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EXTENSIONS OF THEOREMS OF GAGLIARDO AND MARCUS AND MIZEL TO ORLICZ SPACES

GRAHAME HARDY

In 1958, Gagliardo showed that if u is a locally integrable function on a domain Ω satisfying the cone condition, with all weak derivatives belonging to the Lebesgue space $L_p(\Omega)$ $(1 \le p < \infty)$, then u belongs to $L_p(\Omega)$ also. We extend this result to Orlicz spaces, and use it to extend a result of Marcus and Mizel on Nemitsky operators between Sobolev spaces to Orlicz-Sobolev spaces.

1. Introduction

Let Ω be a domain (that is, an open and connected set) in R_n , and g a function from $\Omega \times \mathrm{R}_m$ into R . In Marcus and Mizel [8], it is shown that, under suitable assumptions, g determines a mapping from $X = W_{1,q_1}(\Omega) \times \ldots \times W_{1,q_m}(\Omega)$ into $W_{1,p}(\Omega)$. This mapping associates with every $u = \left(u_1, \ldots, u_m\right) \in X$, a function $G \in W_{1,p}$ defined by $G \circ u(x) = g\left(x, u_1(x), \ldots, u_m(x)\right)$. Two cases are considered separately in [8]:

(i) $p > 1$ (Theorem 2.1 of [8] and its consequences); and

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(ii) $p = 1$ (Theorem 3.1 of [8] and its consequences).

In both cases, critical use is made of the following theorem, essentially contained in Gagliardo [5]. (The notations ∂'_{x_i} and A' used below will be defined fully in the next section. Roughly, $A'(\Omega)$ is the class of functions f almost everywhere equal to a function $\tilde{f}$ on Ω which is absolutely continuous on almost all line segments parallel to the axes, and $\partial'_{x_i} f$ is a function almost everywhere equal to $\partial\tilde{f}/\partial x_i$. $\partial_{x_i} f$ denotes a weak derivative.)

THEOREM 1.1. *Let* $1 \le p < \infty$, *and suppose* Ω *is a bounded domain in* R_n *with the cone property. Then* $f : \Omega \to R$ *belongs to* $W_{1,p}(\Omega)$ *if and only if*

(i) $f \in A'(\Omega)$,

(ii) $\partial'_{x_i} f \in L_p(\Omega)$, $i = 1, \ldots, n$.

Moreover, if $f \in W_{1,p}(\Omega)$, *then* $\partial'_{x_i} f = \partial_{x_i} f$ *almost everywhere in* Ω , $i = 1, \ldots, n$.

(Lemma 1.4 in [8] gives a slightly more general form of the above.)

We shall show how the case $1 < p < \infty$ of Theorem 1.1 remains true if the Lebesgue space $L_p(\Omega)$ is replaced by an Orlicz space $L_P(\Omega)$, where P now denotes an N-function. We shall then show how this result may be used to generalise Theorem 2.1 of [8] to a class of Orlicz-Sobolev spaces containing the original Sobolev spaces.

2. Preliminaries

2.1. ORLICZ SPACES. We shall use the properties of N-functions and Orlicz spaces as given in Krasnosel'skiĭ and Rutickiĭ [7]. We shall only need to consider Orlicz spaces defined on bounded domains $\Omega \subset R_n$. For our purposes, it is convenient to use the characterisation of Orlicz spaces given below.

(i) Let M be an N-function. Then a measurable function $u : \Omega \to R$ belongs to the Orlicz space $L_M(\Omega)$ if and only if there exists a constant

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$k > 0$ such that $\int_\Omega M[ku(x)]dx < \infty$.

Throughout we shall use the Luxemburg norm, denoted here by $\|\cdot\|_{M(\Omega)}$. With this norm, Hölder's inequality takes the form

(ii) $\int_\Omega uv \le 2\|u\|_{M(\Omega)}\|v\|_{\tilde{M}(\Omega)}$, where $\tilde{M}$ denotes the N-function complementary to M .

For convenience, a few other properties are given below.

(iii) If M is an N-function and $u \in \mathrm{R}$, then

(a) $M(\alpha u) \le \alpha M(u)$ if $0 \le \alpha \le 1$; and

(b) $M(\alpha u) \ge \alpha M(u)$ if $\alpha \ge 1$.

(iv) Suppose P, Q and $Q^\dagger$ are N-functions, and there exist complementary N-functions R and $\tilde{R}$ such that the inequalities

$$R(u) \le P^{-1}[Q(\alpha u)] ,$$

$$\tilde{R}(u) \le P^{-1}[Q^\dagger(\beta u)] ,$$

are satisfied for all $u \ge u_0$, where α, β and u_0 are constants. Then there exists a constant k such that

$$\|uw\|_P \le k\|u\|_Q\|w\|_{Q^\dagger} .$$

If P and R are N-functions, $Q = P \circ R$ and $Q^\dagger = P \circ \tilde{R}$ are N-functions, and it is evident that R, P, Q and $Q^\dagger$ satisfy the conditions in (iv). For use in §5, we note that it is also possible to choose P and Q such that $P \prec Q$ and such that both P and Q satisfy the Δ_2 condition. For example, we can take $P(u) = |u|^p$, $p > 1$, and $Q(u) = [(1+|u|)\ln(1+|u|)-|u|]^p$.

2.2. THE CONE PROPERTY. (i) DEFINITION. A domain $\Omega \subset \mathrm{R}_n$ is said to have the *cone property* if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C .

The following may be proved (see Adams [1], Theorem 4.8):

(ii) Let Ω be a bounded domain in R_n having the cone property. For each $\rho > 0$ there exists a finite collection $\{\Omega_1, \Omega_2, \ldots, \Omega_m\}$ of open subsets of Ω such that $\Omega = \bigcup_{j=1}^{m} \Omega_j$, and such that to each Ω_j there corresponds a subset A_j of $\overline{\Omega}_j$ having diameter not exceeding ρ, and an open parallelepiped P_j with one vertex at 0, such that $\Omega_j = \bigcup_{x \in A_j} (x+P_j)$.

The parallelepipeds P_j are determined by C, and not by ρ.

2.3. THE CLASS $A(\Omega)$. Let Ω be a domain in R_n. $A(\Omega)$ denotes the class of real measurable functions on Ω such that, for almost every line τ parallel to any coordinate axis, u is locally absolutely continuous on $\tau \cap \Omega$ (that is, u is absolutely continuous on each compact subinterval of $\tau \cap \Omega$). $A'(\Omega)$ denotes the class of functions u such that u coincides almost everywhere in Ω with a function $\tilde{u}$ in $A(\Omega)$. For $u \in A'(\Omega)$, $\partial'_{x_i} u$, the *strong approximate derivative* of u with respect to x_i, denotes any member of the equivalence class of functions measurable on Ω which contains $\partial\tilde{u}/\partial x_i$.

2.4. ORLICZ-SOBOLEV SPACES. (i) We shall use the notation $\partial_{x_i} u(x)$ to denote the ith distribution derivative of $u : \Omega \to R$, for $\Omega \subset R_n$. If M is an N-function, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ denote the classes of functions u for which u and $\partial_{x_i} u \in L_M(\Omega)$ and $E_M(\Omega)$ respectively.

(ii) We shall use the norm

$$\|u\|_{W^1_M(\Omega)} = \|u\|_{M(\Omega)} + \sum_{i=1}^{n} \|\partial_{x_i} u\|_{M(\Omega)} .$$

The following density theorem holds (see [3], Theorem 2.2).

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(iii) If Ω is a bounded domain in R_n , then $C^\infty(\Omega)$ is dense in $W^1E_M(\Omega)$.

3. Extension of Gagliardo's Theorem

The statement of the theorem of Gagliardo (contained in Theorem 1.1), still holds true if the Lebesgue space $L_p(\Omega)$ is replaced by an Orlicz space $L_P(\Omega)$.

THEOREM 3.1. *Let Ω be a bounded domain in R_n with the cone property, and let P be an N-function. Then if $u \in A'(\Omega)$ and $\partial'_{x_i} u \in L_P(\Omega)$, $u \in L_P(\Omega)$ also.*

The proof follows from the sequence of lemmas below.

LEMMA 3.2. *Let Φ be a one-to-one transformation of a domain Ω in R_n onto a domain G in R_n , having inverse Ψ . Suppose Φ and Ψ have continuous derivatives on $\overline{\Omega}$ and $\overline{G}$ respectively, and let*

$$0 < c = \min\{1, \inf_{x\in\Omega} |\det \Phi'(x)|\} , \quad C = \max\{1, \sup_{x\in\Omega} |\det \Phi'(x)|\} .$$

Suppose $u : \Omega \to R$ is measurable, and that the function $Au : G \to R$ is defined by

$$Au(y) = u(\Psi(y)) .$$

Then if P is an N-function,

$$c\|u\|_{P(\Omega)} \le \|Au\|_{P(G)} \le C\|u\|_{P(\Omega)} .$$

Proof. For $\lambda > 0$, 2.1 (iii) (a) gives

$$\int_\Omega P[u(x)/\lambda]dx \le \int_\Omega cP[u(x)/c\lambda]dx \le \int_\Omega P[u(x)/c\lambda]|\det \Phi'(x)|dx$$
$$= \int_G P[Au(y)/c\lambda]dy ;$$

whence, from 2.1 (iii) (a),

$$\|Au/c\|_{P(G)} \le \|u\|_{P(\Omega)} ,$$

which gives the first inequality.

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A similar proof, using 2.1 (iii) (b), gives the second inequality.

LEMMA 3.3. *Let* Φ *be a non-singular linear transformation of a domain* $\Omega \subset R_n$ *onto a domain* $G \subset R_n$. *Then if* u *has weak derivatives*

$\partial_{x_i} u(x)$, $i = 1, \ldots, n$, *for* $x \in \Omega$, $u \circ \Phi^{-1}$ *has weak derivatives*

$\partial_{y_i}\left[u\left(\Phi^{-1}(y)\right)\right]$, $1 \le i \le n$, *for* $y \in G$.

More general versions of the above are well-known; see, for example, Gilbarg and Trudinger ([6], page 144), or Mihaĭlov ([9], page 124, para. 5).

Lemma 3.3 can also be easily proved directly, using the definition of a weak derivative and the change of variable formula for integrals.

LEMMA 3.4. *Suppose* Ω *is a bounded domain in* R_n *having the cone property. Let* Φ *and* G *be as in Lemma* 3.3. *Then if* $u \in A'(\Omega)$ *and* $\partial'_{x_i} u \in L_1(\Omega)$, $1 \le i \le n$, $u \circ \Psi \in A'(G)$.

Proof. The lemma is an immediate consequence of Lemma 3.3, and the $p = 1$ case of Theorem 1.1.

We shall use the notation $C_l(c)$ to denote a cube in R_n with side of length l , having centre at c . If c is the point $(l/2, l/2, \ldots, l/2)$, so that one vertex is at the origin, we shall denote $C_l(c)$ by C_l .

LEMMA 3.5. *Let* Ω *be a bounded domain in* R_n *having the cone property. Then*

$$\Omega = \bigcup_{j=1}^{m} \Omega_j$$

where each Ω_j *is an open subset of* R_n *having the property (*) stated below:*

(*) *there exists a non-singular linear transformation* T_j *such that*

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$$T_j(\Omega_j) = \bigcup_{c\in B_j} C_1(c)$$

where diam $B_j < 1/8$.

Proof. Let P_j , $1 \le j \le k$, be the parallelepipeds which occur in 2.2 (ii). Let T_j , $1 \le j \le k$, be linear transformations which map P_j onto C_1 , and let $\pi_j = T_j^{-1}\left(C_{1/8\sqrt{n}}\right)$. Let d_j be the minimum distance between opposite faces of π_j , and let $\rho = \min\{d_1, \ldots, d_k\}$.

By 2.2 (ii), we may write

$$\Omega = \bigcup_{j=1}^{k} \Omega_j ,$$

where

$$\Omega_j = \bigcup_{a\in A_j} \left(a+P_j\right)$$

and diam $A_j < \rho$. Thus

$$T_j(\Omega_j) = \bigcup_{a\in T_j(A_j)} \left(a+C_1\right) .$$

Each A_j may be enclosed in a translate of π_j , and $T_j(A_j)$ is a subset of a cube of side $1/8\sqrt{n}$, so that

$$T_j(\Omega_j) = \bigcup_{c\in B_j} C_1(c)$$

where diam $B_j < 1/8$.

LEMMA 3.6. *Let* $\Omega \subset R_n$ *, and suppose* $\Omega = \bigcup_{i=1}^{m} \Omega_i$. *Let* P *be an* N*-function, and let* $u : \Omega \to R$ *be measurable. Then if* $\|u\|_{P(\Omega_i)} < \infty$, $1 \le i \le m$, $\|u\|_{P(\Omega)} < \infty$ *also.*

Lemma 3.6 is easily proved from 2.1 (i).

Lemmas 3.4, 3.5 and 3.6 show that it is sufficient to prove Theorem

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3.1 under the assumption that Ω is of the form

$$\Omega = \bigcup_{c \in B} C_1(c)$$

where $\operatorname{diam} B < 1/8$.

LEMMA 3.7. *Suppose* Ω *is a domain in* R_n *of the form (*), that is,*

$$\Omega = \bigcup_{c \in B} C_1(c)$$

where $B \subset R_n$, *and* $\operatorname{diam} B < 1/8$. *Then* Ω *has the property (**) below:*

(**) *there exists an open set* D *of positive measure, where* $D \subset \Omega$, *such that if* $\alpha = (\alpha_1, \ldots, \alpha_n) \in D$ *and if* $x = (x_1, \ldots, x_n)$ *is any point of* Ω , α *and* x *can be joined by a path consisting of* n *or less straight line segments* $S_1, S_2, \ldots, S_n$, *parallel to the axes, joining the points* $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *to* $(x_1, \alpha_2, \ldots, \alpha_n)$, $(x_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ *to* $(x_1, x_2, \alpha_3, \ldots, \alpha_n)$, ... , *and* $(x_1, \ldots, x_{n-1}, \alpha_n)$ *to* $(x_1, \ldots, x_{n-1}, x_n)$ *respectively, where each line segment* S_i *lies in* Ω *and has length less than* 1 .

Proof. Let $\gamma \in B$. Since $\operatorname{diam} B < 1/8$, it follows that $C_{\frac{1}{2}}(\gamma) \subset \bigcap_{c \in B} C_1(c)$. Let $\alpha \in C_{\frac{1}{2}}(\gamma)$, and let $x \in \Omega$. Since x belongs to some cube $C_1(\delta)$, where $\delta \in B$, and $\alpha \in C_1(\delta)$ also, α and x may be joined by a path of the form required. Hence we may take $D = C_{\frac{1}{2}}(\gamma)$.

Thus we need only prove Theorem 3.1 for domains having the property (**). We do this in the final lemma.

For $\Omega \subset R_n$, we shall use the notation $\Omega(x_i, \ldots, x_n)$ to denote the set of points $(x_1, \ldots, x_{i-1})$ such that $x = (x_1, \ldots, x_n) \in \Omega$, and $\Omega_{1,2,\ldots,i}$ to denote the projection of Ω on the hyperplane $x_1 = 0, \ldots, x_i = 0$. Note that

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$$[\Omega(\alpha_{i+1}, \ldots, \alpha_n)](x_1, \ldots, x_{i-1}) = \Omega(x_1, \ldots, x_{i-1}, \alpha_{i+1}, \ldots, \alpha_n) .$$

LEMMA 3.8. *Suppose* Ω *is a domain in* R_n *having the property (**) stated in Lemma* 3.7. *Suppose* $u \in A'(\Omega)$, *and that* P *is an* N*-function. Then if* $\partial'_{x_i} u \in L_P(\Omega)$, $1 \le i \le n$, $u \in L_P(\Omega)$ *also.*

Proof. By 2.1 (i), for each i , $1 \le i \le n$, there exists a $k_i > 0$ such that

$$\text{(i)} \qquad \int_\Omega P\left[(n+1)k_i\partial'_{x_i} u\right] < \infty .$$

Since P is an increasing function, (i) still holds if we replace k_i by $k = \min(k_1, \ldots, k_n)$.

Let $\tilde{u} \in A(\Omega)$ be such that $\tilde{u} = u$ almost everywhere in Ω and $\partial\tilde{u}/\partial x_i = \partial'_{x_i} u$ almost everywhere in Ω , so that

$$\text{(ii)} \qquad \int_\Omega P\left[(n+1)k\left(\partial\tilde{u}/\partial x_i\right)\right] < \infty$$

also. Since $\tilde{u} \in A(\Omega)$, there exists a null subset N_0 of Ω such that $\tilde{u}(\alpha)$ is finite for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega - N_0$. Further, since $P \ge 0$, we may write (ii) in the form

$$\text{(iii)} \quad \int_{\Omega_{1,\ldots,i}} dx_{i+1}, \ldots, dx_n \int_{\Omega(x_{i+1},\ldots,x_n)} \times P\left[(n+1)k\left(\partial\tilde{u}/\partial x_i\right)(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n)\right]dx_1 \ldots dx_i < \infty$$

for each i , $1 \le i \le n$. (iii) shows that there exists a null subset $\tilde{N}_i$ of each $\Omega_{1,\ldots,i}$ such that

$$\text{(iv)} \quad \int_{\Omega(\alpha_{i+1},\ldots,\alpha_n)} \times P\left[(n+1)k\left(\partial\tilde{u}/\partial x_i\right)(x_1, \ldots, x_i, \alpha_{i+1}, \ldots, \alpha_n)\right]dx_1 \ldots dx_i < \infty$$

provided $(\alpha_{i+1}, \ldots, \alpha_n) \in \Omega_{1,\ldots,i} - \tilde{N}_i$. We may then choose a null set $N_i \subset \Omega$ such that (iv) holds for $\alpha \in \Omega - N_i$. Finally we choose a null

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set $N_{\tilde{u}}$ such that $\tilde{u}$ is locally absolutely continuous on line segments in $\Omega - N_{\tilde{u}}$ parallel to each axis. Put $N = N_{\tilde{u}} \cup \left(\bigcup_{i=0}^{n} N_i\right)$. Let $\alpha \in D - N$, where D is as in Lemma 3.7. For any $x \in \Omega$, we may connect α to x by straight line segments joining the points

$$\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right), \left(x_1, \alpha_2, \ldots, \alpha_n\right), \ldots, \left(x_1, \ldots, x_{n-1}, \alpha_n\right), \left(x_1, \ldots, x_n\right),$$

and since $\tilde{u}$ is absolutely continuous on these line segments,

$$\tilde{u}(x) = \tilde{u}(\alpha) + \sum_{i=1}^{n} \int_{\alpha_i}^{x_i} \left(\partial\tilde{u}/\partial x_i\right)\left(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n\right) dt .$$

Let J_i denote the closed interval with end points α_i, x_i, and let $|J_i|$ denote its length. From the convexity of P,

(v) $$P[k\tilde{u}(x)] \le 1/(n+1) P[(n+1)k\tilde{u}(\alpha)] + 1/(n+1) \times \sum_{i=1}^{n} P\left\{\int_{J_i} \left[(n+1)k\left(\partial\tilde{u}/\partial x_i\right)\left(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n\right)\right] dt\right\} .$$

For $\alpha_i \ne x_i$, Jensen's inequality shows that

$$\begin{aligned} P\left\{\left(\int_{J_i} \left[|J_i|(n+1)k\left(\partial\tilde{u}/\partial x_i\right)\left(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n\right)\right] dt\right)/|J_i|\right\} \\ \le 1/|J_i| \int_{J_i} P\left[|J_i|(n+1)k\left(\partial\tilde{u}/\partial x_i\right)\left(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n\right)\right] dt \\ \le \int_{J_i} P\left[(n+1)k\left(\partial\tilde{u}/\partial x_i\right)\left(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n\right)\right] dt \end{aligned}$$

on using 2.1 (iii) (a). Since the case $\alpha_i = x_i$ is trivial, we have

$$\text{(vi)}\quad P\left\{\int_{J_i}[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt\right\}$$

$$\leq \int_{J_i} P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt$$

$$\leq \int_{\Omega(x_1,\ldots,x_{i-1},\alpha_{i+1},\ldots,\alpha_n)} \times P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt .$$

Substituting (vi) and (v) and then integrating over Ω , we obtain

$$\text{(vii)}\quad \int_\Omega P[ku(x)]dx \leq [1/(n+1)]P[(n+1)k\tilde{u}(\alpha)]|\Omega| + [1/(n+1)] \sum_{i=1}^{n} \times \int_{\Omega_{i,\ldots,n}} dx_1, \ldots, dx_{i-1} \int_{\Omega(x_1,\ldots,x_{i-1})} dx_i, \ldots, dx_n$$

$$\times \int_{\Omega(x_1,\ldots,x_{i-1},\alpha_{i+1},\ldots,\alpha_n)} \times P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt .$$

The first term on the right-hand side of (vii) is less than ∞ , because Ω is bounded. Moreover, there exists $K_i < \infty$ such that

$$\int_{\Omega(x_1,\ldots,x_{i-1})} dx_i, \ldots, dx_n \leq K_i \quad \text{for any} \quad (x_1, \ldots, x_{i-1}) \in \Omega_{i,\ldots,n} ,$$

again because Ω is bounded. Now consider the ith term, T_i say, inside the summation sign in (vii). We may write

$$T_i = \int_{\Omega_{i,\ldots,n}} dx_1, \ldots, dx_{i-1} \int_{[\Omega(\alpha_{i+1},\ldots,\alpha_n)](x_1,\ldots,x_{i-1})} \times P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt \times \int_{\Omega(x_1,\ldots,x_{i-1})} dx_i, \ldots, dx_n$$

$$\leq K_i \int_{\Omega_{i,\ldots,n}} dx_1, \ldots, dx_{i-1} \int_{[\Omega(\alpha_{i+1},\ldots,\alpha_n)](x_1,\ldots,x_{i-1})} \times P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt$$

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$$= K_i \int_{[\Omega(\alpha_{i+1},\ldots,\alpha_n)]_{i,\ldots,n}} dx_1, \ldots, dx_{i-1}$$
$$\times \int_{[\Omega(\alpha_{i+1},\ldots,\alpha_n)](x_1,\ldots,x_{i-1})}$$
$$\times P[(n+1)k(\partial\tilde{u}/\partial x_i)(x_1, \ldots, x_{i-1}, t, \alpha_{i+1}, \ldots, \alpha_n)]dt ,$$

where we have used the fact that $[\Omega(\alpha_{i+1}, \ldots, \alpha_n)](x_1, \ldots, x_{i-1}) = \emptyset$ if $(x_1, \ldots, x_{i-1}) \in \Omega_{i,\ldots,n} - [\Omega(\alpha_{i+1}, \ldots, n)]_{i,\ldots,n}$. From (iv), $T_i < \infty$, $1 \le i \le n$, so that we have shown that there exists $k > 0$ such that $\int_\Omega P[ku(x)]dx < \infty$, whence $\|u\|_{P(\Omega)} < \infty$, by 2.1 (i).

4. Two theorems on Orlicz-Sobolev spaces

The following two theorems will be needed in §5. For the corresponding results in Sobolev spaces, see Lemmas 1.5 and 1.6 in Marcus and Mizel [8]. We shall use the following notation.

(i) If Ω is a domain in R_n, Ω_v denotes the translate of Ω by the vector $v \in R_n$; and for $\Omega' \subset R_n$, $\Omega' \subset\subset \Omega$ means that $\bar{\Omega}'$ is a compact subset of Ω. $\partial\Omega$ denotes the boundary of Ω.

(ii) For $h > 0$, e_i, $1 \le i \le n$, the standard basis for R_n, and $x \in R_n$,

$$\delta_h^i u(x) = (u(x+he_i)-u(x))/h .$$

THEOREM 4.1. *Suppose that* Ω *is a bounded domain in* R_n, *that* Ω' *is an open set such that* $\Omega' \subset\subset \Omega$, *and that* P *is an N-function. Then if* $0 < h < \operatorname{dist}(\Omega', \partial\Omega)$, *and if* $u \in W^1E_P(\Omega)$,

$$\left\|\delta_h^i u\right\|_{P(\Omega')} \le \|\partial_{x_i} u\|_{P(\Omega)} .$$

Proof. For $u \in C^1(\Omega)$,

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$$\left|\delta_h^i u(x)\right| \leq \int_0^1 |(\partial/\partial x_i) u(x+he_i t)|$$

and so for $\lambda > 0$, an application of Jensen's inequality gives

$$\int_{\Omega'} P\left[\delta_h^i u(x)/\lambda\right] dx \leq \int_{\Omega'} \int_0^1 P\left((\partial/\partial x_i) u(x+he_i t)/\lambda\right) dt dx$$

$$= \int_0^1 dt \int_{\Omega'} P\left((\partial/\partial x_i) u(x+he_i t)/\lambda\right) dx$$

$$= \int_0^1 dt \int_{\Omega'_{-he_i t}} P\left((\partial/\partial x_i) u(x)/\lambda\right) dx$$

$$\leq \int_0^1 dt \int_{\Omega} P\left((\partial/\partial x_i) u(x)/\lambda\right) dx$$

$$= \int_{\Omega} P\left((\partial/\partial x_i) u(x)/\lambda\right) dx .$$

Taking the infinium of all $\lambda > 0$ such that the right-hand side is less than or equal to 1 gives

$$\left\|\delta_h^i u\right\|_{P(\Omega')} \leq \|\partial u/\partial x_i\|_{P(\Omega)} .$$

By 2.4 (iii) for any $u \in W^1 E_P(\Omega)$, there exists a sequence u_n of $C^\infty(\Omega)$ functions such that $u_n \to u$ in $W^1 L_p(\Omega)$. Replacing u by u_n in the last inequality and letting $n \to \infty$ gives the result.

THEOREM 4.2. *Suppose that* Ω *is a domain in* R_n, *and* $u \in L_P(\Omega)$, *where* P *is an N-function. Then if there exists a number* C *such that*

$$\left\|\delta_h^i u\right\|_{P(\Omega')} \leq C$$

for every open $\Omega' \subset\subset \Omega$ *and* $|h|$ *sufficiently small,* $\partial_{x_i} u \in L_P(\Omega)$ *and* $\|\partial_{x_i} u\|_{P(\Omega)} \leq C$.

We omit the proof, as it is almost identical to that for the Lebesgue L_p spaces, as given in, say, Agmon [2] and Friedmann [4].

5. A theorem on Nemitsky operators

We now have all the material necessary to extend Theorem 2.1 in Marcus and Mizel [8] from Lebesgue to Orlicz spaces. For convenience, we shall repeat some of the definitions from [8]. As before, Ω is a domain in R_n .

DEFINITIONS AND NOTATION 5.1. A function $g : \Omega \times R_m \to R$ is said to be a *generalised locally absolutely continuous Caratheodory function* if

(i) there exists a null subset N_g of Ω such that if

$x \in \Omega - N_g$,

(a) $g(x, \cdot)$ is continuous in each variable separately in R_m ,

(b) for every line τ parallel to one of the axes in R_m , $g(x, \cdot)|_\tau$ is locally absolutely continuous;

(ii) for every fixed $t \in R_m$, $g(\cdot t) \in A'(\Omega)$.

If "continuous in each variable separately" in (a) is replaced by "continuous", the above then defines a *locally absolutely continuous Caratheodory function.*

An operator G on vector valued functions $u = (u_1, \ldots, u_m)$ measurable on Ω , defined by

$$Gu(x) = g(x, u(x)) = (g \circ u)(x)$$

is called a *Nemitsky operator.*

Given $u = (u_1, \ldots, u_m) : \Omega \to R_m$, and N-functions $Q_1, \ldots, Q_m$, we shall use the notation

$$u \in W^1L_{\tilde{Q}}(\Omega)$$

to mean that $u_i \in W^1L_{Q_i}(\Omega)$, $1 \le i \le m$.

THEOREM 5.2. *Let* Ω *be a bounded domain in* R_n *having the cone property, and let* g *be a generalised locally absolutely continuous*

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Caratheodory function in $\Omega \times R_m$. *Let* P, Q_i *and* $Q_i^\dagger$, $1 \le i \le m$, *be N-functions having the following properties:*

(i) P *and* Q_i , $1 \le i \le m$, *satisfy the* Δ_2 *condition;*

(ii) $P \prec Q_i$, $1 \le i \le m$;

(iii) *there exist complementary N-functions* R_i *and* $\tilde{R}_i$ *such that the inequalities*

$$R_i(u) \le P^{-1}[Q_i(\alpha_i u)]$$

and

$$\tilde{R}_i(u) \le P^{-1}\left[Q_i^\dagger(\beta_i u)\right]$$

are satisfied for $u \ge u_i$, *where* α_i, β_i, u_i , $1 \le i \le m$, *are constants.*

Suppose a, b, a_k, $b_{k,j}$ *are functions such that*

I. *for every fixed* $t \in R_m$,

$$|\partial'_{x_i} g(x, t)| \le a(x) + b(t) \text{ almost everywhere in } \Omega , \quad i = 1, \ldots, n ;$$

II. *the inequality*

$$|\partial g(x, t)/\partial t_k| \le a_k(x) + \sum_{j=1}^{m} b_{k,j}(t_j) , \quad k = 1, \ldots, m ,$$

holds at every point $(x, t) \in (\Omega - N_g) \times R_n$ *at which the derivative exists in the classical sense.*

Furthermore, a, b, a_k *and* $b_{k,j}$ *have the properties (iv)-(viii) listed below:*

(iv) $0 \le a \in L_P(\Omega)$;

(v) b *is non-negative and separately continuous in* R_m ;

(vi) $0 \le a_k \in L_{Q'_k}(\Omega)$, $1 \le k \le m$;

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(vii) $0 \le b_{k,j}$ *is an extended real valued Borel function on* R , $k, j = 1, \ldots, m$;

(viii) $b_{k,k} \in L_1^{loc}(\mathrm{R})$, $k = 1, \ldots, m$.

Let $u = (u_1, \ldots, u_m) \in W^1L_{\tilde{Q}}(\Omega)$, *and suppose that*

(ix) $b \circ u \in L_P(\Omega)$,

(x) $b_{k,j} \circ u_j \in L^{\dagger}_{Q_k}(\Omega)$, $k, j = 1, \ldots, m$, $k \ne j$,

(xi) $[b_{k,k} \circ u_k]\partial_{x_i}u_k \in L_P(\Omega)$, $k = 1, \ldots, m$, $i = 1, \ldots, n$,

where the product is to be interpreted as zero whenever $\partial_{x_i}u_k = 0$.

Then $v = g \circ u$ *belongs to* $W^1L_P(\Omega)$.

Proof. We first observe that, using Theorem 3.1, we can obtain the following version of Lemma 1.4 in [8]:

(i) Let P be an N-function, and let Ω be a bounded domain in R_n having the cone property. Then a function $f : \Omega \to \mathrm{R}$ belongs to $W^1L_P(\Omega)$ if and only if

(a) $f \in A'(\Omega)$,

(b) $\partial'_{x_i} f \in L_P(\Omega)$, $i = 1, \ldots, n$.

Moreover, if $f \in W^1L_P(\Omega)$, $\partial'_{x_i} f = \partial_{x_i} f$ almost everywhere in Ω , $i = 1, \ldots, n$.

Using the above instead of Lemma 1.4 in [8], the proof of Corollary 1.3 in [8] yields

(ii) Let $g : \mathrm{R} \to \mathrm{R}$ be a locally absolutely continuous function and let Ω be a bounded domain in R_n having the cone property. Suppose $u \in W_{1,1}(\Omega)$, and let $v = g \circ u$. Then $v \in W^1L_P(\Omega)$ if and only if

$$(*) \qquad v_i = [g' \circ u]\partial_{x_i} u \in L_P(\Omega) \ , \quad i = 1, \ldots, n \ ,$$

the product being interpreted as zero wherever $\partial_{x_i} u = 0$. Moreover, if (*) holds, $v_i = \partial_{x_i} v$ almost everywhere in Ω, $i = 1, \ldots, n$.

If we now repeat the proof of Theorem 2.1 in [8], using Theorem 5.2 *(i)*, Theorem 5.2 *(ii)*, 2.1 (iv), Theorem 4.1 and Theorem 4.2 instead of Lemma 1.4, Corollary 1.3, $\|uw\|_p \le \|u\|_{q_i} \|w\|_{q_i'}$ (for suitable u and w), Lemma 1.5, and Lemma 1.6 of [8] respectively, we obtain Theorem 5.2 above.

5.3. A PARTICULAR CASE. Suppose we choose $p > 1$, $q_k \ge p$, and q_k' such that

$$1/q_k' + 1/q_k = 1/p \ , \quad 1 \le k \le m \ ,$$

and let

$$P(u) = |u|^p \ ,$$

$$Q_k(u) = |u|^{q_k} \ ,$$

$$Q_k^\dagger(u) = |u|^{q_k'} \ ,$$

$$R_k(u) = (p/q_k)|u|^{q_k/p} \ ,$$

and

$$\tilde{R}_k(u) = (p/q_k')|u|^{q_k'/p} \ .$$

Then P, Q_k, $Q_k^\dagger$, R_k and $\tilde{R}_k$ satisfy conditions *(i)*, *(ii)* and *(iii)* in Theorem 5.2. It follows that Theorem 5.2 contains Theorem 2.1 in [8] as a special case.

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Division of External Studies,
University of Queensland,
St Lucia,
Queensland 4067,
Australia.