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ON Y. C. WONG'S CONJECTURE

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Let M be an n-dimensional connected C^∞ manifold with a linear connection Γ. M is said to be of recurrent curvature with respect to Γ if the corresponding curvature tensor R satisfies [1], [4]

$$\nabla R = W \otimes R, \quad W \neq 0,$$

where ∇ denotes covariant derivative with respect to Γ and W is a nonzero co-vector called the recurrence co-vector. Let T be the torsion of Γ. Then the conjecture of Y. C. Wong as given in [5] is as follows:

CONJECTURE. For every linear connection with $T=0$, $\nabla R = W \otimes R$ and $W \neq 0$, the tensor ∇W is (everywhere) symmetric if the Ricci tensor is (everywhere) symmetric.

Attempts have been made to prove or disprove this conjecture [2], [3] but to the best knowledge of the present author no successful proof has been given as yet. It is proved here that the conjecture is true for the case where the linear connection with recurrent curvature admits a suitable projective transformation.

Throughout this paper it will be assumed that $T=0$. Also tensor fields and vector fields on M will be simply referred to as tensors and vectors respectively. Whenever a tensor or a vector is expressed in terms of components it will be understood that they are being expressed in terms of a local co-ordinate system.

Thus in terms of a local co-ordinate system the condition for recurrent curvature can be written as

$$\nabla_m R^h_{ijk} = W_m R^h_{ijk}, \tag{1}$$

where R^h_{ijk} are the components of R in this co-ordinate system. The condition $T=0$ is expressed locally as $\Gamma^h_{ij} = \Gamma^h_{ji}$.

For a linear connection with recurrent curvature it follows that

$$\nabla_i R_{[jk]} = W_i R_{[jk]}, \tag{2}$$

where $R_{jk} = R^i_{ijk}$, $R_{[jk]} = \frac{1}{2}(R_{jk} - R_{kj})$. The Bianchi identities for curvature also imply

$$W_i R_{[jk]} + W_k R_{[ij]} + W_j R_{[ki]} = 0. \tag{3}$$

We shall say that a vector field v induces a $W(v)$ projective change of Γ if

$$L_v \Gamma^i_{jk} = \delta^i_j W_k + \delta^i_k W_j \tag{4}$$

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where L_v denotes the Lie derivative with respect to v. It follows that

(5) $$L_v\Gamma^i_{ik} = (n+1)W_k$$

and in view of the formula

$$L_vR^h_{ijk} = \nabla_i L_v\Gamma^h_{jk} - \nabla_j L_v\Gamma^h_{ik} \quad [7, \text{p. } 17]$$

together with (4) and (5), we find

(6) $$L_vR_{[jk]} = -(n+1)(\nabla_{[j}W_{k]}).$$

Also from the commutation relation between Lie and covariant derivatives [7, p. 16]

$$L_v\nabla_i R_{[jk]} - \nabla_i L_v R_{[jk]} = -L_v\Gamma^r_{ij}R_{[rk]} - L_v\Gamma^r_{ik}R_{[jr]},$$

together with (2), (3) and (4), we obtain

(7) $$W_iL_vR_{[jk]} - \nabla_i L_vR_{[jk]} = -(L_vW_i + 3W_i)R_{[jk]}.$$

THEOREM. *If a linear connection with* $T=0$, $\nabla R = W\otimes R$ *and* $W\neq 0$ *admits a* $W(v)$ *projective change, then* R_{ij} *is symmetric if and only if* $\nabla_i W_j$ *is symmetric.*

Proof. The "only if" part follows from (6). Conversely, Yamaguchi [6] has proved that if such a connection admits a $\phi(v)$ projective change then exactly one of the following holds: (i) $\phi=0$, (ii) $P^h_{ijk}=0$ and (iii) $L_vW_i+4\phi_i=0$. Here $\phi=W\neq 0$ so case (i) is excluded. In case (ii) R_{ij} is symmetric since [7, p. 132]

$$R_{[ij]} = -(n+1)P_{[ij]}.$$

In case (iii) the result follows from (7).

This establishes the conjecture on the assumption that such a transformation is possible.

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