Adv. Appl. Prob. **20,** 483–487 (1988)
Printed in N. Ireland

ON A MULTICLASS BATCH ARRIVAL RETRIAL QUEUE

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Abstract

Kulkarni (1986) derived expressions for the expected waiting times for customers of two types who arrive in batches at a single-channel repeated orders queueing system. We propose another method of solving this problem and extend Kulkarni's result to the case of $N \geqq 2$ classes of customers.

RETRIALS; BATCH ARRIVALS

In the context of local area computer networks, Kulkarni (1986) considered the following queueing system. There is a single channel, and arriving customers belong to n different types. The arrival times of demands of the ith type (i-demands) form a Poisson process with rate λ_i; at every arrival epoch with certain probability c_{ik} exactly k i-demands arrive. These demands we call primary calls. If an arriving batch of i-customers finds the channel free, one of the batch members immediately occupies the channel and the rest of the customers in that batch form the sources of repeated i-calls (i-sources). Every such source produces a Poisson process of repeated calls with intensity $\mu_i > 0$. If an incoming repeated call finds a free line it is served and leaves the system after service. Otherwise, if the channel is engaged, the system state does not change. Service times, for primary and for repeated i-calls, have the same distribution function $B_i(x)$. As usual we suppose that interarrival period, batch sizes, retrial times and service times are mutually independent.

Let $b_i(x) = B_i'(x)/(1 - B_i(x))$ be the instantaneous service intensity of i-calls, $\beta_i(S) = \int_0^\infty \exp(-sx)\, dB_i(x)$ be the Laplace–Stieltjes transform of the service time distribution function $B_i(x)$, $\beta_{ik} = (-1)^k \beta_i^{(k)}(0)$ be the kth initial moment of the i-calls service time, $c_i(z) = \sum_{k=1}^\infty c_{ik} z^k$, $\bar{c}_i$, σ_i^2 be respectively the generating function, the mean and the variance of batch size of i-calls, $\lambda = \lambda_1 + \cdots + \lambda_n$, $\rho_i = \lambda_i \bar{c}_i \beta_{i1}$ be the system load due to primary i-calls.

$$\beta(s) = \int_0^\infty \exp(-sx)\, dB(x), \qquad \beta_k(-1)^k \beta^{(k)}(0), \qquad \rho = \sum_{i=1}^n \rho_i.$$

Let $c(t) = 0$ if at time t the channel is free; $c(t) = i$ if at time t the channel is occupied by some i-call; $N_i(t)$ is the number of i-sources at time t. If $c(t) \neq 0$ then $\xi(t)$ is the time during which the channel has been serving the call which occupies the channel at time t.

We shall consider the system in steady state, which exists if and only if $\rho < 1$, so the condition $\rho < 1$ is assumed to hold from now on. Our goal consists of finding mean queue lengths $N_i = EN_i(t)$, $1 \leqq i \leqq n$ as well as the variance–covariance matrix of the $(N_1(t), \ldots, N_n(t))$. Kulkarni (1986) proposed a method of solving the problem and in the case of two types of customers obtained formulas for N_1, N_2. In this note we describe another approach to the problem and obtain a solution in the general case.

Theorem 1. The expected number of retrial customers of type i in steady state is

$$N_i = \frac{\lambda_i(\rho + \bar{c}_i - 1)}{\mu_i(1 - \rho)} + \frac{\lambda_i \bar{c}_i}{2} x_i,$$

Received 22 April 1987; revision received 23 November 1987.

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https://doi.org/10.2307/1427403 Published online by Cambridge University Press

where the values x_i can be found as the solution of the system of linear equations

$$\sum_{j=1}^{n} \frac{\mu_j \rho_j}{\mu_i + \mu_j}(x_i + x_j) = x_i - \sum_{j=1}^{n} \lambda_j \bar{c}_j \beta_{j2} - \frac{\beta_{i1}}{\bar{c}_i}(\sigma_i^2 + \bar{c}_i^2 - \bar{c}_i).$$

Proof. Let $m = (m_1, \cdots, m_n)$, $z = (z_1, \cdots, z_n)$, $e_i = (0, \cdots, 1, \cdots, 0)$ be the n-dimensional vector with ith coordinate equal to 1 and the rest equal to 0, and $e = (1, \cdots, 1)$ be the n-dimensional vector which has all coordinates equal to 1.

Consider the system in steady state and write:

$$p_0(m) = P\{c(t) = 0, N_1(t) = m_1, \cdots, N_n(t) = m_n\}$$

$$p_i(m, x)\,dx = P\{c(t) = i, x < \xi(t) < x + dx, N_1(t) = m_1, \cdots, N_n(t) = m_n\}, \qquad i = 1, \cdots, n.$$

In a general way we obtain the equations of statistical equilibrium:

$$\Big(\lambda + \sum_{i=1}^{n} \mu_i m_i\Big) p_0(m) = \sum_{i=1}^{n} \int_0^\infty p_i(m, x) b_i(x)\,dx,$$

$$\frac{d}{dx} p_j(m, x) = -[\lambda + b_j(x)] p_j(x) + \sum_{i=1}^{n} \lambda_i \sum_{k=1}^{m_i} c_{ik} p_j(m - ke_i, x),$$

$$p_j(m, 0) = \lambda_j \sum_{k=1}^{m_j+1} c_{jk} p_0(m - (k-1)e_j) + \mu_j(m_j + 1) p_0(m + e_j).$$

For the generating functions

$$\varphi_0(z) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} z_1^{m_1} \cdots z_n^{m_n} p_0(m),$$

$$\varphi_i(z, x) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} z_1^{m_1} \cdots z_n^{m_n} p_i(m, x)$$

these equations give

(1) $$\lambda \varphi_0(z) + \sum_{i=1}^{n} \mu_i z_i \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^{n} \int_0^\infty \varphi_i(z, x) b_i(x)\,dx,$$

(2) $$\frac{\partial}{\partial x} \varphi_j(z, x) = -\Big(\sum_{i=1}^{n} \lambda_i(1 - c_i(z_i)) + b_j(x)\Big) \varphi_j(z, x),$$

(3) $$\varphi_j(z, 0) = \lambda_j \frac{c_j(z_j)}{z_j} \varphi_0(z) + \mu_j \frac{\partial \varphi_0(z)}{\partial z_j}.$$

From (2) we find that $\varphi_j(z, x)$ depends upon x as follows:

(4) $$\varphi_j(z, x) = \varphi_j(z, 0)[1 - B_j(x)] \exp(-sx),$$

where we have denoted $\sum_{i=1}^{n} \lambda_i(1 - c_i(z_i))$ by s.

From (4) it follows that

(5) $$\varphi_j(z) = \int_0^\infty \varphi_j(z, x)\,dx = \varphi_j(z, 0) \frac{1 - \beta_j(s)}{s}.$$

Now with the help of (4) and (5), Equations (1) and (3) can be rewritten as follows:

$$\lambda \varphi_0(z) + \sum_{i=1}^{n} \mu_i z_i \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^{n} \frac{s\beta_i(s)}{1-\beta_i(s)} \varphi_i(z) \tag{6}$$

$$\lambda_j \frac{c_j(z_j)}{z_j} \varphi_0(z) + \mu_j \frac{\partial \varphi_0(z)}{\partial z_j} = \frac{s}{1-\beta_j(s)} \varphi_j(z). \tag{7}$$

In order to find the distribution of the channel state we multiply (7) by z_j, then sum over $j = 1, \cdots, n$ and subtract from (6); after some transformations we get

$$\varphi_0(z) = \sum_{i=1}^{n} \varphi_i(z) \frac{\beta_i(s) - z_i}{1-\beta_i(s)}. \tag{8}$$

Fixing some j and putting $z_i = 1$ for all $i \neq j$,

$$\varphi_0(z) + \sum_{i=1}^{n} \varphi_i(z) = \frac{1-z_j}{1-\beta_j(\lambda_j - \lambda_j c_j(z_j))} \varphi_j(z).$$

Setting $z_j = 1$ and taking into account the normalization condition $\sum_{i=0}^{n} \varphi_i(e) = 1$ we get

$$\begin{aligned} &\varphi_j(e) = \rho_j \\ &\varphi_0(e) = 1 - \sum_{j=1}^{n} \varphi_j(e) = 1 - \rho. \end{aligned} \tag{9}$$

Also, with $z = e$ (7) and (9) yield

$$\frac{\partial \varphi_0(e)}{\partial z_j} = \frac{\lambda_j(\rho - 1 + \bar{c}_j)}{\mu_j}. \tag{10}$$

Summing up (7) over $j = 1, \cdots, n$ and subtracting from (6) we have

$$\sum_{i=1}^{n} \lambda_i \left[c_i(z_i) - \frac{c_i(z_i)}{z_i} \right] \varphi_0(z) + \sum_{i=1}^{n} \mu_i (z_i - 1) \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^{n} \lambda_i (c_i(z_i) - 1) N(z), \tag{11}$$

where $N(z) = \sum_{i=0}^{n} \varphi_i(z)$.

Differentiating (11) with respect to $z_i z_j$ at the point $z = e$ we obtain, after some algebra,

$$\begin{aligned} (\mu_i + \mu_j) \frac{\partial^2 \varphi_0(e)}{\partial z_i \, \partial z_j} &= \lambda_i \bar{c}_i N_j + \lambda_j \bar{c}_j N_i - \lambda_i \lambda_j \left(\frac{\rho - 1 + \bar{c}_j}{\mu_j} + \frac{\rho - 1 + \bar{c}_i}{\mu_i} \right) \\ &\quad + \delta_{i,j} \lambda_i [\sigma_i^2 + \bar{c}_i^2 - 2(\bar{c}_i - 1)(1-\rho) - \bar{c}_i]. \end{aligned} \tag{12}$$

Now differentiate (7) with respect to z_i at the point $z = e$:

$$\begin{aligned} \frac{1}{\beta_{j1}} \frac{\partial \varphi_j(e)}{\partial z_i} &= \mu_j \frac{\partial^2 \varphi_0(e)}{\partial z_i \, \partial z_j} + \frac{\lambda_j \lambda_i (\rho - 1 + \bar{c}_i)}{\mu_i} + \lambda_i \lambda_j \bar{c}_i \bar{c}_j \frac{\beta_{j2}}{2\beta_{j1}} \\ &\quad + \delta_{i,j} \lambda_j [c_j - 1)(1-\rho). \end{aligned} \tag{13}$$

Then multiply (13) by β_{j1} and sum up over $j = 1, \cdots, n$:

$$\begin{aligned} \sum_{j=1}^{n} \mu_j \beta_{j1} \frac{\partial^2 \varphi_0(e)}{\partial z_i \, \partial z_j} &= N_i - \frac{\lambda_i (\rho - 1 + \bar{c}_i)}{\mu_i} (1 + \lambda \beta_1) - \frac{\lambda_i \bar{c}_i}{2} \sum_{j=1}^{n} \lambda_j \bar{c}_j \beta_{j2} \\ &\quad - \lambda_i \beta_{i1} (c_i - 1)(1-\rho). \end{aligned}$$

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Using (12) we obtain from this equality:

$$\sum_{j=1}^{n} \frac{\mu_j\beta_{j1}}{\mu_i+\mu_j}(\lambda_i\bar{c}_iN_j+\lambda_j\bar{c}_jN_i) = N_i - \frac{\lambda_i\bar{c}_i}{2}\sum_{j=1}^{n}\lambda_j\bar{c}_j\beta_{j2} - \frac{\lambda_i\beta_{i1}}{2}(\sigma_i^2+\bar{c}_i^2-\bar{c}_i) - \frac{\lambda_i}{\mu_i}(\rho-1+\bar{c}_i+\lambda\beta_1\bar{c}_i) + \sum_{j=1}^{n}\mu_j\beta_{j1}\frac{\lambda_i\lambda_j}{\mu_i+\mu_j}\left(\frac{\bar{c}_j}{\mu_j}+\frac{\bar{c}_i}{\mu_i}\right). \tag{14}$$

Introducing the new variables x_i by the formula

$$N_i = \frac{\lambda_i(\rho+\bar{c}_i-1)}{\mu_i(1-\rho)} + \frac{\lambda_i\bar{c}_i}{2}x_i$$

completes the proof.

For every concrete n it is easy to obtain the solution in explicit form. For example, if $n=2$ then we have the system of two linear equations with two unknown variables and so after some algebra we get the main results of Kulkarni (1986):

$$N_1 = \frac{\lambda_1(\rho+\bar{c}_1-1)}{\mu_1(1-\rho)} + \frac{\lambda_1\bar{c}_1}{2}\frac{[\mu_2+(1-\rho)\mu_1]A+[(1-\rho)\mu_1+(1-\rho_2)\mu_2]B_1+\mu_2\rho_2B_2}{(1-\rho)[(1-\rho_1)\mu_1+(1-\rho_2)\mu_2]},$$

$$N_2 = \frac{\lambda_2(\rho+\bar{c}_2-1)}{\mu_2(1-\rho)} + \frac{\lambda_2\bar{c}_2}{2}\frac{[\mu_1+(1-\rho)\mu_2]A+[(1-\rho)\mu_2+(1-\rho_1)\mu_1]B_2+\mu_1\rho_1B_1}{(1-\rho)[(1-\rho_1)\mu_1+(1-\rho_2)\mu_2]}$$

where

$$A = \sum_{j=1}^{n}\lambda_j\bar{c}_j\beta_{j2}, \qquad B_i = \frac{\beta_{i1}}{\bar{c}_i}(\sigma_i^2+\bar{c}_i^2-\bar{c}_i).$$

It is, of course, generally convenient to use a computer to carry out the calculations.

Our method allows us to obtain second moments of queue lengths

$$N_{ij} = \frac{\partial N(e)}{\partial z_i\,\partial z_j} = EN_i(t)N_j(t) - \delta_{i,j}\,.\,EN_i(t).$$

For lack of space we consider only the case $c_i(z)\equiv z$, $i=1,\cdots,n$, i.e. singleton arrivals.

Theorem 2. The second moments of queue lengths in the steady state are

$$N_{ij} = \lambda_i\lambda_jx_{ij} + \frac{\lambda_i\lambda_j}{2}\frac{x_i+x_j}{\mu_i+\mu_j} + \frac{\lambda_i\lambda_j}{\mu_i\mu_j}\frac{\rho^2}{(1-\rho)^2},$$

where the values x_i were defined in Theorem 1 and the values x_{ij} can be found as the solution

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of the system of linear equations:

$$\sum_{k=1}^{n} \mu_k \rho_k \frac{x_{ij} + x_{ik} + x_{kj}}{\mu_i + \mu_j + \mu_k} = x_{ij} - \frac{\rho}{2} \frac{x_i + x_j}{\mu_i + \mu_j} - \frac{\lambda \beta_3}{3} - \frac{\lambda \beta_2}{2} \cdot \frac{\rho}{1-\rho} \cdot \frac{\mu_i + \mu_j}{\mu_i \,.\, \mu_j} - \frac{1}{4} \sum_{k=1}^{n} \lambda_k \mu_k \beta_{k2} \left(\frac{x_i + x_k}{\mu_i + \mu_k} + \frac{x_j + x_k}{\mu_j + \mu_k} \right).$$

The proof is along the lines of Theorem 1, but now we have to differentiate (11) with respect to $z_i z_j z_k$ (instead of differentiating it with respect to $z_i z_j$ as we did earlier in obtaining (12)) and differentiate (7) with respect to $z_i z_k$ (instead of differentiating it with respect to z_i as we did earlier in obtaining (14)).

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https://doi.org/10.2307/1427403 Published online by Cambridge University Press