf{i})w$, where w is a primitive vector in Λ of squared norm -2. The fundamental domain of $\mathbb{P}\,\mathrm{Isom}(L_4)$ therefore has one discriminant wall, and octics parametrized by $\mathbb{RH}^5_{[\chi_4]}$ can deform to nodal ones.
- Antipodal octics can only deform to octics which are more singular than the nodal ones. (See the proof of Lemma 3.25.)

It is now clear that χ_4 is induced by real binary octics of type 4.

Remark B.2 As already mentioned in subsection B.4, we have

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi]}) = \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi), \quad \text{for } \chi = \chi_0, \chi_1, \chi_4.$$

It remains to determine, for $\chi = \chi_2, \chi_3$, whether $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi]})$ is equal to $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi)$, or is isomorphic to $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi) \rtimes (\mathbb{Z}/2\mathbb{Z})$.

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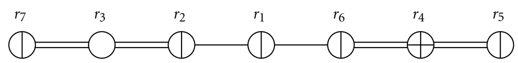
B.7 Comparing $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi_2]})$ and $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_2)$

Recall that the Vinberg diagram for $\mathbb{P}\,\mathrm{Isom}(\mathrm{Fix}\,\chi_2)$ is



where the roots $r_1, r_2, \ldots, r_7$ are labeled according to order of appearance in the Vinberg Algorithm. The above diagram has only one symmetry (ignoring norms of roots): it is the $(\mathbb{Z}/2\mathbb{Z})$-symmetry determined by exchanging the following 1-dimensional subspaces:

$$\mathbb{R}\cdot r_1 \longleftrightarrow \mathbb{R}\cdot r_1, \quad \mathbb{R}\cdot r_2 \longleftrightarrow \mathbb{R}\cdot r_6, \quad \mathbb{R}\cdot r_3 \longleftrightarrow \mathbb{R}\cdot r_4, \quad \mathbb{R}\cdot r_5 \longleftrightarrow \mathbb{R}\cdot r_7.$$

Recall also that the natural identification map (induced by projectivizing over $\mathbb{C}$) from $\mathrm{Fix}(\mathbf{i}\chi_2)\otimes_{\mathbb{Z}}\mathbb{R}$ back to $\mathrm{Fix}(\chi_2)\otimes_{\mathbb{Z}}\mathbb{R}$ is given by scalar multiplication by $1-\mathbf{i}$. Taking all the above observations into account, we see that the group $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]})$ admits elements of type II if and only if the following conditions define an element $T \in \mathrm{Isom}(\Lambda)$ such that $[T]$ is a type II element of $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]})$:

$$(1-\mathbf{i})T(r_1) = \pm\sqrt{2}r_1, \tag{B.2}$$

$$(1-\mathbf{i})T(r_2) = \pm\sqrt{2}r_6, \quad (1-\mathbf{i})T(r_6) = \pm\sqrt{2}r_2, \tag{B.3}$$

$$(1-\mathbf{i})T(r_3) = \pm\frac{1}{\sqrt{2}}r_4, \quad (1-\mathbf{i})T(r_4) = \pm 2\sqrt{2}r_3, \tag{B.4}$$

$$(1-\mathbf{i})T(r_5) = \pm\sqrt{2}r_7, \quad (1-\mathbf{i})T(r_7) = \pm\sqrt{2}r_5, \tag{B.5}$$

where the signs of the right-hand-sides must be either all positive or all negative. Either case leads to a contradiction, which shows that $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]})$ has no type II elements. We derive the contradiction for only the first case, the other case being completely analogous. We now make the following:

Claim There exists no such $T \in \mathrm{Isom}(\Lambda)$.

When expressed in the "standard" basis of Λ, the roots $r_1, \ldots, r_7$ are given, respectively from left to right, by the column vectors of the following matrix:

$$\begin{bmatrix} -1 & 0 & 0 & -1+\mathbf{i} & 0 & 1 & 2-\mathbf{i} \\ -1 & 0 & 0 & -1-\mathbf{i} & 0 & 1 & 2+\mathbf{i} \\ 1 & 0 & -1 & 1-\mathbf{i} & -1+\mathbf{i} & 1 & -1 \\ 1 & 0 & -1 & 1+\mathbf{i} & -1-\mathbf{i} & 1 & -1 \\ -1+\mathbf{i} & 1-\mathbf{i} & 0 & 0 & 0 & 0 & 1-\mathbf{i} \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that $r_7 = r_2 + 2r_3 - r_4 - r_5 + r_6$. Hence, condition (B.5) implies

$$\begin{aligned}\sqrt{2}r_5 = (1-\mathbf{i})T(r_7) &= (1-\mathbf{i})T(r_2 + 2r_3 - r_4 - r_5 + r_6)\\ &= \sqrt{2}r_6 + \sqrt{2}r_4 - 2\sqrt{2}r_3 - \sqrt{2}r_7 + \sqrt{2}r_2,\end{aligned}$$

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which yields this alternative expression for r_7: $r_7 = r_2 - 2r_3 + r_4 - r_5 + r_6$. Comparing with the original expression for r_7 in terms of $r_2, \dots, r_6$, we see that

$$r_2 + 2r_3 - r_4 - r_5 + r_6 = r_7 = r_2 - 2r_3 + r_4 - r_5 + r_6 \implies 2r_3 = r_4,$$

which is a contradiction, since r_3 and r_4 are linearly independent over $\mathbb{Z}[\mathbf{i}]$, in particular, over $\mathbb{Z}$. The claim is proved. By Corollary 4.13, we may now conclude that $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]})$ has no elements of type II, and $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}^5_{[\chi_2]}) = \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_2)$.

B.8 Comparing $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi_3]})$ and $\mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_3)$

Recall the Vinberg diagram in this case from Figure B.1. Again, the roots $r_1, r_2, \dots, r_8$ are labeled according to order of appearance in the Vinberg Algorithm. In terms of the "standard" basis for Λ, these roots are given, respectively from left to right, by the column vectors of the following matrix:

$$\begin{bmatrix} 1 & -1 & -1+\mathbf{i} & 0 & 0 & 1-\mathbf{i} & 1 & 1-\mathbf{i} \\ 1 & -1 & -1-\mathbf{i} & 0 & 0 & 1+\mathbf{i} & 1 & 1+\mathbf{i} \\ -\mathbf{i} & 0 & 1-\mathbf{i} & 0 & -1+\mathbf{i} & 2\mathbf{i} & 2\mathbf{i} & 2 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 2 \\ -1+\mathbf{i} & 0 & 0 & 0 & 0 & 2-2\mathbf{i} & 2-2\mathbf{i} & 2-2\mathbf{i} \\ 0 & 0 & 0 & -1-\mathbf{i} & 1+\mathbf{i} & 1 & 1 & 1-\mathbf{i} \end{bmatrix}.$$

The only symmetry (ignoring norms of roots) here is the $(\mathbb{Z}/2\mathbb{Z})$-symmetry determined by exchanging the following 1-dimensional subspaces:

$$\mathbb{R}\cdot r_1 \longleftrightarrow \mathbb{R}\cdot r_4, \quad \mathbb{R}\cdot r_2 \longleftrightarrow \mathbb{R}\cdot r_5, \quad \mathbb{R}\cdot r_3 \longleftrightarrow \mathbb{R}\cdot r_6, \quad \mathbb{R}\cdot r_7 \longleftrightarrow \mathbb{R}\cdot r_8.$$

Therefore, $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi_3]})$ has elements of type II if and only if the following conditions define an element $T \in \mathrm{Isom}(\Lambda)$ such that $[T]$ is an element of $\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi_3]})$ of type II:

$$(1-\mathbf{i})T(r_1) = r_4, \quad (1-\mathbf{i})T(r_2) = r_5, \quad (1-\mathbf{i})T(r_3) = r_6, \quad (1-\mathbf{i})T(r_7) = r_8,$$
$$(1-\mathbf{i})T(r_4) = 2r_1, \quad (1-\mathbf{i})T(r_5) = 2r_2, \quad (1-\mathbf{i})T(r_6) = 2r_3, \quad (1-\mathbf{i})T(r_8) = 2r_7.$$

Straightforward calculations now show that the above (overdetermined) set of conditions indeed defines such a $T \in \mathrm{Isom}(\Lambda)$ and we conclude that

$$\mathrm{Stab}_{\mathbb{P}\,\mathrm{Isom}\,\Lambda}(\mathbb{RH}_{[\chi_3]}) = \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_3) \rtimes \langle [T] \rangle \cong \mathbb{P}\,\mathrm{Stab}_{\mathrm{Isom}\,\Lambda}(\mathrm{Fix}\,\chi_3) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

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https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press

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Department of Mathematics, University of Utah, Salt Lake City, Utah, USA
e-mail: chu@math.utah.edu

https://doi.org/10.4153/CJM-2011-026-1 Published online by Cambridge University Press