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THE NONABELIAN TENSOR SQUARE OF A FINITE SPLIT METACYCLIC GROUP

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Given any group G, its tensor square $G \otimes G$ is defined by the following presentation (see [**3**]):

$$\text{generators: } g \otimes h,$$

$$\text{relations: } \begin{aligned} gg' \otimes h &= ({}^{g}g' \otimes {}^{g}h)(g \otimes h), \\ g \otimes hh' &= (g \otimes h)({}^{h}g \otimes {}^{h}h') \end{aligned} \tag{1}$$

where g, g', h, h' range independently over G, and ${}^{g}h = ghg^{-1}$. In what follows, ${}^{g}g' \otimes {}^{g}h$ is often written in the abbreviated form ${}^{g}(g' \otimes h)$.

Among the groups G for which $G \otimes G$ is computed in [**2**] is the metacyclic group

$$G = \langle x, y \mid y^n = e, x^m = e, xyx^{-1} = y^l \rangle,\ l^m \equiv 1 \pmod n \tag{2}$$

in the favourable special case when n is odd. It is the direct product of four cyclic groups, of orders $m, (n, l-1)$, $(n, l-1, 1+l+\cdots+l^{m-1})$, and $(n, 1+l+\cdots+l^{m-1})$, respectively. Our purpose here is to remedy this deficiency by evaluating $G \otimes G$ for even n. The following preliminary results, valid for all G, are stated in [**3**] and some proofs are given in [**2**].

The mapping

$$\begin{aligned} \kappa: G \otimes G &\to G' \\ g \otimes h &\mapsto [g, h] = ghg^{-1}h^{-1} \end{aligned} \tag{3}$$

is an epimorphism. Its kernel is denoted by $J_2(G)$; $J_2(G)$ is G-trivial and lies in the centre of $G \otimes G$.

It follows from (1) that, for all $g; g', h, h' \in G$,

$$[g \otimes h, g' \otimes h'] = [g, h] \otimes [g', h']. \tag{4}$$

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The mapping

$$\tau: G \otimes G \to G \otimes G$$
$$g \otimes h \mapsto (h \otimes g)^{-1} \tag{5}$$

is an automorphism.

There is an exact sequence

$$H_3(G) \to \Gamma G^{ab} \xrightarrow{\psi} J_2(G) \to H_2(G) \to 0, \tag{6}$$

where Γ is Whitehead's quadratic functor (see [4]), and $\operatorname{Im}\psi$ is generated by the elements $g \otimes g, g \in G$.

We assume henceforth that G is the metacyclic group given by (2). The calculation now proceeds in a number of steps.

(7). We first note two consequences of (3). Being an extension of (the central) $J_2(G)$ by (the cyclic) G', $G \otimes G$ is abelian. Secondly, x and y fix each of $x \otimes x$, $y \otimes y$, $(x \otimes y)(y \otimes x)$. It is clear from (1) that these three elements, together with $x \otimes y$, generate $G \otimes G$ qua G-module. Our first main aim is to show that they generate $G \otimes G$ as a group.

$$(y \otimes y)^n = e = a^2, \text{ where } a = (y \otimes y)^{l-1}. \tag{8}$$

First, it follows from (4) and (1) that

$$(y \otimes y)^{(l-1)^2} = y^{l-1} \otimes y^{l-1} = [x, y] \otimes [x, y] = [x \otimes y, x \otimes y] = e.$$

Next, by (7) and (2),

$$y \otimes y = {}^x(y \otimes y) = y^l \otimes y^l = (y \otimes y)^{l^2}.$$

Finally, (1) implies that

$$(y \otimes y)^n = y \otimes y^n = y \otimes e = e.$$

These three equations together yield (8). Substitution of e for a in what follows gives a replica of the calculation in [2] for n odd.

$${}^y(x \otimes y) = (x \otimes y)a. \tag{9}$$

Using (2), (1) and (8),

$${}^y(x \otimes y) = ({}^yx) \otimes y = (y^{1-l}x) \otimes y = {}^{y^{1-l}}(x \otimes y)(y^{1-l} \otimes y)$$
$$= {}^{y^{1-l}}(x \otimes y)(y \otimes y)^{1-l} = {}^{y^{1-l}}(x \otimes y)a.$$

Thus y^l acts on $x \otimes y$ as multiplication by a. Hence the action of $y = y^{l^m} = (y^l)^{l^{m-1}}$ multiplies $x \otimes y$ by $a^{l^{m-1}}$. Since n is even, (2) implies that l is odd, and the result follows from (8).

$$ {}^{x}(x \otimes y) = (x \otimes y)^l a^{(l-1)/2}. \tag{10}$$

Using (9) and the second relation of (1) $l-1$ times each, together with (8) and the centrality of a (7),

$$\begin{aligned} {}^{x}(x \otimes y) &= x \otimes {}^{x}y = x \otimes y^l \\ &= (x \otimes y)^{y}(x \otimes y^{l-1}) = \cdots \\ &= \prod_{k=0}^{l-1} {}^{y^k}(x \otimes y) = (x \otimes y)^l a^{(l-1)/2}. \end{aligned}$$

This achieves our first objective: $G \otimes G$ is four-generated.

$$x^p \otimes y^q = (x \otimes y)^{q \cdot p} a^r, \quad y^q \otimes x^p = (y \otimes x)^{q \cdot p} a^r, \tag{11}$$

where

$$r = (pq(q-2) + q \,.\, p)/2 \text{ and } q \,.\, p = q(1 + l + \cdots + l^{p-1}).$$

Repeated use of the relations (1) gives

$$\begin{aligned} x^p \otimes y^q &= {}^{x}(x^{p-1} \otimes y^q)(x \otimes y^q) \\ &= \prod_{k=p-1}^{0} {}^{x^k}(x \otimes y^q) \end{aligned}$$

and

$$x \otimes y^q = \prod_{k=0}^{q-1} {}^{y^k}(x \otimes y) = (x \otimes y)^q a^{q(q-1)/2}$$

by (7). Using (10),

$$x^p \otimes y^q = \prod_{k=p-1}^{0} {}^{x^k}(x \otimes y)^q a^{q(q-1)/2} = (x \otimes y)^{q \cdot p} a^r,$$

where

$$r = pq(q-1)/2 + \sum_{k=p-1}^{0} q(l^k - 1)/2 = pq(q-1)/2 + (q \,.\, p - pq)/2,$$

as claimed. The second formula now follows by applying the automorphism τ of (5) to the first.

$$y^p x^q \otimes y^r x^s = (x \otimes x)^{qs} (y \otimes y)^{pr} (x \otimes y)^{r \cdot q} (y \otimes x)^{p \cdot s} a^t, \tag{12}$$

where

$$t = pr(q+s) + (1/2)(qr(r-2) + sp(p-2) + r.q + p.s).$$

Using (1),

$$y^p x^q \otimes y^r x^s = {}^{y^p}(x^q \otimes y^r x^s)(y^p \otimes y^r x^s)$$
$$= {}^{y^p}(x^q \otimes y^r)^{y^{p+r}}(x^q \otimes x^s)(y^p \otimes y^r)^{y^r}(y^p \otimes x^s),$$

and the result follows from (11), (9) and $\tau(9)$, since $p(r \cdot q) + r(p \cdot s) \equiv pr(q+s) \pmod 2$.

$$(y \otimes y)^n = e = (x \otimes x)^m,$$
$$(y \otimes x)^n = e = (x \otimes y)^n \tag{13}$$
$$(y \otimes x)^{1+l+\cdots+l^{m-1}} = a^{(l-1)m(m-1)/4} = (x \otimes y)^{1+l+\cdots+l^{m-1}}$$

These are immediate consequences of the fact that the right-hand side of (12) must be independent of the choices of p and r mod n and of q and s mod m.

$$(x \otimes y)^{l-1}(y \otimes x)^{l-1} = e, \; a^m = e. \tag{14}$$

Because of (3),

$$x \otimes x = {}^y(x \otimes x) = {}^y x \otimes {}^y x = y^{1-l}x \otimes y^{1-l}x$$
$$= {}^{y^{1-l}}(x \otimes y^{1-l})^{y^{2(1-l)}}(x \otimes x)(y^{1-l} \otimes y^{1-l})^{y^{1-l}}(y^{1-l} \otimes x)$$
$$= {}^{y^{1-l}}((x \otimes y^{1-l})(y^{1-l} \otimes x))(x \otimes x)(y \otimes y)^{(1-l)^2},$$

so that

$$(x \otimes y^{1-l})(y^{1-l} \otimes x) = e,$$

using (8). The first relation now follows from (11). For the second, y fixes $(y \otimes x)^{1+l+\cdots+l^{m-1}}$ by (13), but (by $\tau(9)$) multiplies it by $a^{1+l+\cdots+l^{m-1}} = a^m$, since l is odd and $a^2 = e$.

(15). The relations (8), (13) and (14) now define $G \otimes G$ as an abelian group on the generators $y \otimes y$, $x \otimes x$, $x \otimes y$ and $y \otimes x$. For, let $Y = y \otimes y$, $X = x \otimes x$, $T = x \otimes y$, $Z = (x \otimes y)(y \otimes x)$, and retain the abbreviation $A = a$ for convenience. Then these

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relations are equivalent to the following:

$$Y^n = e,\ Y^{l-1} = A,\ A^2 = e = A^m,\ X^m = e,$$

$$T^n = e,\ T^{1+l+\cdots+l^{m-1}} = A^{(l-1)(m-1)m/4},$$

$$Z^{l-1} = Z^n = Z^{1+l+\cdots+l^{m-1}} = e.$$

On the other hand, the mapping

$$\gamma: y^p x^p \otimes y^r x^s \mapsto X^{qs} Y^{pr} Z^{p.s} T^{r.q-p.s} A^{pr(q+s)+(1/2)(qr(r-2)+sp(p-2)+r.q+p.s)}$$

preserves the relations (1), by tedious and omitted checking. (Note that, since $\gamma\tau = \tau\gamma$ on the generators, we only need to check one of the two relations).

(16) **Proposition.** *If G is the metacyclic group given by* (2), *then $G \otimes G$ is the abelian group with generators T, X, Y, Z, A and relations* (15). *Furthermore, $M(G)$ is cyclic of order* $(n, l-1)(n, 1+l+\cdots+l^{m-1})/n$, (cf. [1]).

Proof. The first assertion is proved above. For the second, note that $M(G)$ is cyclic since G has deficiency $\geqq -1$, and its order is given by (6) as

$$|J_2(G) : \operatorname{Im}\psi| = \frac{|G \otimes G|}{|G'||\langle g \otimes g, g \in G\rangle|},$$

by (3). By (12), $\operatorname{Im}\psi = \langle g \otimes g, g \in G\rangle = \langle X, Y, Z\rangle$. By (15), $\langle X, Y, Z\rangle$ has index $(n, 1+l+\cdots+l^{m-1})$ in $G \otimes G$, and G' has index $m(n, l-1)$ in G from (2).

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