ON THE ZARISKI TOPOLOGY ON ENDOMORPHISM MONOIDS OF OMEGA-CATEGORICAL STRUCTURES

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Abstract. The endomorphism monoid of a model-theoretic structure carries two interesting topologies: on the one hand, the topology of pointwise convergence induced externally by the action of the endomorphisms on the domain via evaluation; on the other hand, the Zariski topology induced within the monoid by (non-)solutions to equations. For all concrete endomorphism monoids of ω-categorical structures on which the Zariski topology has been analysed thus far, the two topologies were shown to coincide, in turn yielding that the pointwise topology is the coarsest Hausdorff semigroup topology on those endomorphism monoids.

We establish two systematic reasons for the two topologies to agree, formulated in terms of the model-complete core of the structure. Further, we give an example of an ω-categorical structure on whose endomorphism monoid the topology of pointwise convergence and the Zariski topology differ, answering a question of Elliott, Jonušas, Mitchell, Péresse, and Pinsker.

§1. Introduction.

1.1. Motivation. Given a model-theoretic (relational) structure $\mathbb{A}$ with domain A, the set $\mathrm{End}(\mathbb{A})$ of all endomorphisms of $\mathbb{A}$ is closed under composition of functions and thus forms a semigroup (even a monoid). Inheriting the subspace topology of the product topology on A^A where each copy of A is equipped with the discrete topology, $\mathrm{End}(\mathbb{A})$ additionally carries a topological structure which turns out to be Polish, i.e., separable and completely metrisable, in particular Hausdorff. In this topology, a sequence $(f_n)_{n\in\mathbb{N}}$ converges to f if and only if for every $a \in A$, the sequence of evaluations $(f_n(a))_{n\in\mathbb{N}}$ converges to $f(a)$ in the discrete topology, i.e., if it is eventually constant with value $f(a)$. For this reason, the topology is called the *topology of pointwise convergence* or *pointwise topology* for brevity. These two types of structures are compatible in the sense that the composition operation is continuous with respect to the pointwise topology; one says that the topology is a *semigroup topology*. For many model-theoretic structures $\mathbb{A}$ on a countable domain, the algebraic (semigroup) structure and the topological (Polish) structure turn out to be so deeply intertwined that the pointwise topology is the *unique* Polish semigroup topology on $\mathrm{End}(\mathbb{A})$. Examples include the structure without relations (whose endomorphism monoid is the full transformation monoid) [9]; the random (di-)graph, the random strict partial order and the equivalence relation with either

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finitely or countably many equivalence classes of countably infinite size [10]; as well as the rational numbers with the non-strict order [13].

One obvious step in the proofs of these results is to show that the pointwise topology is the coarsest Polish semigroup topology on $\mathrm{End}(\mathbb{A})$. For this purpose, the authors of [9] transferred a notion from the theory of topological groups to the realm of semigroups, namely the so-called *Zariski topology* (or sometimes *verbal topology*) (see [7, 8, 12]); roughly speaking, the closed sets in this topology are given by solution sets to identities in the language of semigroups. Hence, the Zariski topology is an object associated with the algebraic (semi-)group structure. Considering $\mathrm{End}(\mathbb{A})$ as an abstract semigroup, the Zariski topology can thus be regarded as an "internal" object. The pointwise topology, in contrast, is defined from the evaluations at elements of the domain of $\mathbb{A}$ and is thus an "external" object with respect to the abstract semigroup structure of $\mathrm{End}(\mathbb{A})$—precisely speaking, the pointwise topology is associated with the semigroup action of $\mathrm{End}(\mathbb{A})$ on A.

As it turns out, the Zariski topology is necessarily coarser than any Hausdorff semigroup topology on a given semigroup. In particular, the pointwise topology on $\mathrm{End}(\mathbb{A})$ is always finer than the Zariski topology. If one manages to show that the Zariski topology on $\mathrm{End}(\mathbb{A})$ even coincides with the pointwise topology for some structure $\mathbb{A}$, one can draw two conclusions: on the one hand, the pointwise topology can also be understood as an "internal" object with respect to the abstract semigroup structure; on the other hand, the pointwise topology then indeed is the coarsest (in particular) Polish semigroup topology on $\mathrm{End}(\mathbb{A})$. This technique was used in [9, 10] as well as, implicitly, in [13]. In each instance, however, the proof that the topologies coincide has not been particularly systematic but tuned to the specific situation being considered, based on two sets of rather technical sufficient conditions established in [9] and the ad hoc notion of so-called *arsfacere* structures introduced in [10] for which these conditions always hold. This raises the question whether there are systematic reasons for equality of the topologies, in other words general and more structural properties to require for $\mathbb{A}$ which yield that the pointwise topology and the Zariski topology on $\mathrm{End}(\mathbb{A})$ coincide.

Furthermore, for each ω-categorical structure $\mathbb{A}$ explicitly considered thus far, it was possible to show that the pointwise topology and the Zariski topology on $\mathrm{End}(\mathbb{A})$ coincide, leading to the authors of [10] asking the following question which formed another essential motivation for the present work:

QUESTION 1.1 [10, Question 3.1]. *Is there an ω-categorical relational structure $\mathbb{A}$ such that the topology of pointwise convergence on* $\mathrm{End}(\mathbb{A})$ *is strictly finer than the Zariski topology*?

1.2. Our work. We establish two new sets of sufficient conditions on a structure $\mathbb{A}$ under which the Zariski topology and the pointwise topology on $\mathrm{End}(\mathbb{A})$ coincide—so, in particular, under which the pointwise topology is the coarsest Polish (Hausdorff) semigroup topology on $\mathrm{End}(\mathbb{A})$. To this end, we give a new application of so-called *model-complete cores* which have proved to be a helpful tool not only in the algebraic theory of constraint satisfaction problems [3] but also—of independent purely mathematical interest—in the universal algebraic study of polymorphism clones of ω-categorical structures [1, 4] as well as in the Ramsey-theoretic analysis of

ω-categorical structures [6]. We introduce *structures with mobile core*—a weakening of the standard notion of transitive structures—and show that for an ω-categorical structure without algebraicity whose core is mobile such that the model-complete core of the structure is either finite or has no algebraicity itself, the Zariski topology and the pointwise topology on its endomorphism monoid coincide.

These two cases leave a middle ground open—namely structures whose model-complete core is infinite but has algebraicity. Thus, this is where a positive answer to Question 1.1 could be found. And indeed, we give an example of an ω-categorical structure without algebraicity whose core is mobile for which the pointwise topology on the endomorphism monoid is strictly finer than the Zariski topology. Being transitive as well as homogeneous in a finite relational language, this structure shows that even these additional standard *well-behavedness* assumptions are insufficient to guarantee that the two topologies coincide. This indicates that the structure of the model-complete core really contains the systematic reason for the two topologies to be equal.

In Section 2, we formally introduce the relevant notions, in particular the Zariski topology as well as model-complete cores. In Section 3, we prove the positive results about finite cores and cores without algebraicity stated above. Finally, Section 4 contains our counterexample.

§2. Preliminaries.

2.1. Structures, homomorphisms, embeddings, and automorphisms. For a function $f: A \to B$ between arbitrary sets A, B and a tuple $\bar{a} = (a_1, \dots, a_n)$ in A, we denote the tuple[1] $(f(a_1), \dots, f(a_n))$ of evaluations by $f(\bar{a})$ for notational simplicity. A *(relational) structure* $\mathbb{A} = \langle A, (R_i)_{i\in I}\rangle$ is a domain A (in the following always finite or countably infinite) equipped with m_i-ary relations $R_i \subseteq A^{m_i}$. If $\mathbb{B} = \langle B, (S_i)_{i\in I}\rangle$ is another structure such that S_i also has arity m_i, we call a function $f: A \to B$ a *homomorphism* and write $f: \mathbb{A} \to \mathbb{B}$ if f is compatible with all R_i and S_i, i.e., if $\bar{a} \in R_i$ implies $f(\bar{a}) \in S_i$. A homomorphism $f: \mathbb{A} \to \mathbb{A}$ is called an *endomorphism* of $\mathbb{A}$. We denote the set of all endomorphisms of $\mathbb{A}$ by $\mathrm{End}(\mathbb{A})$; it forms a monoid with the composition operation and the neutral element id_A. An *embedding* of $\mathbb{A}$ into $\mathbb{B}$ is an injective homomorphism $f: \mathbb{A} \to \mathbb{B}$ which is additionally compatible with the complements of R_i and S_i, equivalently if $f(\bar{a}) \in S_i$ also implies $\bar{a} \in R_i$. The set of all *self-embeddings* of $\mathbb{A}$, i.e., of all embeddings of $\mathbb{A}$ into $\mathbb{A}$, is denoted by $\mathrm{Emb}(\mathbb{A})$; it also forms a monoid. An *isomorphism* between $\mathbb{A}$ and $\mathbb{B}$ is a surjective embedding from $\mathbb{A}$ into $\mathbb{B}$. The set of all *automorphisms* of $\mathbb{A}$, i.e., of all isomorphisms between $\mathbb{A}$ and itself, is denoted by $\mathrm{Aut}(\mathbb{A})$; it forms a group with the composition operation, the neutral element id_A, and the inversion operation. In the special case that $\mathbb{A}$ is the structure without any relations, the endomorphism monoid is the full transformation monoid A^A, the self-embedding monoid is the set $\mathrm{Inj}(A)$ of all injective maps $A \to A$, and the automorphism group is the set $\mathrm{Sym}(A)$ of all permutations on A. A weakening of isomorphic structures is given by the following

[1] In contrast to some related works (like [9, 10]), we denote the evaluation of the function f at the element a by $f(a)$ and write compositions of functions from right to left, i.e., $fg := f \circ g := (a \mapsto f(g(a)))$.

notion: Two structures $\mathbb{A}$ and $\mathbb{B}$ are called *homomorphically equivalent* if there exist homomorphisms $g\colon \mathbb{A} \to \mathbb{B}$ and $h\colon \mathbb{B} \to \mathbb{A}$.

If $C \subseteq A$, then the *induced substructure* $\mathbb{C}$ of $\mathbb{A}$ on C is the structure with domain C where each relation R_i is replaced by $R_i \cap C^{m_i}$. If $f : \mathbb{A} \to \mathbb{B}$ is a homomorphism, we will in a slight abuse of notation denote the substructure of $\mathbb{B}$ on the domain $f(A)$ by $f(\mathbb{A})$.

2.2. Topologies. If S is a semigroup, we call a topology $\mathcal{T}$ on S a *semigroup topology* (and $(S, \mathcal{T})$ a *topological semigroup*) if the operation $\cdot\colon S \times S \to S$ is a continuous map with respect to $\mathcal{T}$ (where $S \times S$ carries the product topology).

A natural topology on $\mathrm{End}(\mathbb{A})$ (and also on $\mathrm{Emb}(\mathbb{A}), \mathrm{Aut}(\mathbb{A}), \mathrm{Inj}(A), \mathrm{Sym}(A)$) is given by the subspace topology of the product topology on A^A where each copy of A is equipped with the discrete topology, the so-called *pointwise topology* which we denote by $\mathcal{T}_{pw}$ (or $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{A})}$ etc. if misunderstandings are possible). In the sequel, we will need to consider the topological closure of $\mathrm{Aut}(\mathbb{A})$ with respect to the pointwise topology within A^A (or, equivalently, within $\mathrm{End}(\mathbb{A})$ since the latter is itself closed in A^A) which we will call the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{A})$ for brevity. We remark that for an ω-categorical structure $\mathbb{A}$, this closure consists precisely of the so-called *elementary self-embeddings* of $\mathbb{A}$ (see [11]).

The standard topological basis of $\mathcal{T}_{pw}$ is given by the sets

$$\left\{s \in \mathrm{End}(\mathbb{A}) : s(\bar{a}) = \bar{b}\right\}, \qquad \bar{a}, \bar{b} \text{ finite tuples in } A.$$

It is easy to see that $\mathcal{T}_{pw}$ is a Polish semigroup topology on $\mathrm{End}(\mathbb{A})$.

Now we define the *Zariski topology* central to this paper. For notational simplicity, we will restrict to monoids.

Definition 2.1. Let S be a monoid.

(i) For $k, \ell \in \mathbb{N}$, $\ell < k$, and for $p_0, \dots, p_k, q_0, \dots, q_\ell \in S$ as well as $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \dots s q_0$ (if $\ell = 0$, then $\psi(s) = q_0$ for all $s \in S$), we define

$$M_{\varphi,\psi} := \{s \in S : \varphi(s) \neq \psi(s)\}.$$

(ii) The *Zariski topology* on S, denoted by $\mathcal{T}_{\mathrm{Zariski}}$, is the topology generated by all sets $M_{\varphi,\psi}$. Explicitly, the $\mathcal{T}_{\mathrm{Zariski}}$-basic open sets are the finite intersections of sets $M_{\varphi,\psi}$.

In general, the Zariski topology need not be a Hausdorff topology or a semigroup topology, but suitable weakenings do hold. On the one hand, it always satisfies the first separation axiom T1: every singleton set $\{s_0\}$ is $\mathcal{T}_{\mathrm{Zariski}}$-closed (pick $\varphi(s) = s = 1s1$, where 1 denotes the neutral element of S, and $\psi(s) = s_0$). On the other hand, the left and right *translations*, $\lambda_t\colon S \to S$, $s \mapsto ts$ and $\rho_t\colon S \to S$, $s \mapsto st$ (where $t \in S$ is fixed) are continuous with respect to the Zariski topology: To see this, take arbitrary $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \dots s q_0$ as above and note that $\lambda_t^{-1}(M_{\varphi,\psi}) = M_{\tilde{\varphi},\tilde{\psi}}$ where $\tilde{\varphi}(s) := (p_k t)s(p_{k-1}t)s \dots s(p_1 t)s(p_0)$ and $\tilde{\psi}(s) := (q_\ell t)s(q_{\ell-1}t)s \dots s(q_1 t)s(q_0)$; similarly for ρ_t.

By a straightforward argument, the Zariski topology is coarser than any Hausdorff semigroup topology $\mathcal{T}$ on S: One has to show that $M_{\varphi,\psi}$ is $\mathcal{T}$-open.

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If $s \in M_{\varphi,\psi}$, then $\varphi(s) \neq \psi(s)$, so there exist $U, V \in \mathcal{T}$ with $\varphi(s) \in U$, $\psi(s) \in V$ and $U \cap V = \emptyset$ since $\mathcal{T}$ is Hausdorff. Then $O := \varphi^{-1}(U) \cap \psi^{-1}(V)$ is a $\mathcal{T}$-open set (by continuity of the semigroup operation) such that $s \in O \subseteq M_{\varphi,\psi}$.

2.3. Homogeneity, transitivity, and algebraicity. Several important properties of a structure $\mathbb{A}$ can be defined from the canonical group action of $\mathrm{Aut}(\mathbb{A})$ on A^n for $n \geq 1$ by evaluation which we write as $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$. We will consider the (pointwise) *stabiliser* of a set $Y \subseteq A$ (usually finite), that is $\mathrm{Stab}(Y) := \{\alpha \in \mathrm{Aut}(\mathbb{A}) : \alpha(y) = y \text{ for all } y \in Y\}$. For a tuple $\bar{a} \in A^n$, we further define the *orbit* of $\bar{a}$ under the action, $\mathrm{Orb}(\bar{a}) := \{\alpha(\bar{a}) : \alpha \in \mathrm{Aut}(\mathbb{A})\}$, as well as the *Y-relative orbit* $\mathrm{Orb}(\bar{a}; Y) := \{\alpha(\bar{a}) : \alpha \in \mathrm{Stab}(Y)\}$ where $Y \subseteq A$. By the characterisation theorem due to Engeler, Ryll-Nardzewski, and Svenonius (see [11]), a countable structure $\mathbb{A}$ is ω-categorical if and only if for each $n \geq 1$, the action $\mathrm{Aut}(\mathbb{A}) \curvearrowright A^n$ has only finitely many orbits. We say that $\mathbb{A}$ is a *transitive* structure if the action $\mathrm{Aut}(\mathbb{A}) \curvearrowright A$ has a single orbit. The structure $\mathbb{A}$ is said to have *no algebraicity* if for any finite $Y \subseteq A$ and any $a \in A \setminus Y$, the Y-relative orbit $\mathrm{Orb}(a; Y)$ is infinite. Finally, we say that $\mathbb{A}$ is a *homogeneous* structure if any finite partial isomorphism $m : \bar{a} \mapsto \bar{b}$ on $\mathbb{A}$ can be extended to an automorphism $\alpha \in \mathrm{Aut}(\mathbb{A})$. It is easy to see that a homogeneous structure in a *finite (relational) language*, i.e., $\mathbb{A} = \langle A, (R_i)_{i \in I} \rangle$ with I finite, is automatically ω-categorical.

In the sequel, an important property of ω-categorical structures without algebraicity will be the existence of "almost identical" embeddings/endomorphisms which can be obtained using a standard compactness argument.

LEMMA 2.2 [10, Lemma 3.6]. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity. Then for every $a \in A$, there are f, g in the $\mathcal{T}_{pw}$-closure of* $\mathrm{Aut}(\mathbb{A})$ *such that $f|_{A \setminus \{a\}} = g|_{A \setminus \{a\}}$ and $f(a) \neq g(a)$.*

If f and g are as in the previous lemma, then for any $s \in \mathrm{End}(\mathbb{A})$ we note that $a \in \mathrm{Im}(s)$ if and only if $fs \neq gs$. This yields the following fact which will be at the heart of both proofs in Section 3.

LEMMA 2.3 (Contained in [9, Proof of Lemma 5.3]). Let $\mathbb{A}$ be an ω-categorical structure without algebraicity. Then for every $a \in A$, the set $\{s \in \mathrm{End}(\mathbb{A}) : a \in \mathrm{Im}(s)\}$ is open in the Zariski topology on $\mathrm{End}(\mathbb{A})$. ⊣

2.4. Cores. A structure $\mathbb{C}$ is called a *model-complete core* if[2] the endomorphism monoid $\mathrm{End}(\mathbb{C})$ coincides with the $\mathcal{T}_{pw}$-closure of the automorphism group $\mathrm{Aut}(\mathbb{C})$. If $\mathbb{C}$ is finite, this means $\mathrm{End}(\mathbb{C}) = \mathrm{Aut}(\mathbb{C})$. Every ω-categorical structure has a homomorphically equivalent model-complete core structure:

THEOREM 2.4 (Originally [5, Theorem 16], alternative proof in [2, Theorem 5.7]). *Let $\mathbb{A}$ be an ω-categorical structure. Then there exists a model-complete core $\mathbb{C}$ such that $\mathbb{A}$ and $\mathbb{C}$ are homomorphically equivalent. Moreover, $\mathbb{C}$ is either ω-categorical or finite and uniquely determined (up to isomorphism).*

[2]In the case that $\mathbb{C}$ is ω-categorical, this means that any endomorphism of $\mathbb{C}$ is an elementary self-embedding.

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Because of the uniqueness result, $\mathbb{C}$ is commonly referred to as *the* model-complete core of $\mathbb{A}$. We will repeatedly use the following simple property of model-complete cores:

LEMMA 2.5. *Let $\mathbb{A}$ be an ω-categorical structure and let $\mathbb{C}$ be its model-complete core. Then any homomorphism $f: \mathbb{C} \to \mathbb{A}$ is an embedding.*

PROOF. If $g: \mathbb{A} \to \mathbb{C}$ denotes the homomorphism existing by homomorphic equivalence, then gf is an endomorphism of $\mathbb{C}$ and thus contained in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{C})$, in particular a self-embedding. This is only possible if f is an embedding. ⊣

This lemma in particular applies to the homomorphism $h: \mathbb{C} \to \mathbb{A}$ yielded by homomorphic equivalence. Replacing $\mathbb{C}$ by its isomorphic copy $h(\mathbb{C})$, we will subsequently assume that $\mathbb{C}$ is a substructure of $\mathbb{A}$. Note that depending on the structure $\mathbb{A}$, it can but need not be possible to pick the homomorphism $g: \mathbb{A} \to \mathbb{C}$ to be surjective. For instance, the model-complete core of the random graph is the complete graph on countably many vertices, and any bijection from the random graph to the complete graph is a surjective homomorphism.

On the other hand, if $\mathbb{A}$ is given by the rational numbers $\mathbb{Q}$ extended by two elements $\pm\infty$, equipped with the canonical strict order, then the model-complete core of $\mathbb{A}$ is precisely $\langle \mathbb{Q}, < \rangle$ which cannot coincide with any homomorphic image of $\mathbb{A}$ since such an image would have a greatest and a least element. If the model-complete core of $\mathbb{A}$ is finite, however, *any* homomorphism $g: \mathbb{A} \to \mathbb{C}$ is surjective, as can be seen by viewing g as an endomorphism of $\mathbb{A}$ and applying the following lemma we will also use later on:

LEMMA 2.6. *If the model-complete core of an ω-categorical structure $\mathbb{A}$ is finite of size n, then the image of any endomorphism of $\mathbb{A}$ has size at least n.*

PROOF. If $s \in \mathrm{End}(\mathbb{A})$, then $s(\mathbb{A})$ is homomorphically equivalent to $\mathbb{A}$. Hence, $s(\mathbb{A})$ and $\mathbb{A}$ have the same model-complete core which can therefore be regarded as a substructure of $s(\mathbb{A})$. ⊣

§3. Two sets of sufficient conditions. This section is devoted to stating and showing our sufficient conditions, expressed in terms of the model-complete core, for the pointwise topology and the Zariski topology to coincide (see Theorem 3.2).

3.1. Our results. An essential notion for our results is given by *structures with mobile core*:

DEFINITION 3.1. Let $\mathbb{A}$ be an ω-categorical structure. Then $\mathbb{A}$ is said to have a *mobile core* if any element of $\mathbb{A}$ is contained in the image of an endomorphism into the model-complete core. Explicitly, for any $a \in \mathbb{A}$, there ought to exist a substructure $\mathbb{C}$ of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ with the following properties:

(i) $\mathbb{C}$ is a model-complete core homomorphically equivalent to $\mathbb{A}$.
(ii) $a \in g(A) \subseteq C$.

Note that structures with mobile core are a weakening of transitive structures (as introduced in Section 2.3): Let $\mathbb{A}$ be transitive, let $\mathbb{C}$ be its model-complete

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core with homomorphism $g\colon \mathbb{A} \to \mathbb{C}$, and let $a_0 \in A$ be a fixed element. If $a \in A$ is arbitrary, then transitivity yields $\alpha \in \operatorname{Aut}(\mathbb{A})$ such that $\alpha(g(a_0)) = a$. Hence, $\widetilde{\mathbb{C}} := \alpha(\mathbb{C})$ is an isomorphic copy of $\mathbb{C}$ with homomorphism $\tilde{g} := \alpha g\colon \mathbb{A} \to \widetilde{\mathbb{C}}$ such that $a \in \tilde{g}(A) \subseteq \widetilde{C}$. In fact, it suffices to assume that $\mathbb{A}$ is *weakly transitive*, i.e., that for all $a, b \in A$ there exists $s \in \operatorname{End}(\mathbb{A})$ with $s(a) = b$—replacing α in the above argument by s, we still obtain that $s(\mathbb{C})$ is an isomorphic copy of $\mathbb{C}$ by Lemma 2.5.

On the other hand, there exist non-transitive structures which have a mobile core, for instance the disjoint union of two transitive structures where each part gets named by an additional unary predicate (to ascertain that the parts are invariant under any automorphism). Finally, the structure $\langle \mathbb{Q} \cup \{\pm\infty\}, <\rangle$ mentioned after Lemma 2.5 does not have a mobile core: The element $+\infty$ cannot be contained in any copy of the model-complete core $\langle \mathbb{Q}, <\rangle$.

Now we can formally state the main result of this section.

Theorem 3.2. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity which has a mobile core. Then the Zariski topology on* $\operatorname{End}(\mathbb{A})$ *coincides with the pointwise topology if one of the following two conditions holds*:

(i) *EITHER the model-complete core of $\mathbb{A}$ is finite*,
(ii) *OR the model-complete core of $\mathbb{A}$ is infinite and does not have algebraicity.*

The cases (i) and (ii) will be treated separately in Sections 3.2 and 3.3, respectively. Before we get to the proofs, we show how Theorem 3.2 can be used to easily verify that the Zariski topology and the pointwise topology coincide on the endomorphism monoids of a multitude of example structures. Some of them have been treated in [10], but our result applies to many other structures which have not yet been considered, e.g., the random reflexive or irreflexive n-clique-free graph.

Corollary 3.3. *Let $\mathbb{A}$ be one of the following structures*:

(i) $\langle \mathbb{Q}, \leq\rangle$;
(ii) *the random reflexive partial order*;
(iii) *the equivalence relation with either finitely or countably many equivalence classes of countable size* (*for the case of a single class, this includes the complete reflexive graph on countably many vertices*);
(iv) *the random reflexive* (*di-*)*graph*;
(v) *the random reflexive n-clique-free graph*;
(vi) $\langle \mathbb{Q}, <\rangle$;
(vii) *the random strict partial order*;
(viii) *the random tournament*;
(ix) *the irreflexive equivalence relation with either finitely or countably many equivalence classes of countable size* (*for the case of a single class, this includes the complete irreflexive graph on countably many vertices*);
(x) *the random irreflexive* (*di-*)*graph*;
(xi) *the random irreflexive n-clique-free graph.*

Then the pointwise topology and the Zariski topology on $\operatorname{End}(\mathbb{A})$ *coincide. In particular, the pointwise topology is the coarsest Hausdorff semigroup topology on* $\operatorname{End}(\mathbb{A})$.

Proof. It is immediate that all structures in (i)–(xi) are ω-categorical structures without algebraicity which are transitive (in particular, they have a mobile core).

For (i)–(v), the model-complete core of $\mathbb{A}$ is merely a single point with a loop; in particular, the model-complete core is finite. For (vi) and (xi), the structure $\mathbb{A}$ is already a model-complete core, so the model-complete core of $\mathbb{A}$ is just $\mathbb{A}$ itself. For (vii) and (viii), the model-complete core of $\mathbb{A}$ is the structure $\langle \mathbb{Q}, < \rangle$. For (ix) and (x), the model-complete core of $\mathbb{A}$ is the complete graph on countably infinitely vertices. Summarising, the model-complete core of $\mathbb{A}$ has no algebraicity in (vi)–(xi).

In any case, Theorem 3.2 applies and yields the desired conclusion. ⊣

3.2. Finite cores. First, we consider the case that $\mathbb{A}$ has a finite model-complete core.

PROPOSITION 3.4. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity which has a mobile core. If the model-complete core of $\mathbb{A}$ is finite, then the Zariski topology on* $\mathrm{End}(\mathbb{A})$ *coincides with the pointwise topology.*

PROOF. We show that the $\mathcal{T}_{pw}$-generating sets $\{s \in \mathrm{End}(\mathbb{A}) : s(a) = b\}$, $a, b \in A$, are $\mathcal{T}_{\text{Zariski}}$-open by proving that they are $\mathcal{T}_{\text{Zariski}}$-neighbourhoods of each element.

Let $s_0 \in \mathrm{End}(\mathbb{A})$ such that $s_0(a) = b$. Since $\mathbb{A}$ has a mobile core, there exist a copy $\mathbb{C}$ of the model-complete core of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ such that $a \in g(A) \subseteq C$. By Lemma 2.6, we know that $g(A) = C$. We set $n = |C|$ and write $g(\mathbb{A}) = \{a_1, \dots, a_n\}$ where $a_1 = a$. Applying Lemma 2.3, we obtain that the set

$$V := \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_1), \dots, s_0(a_n) \in \mathrm{Im}(s)\} = \bigcap_{j=1}^{n} \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_j) \in \mathrm{Im}(s)\}$$

is open in the Zariski topology. Since the translation $\rho_g \colon s \mapsto sg$ on $\mathrm{End}(\mathbb{A})$ is continuous with respect to the Zariski topology, the preimage

$$U := \rho_g^{-1}(V) = \{s \in \mathrm{End}(\mathbb{A}) : s_0(a_1), \dots, s_0(a_n) \in \mathrm{Im}(sg)\}$$

is $\mathcal{T}_{\text{Zariski}}$-open as well. Again by Lemma 2.6, the images of the endomorphisms $s_0 g$ and sg (for arbitrary $s \in \mathrm{End}(\mathbb{A})$) must both have n elements. Hence, the images $s_0(a_i)$ are pairwise different and, further,

$$U = \{s \in \mathrm{End}(\mathbb{A}) : \mathrm{Im}(sg) = \{s_0(a_1), \dots, s_0(a_n)\}\}.$$

The crucial observation is that $Ug = \{sg : s \in U\}$ is a finite set: Any element sg is determined by the ordered tuple $(s(a_1), \dots, s(a_n))$. Since the unordered set $\{s(a_1), \dots, s(a_n)\}$ is fixed for $s \in U$, there are only finitely many (at most $n!$, to be precise) possibilities for the ordered tuple.

Consequently, the set $M := \{sg : s \in U, s(a) \neq b\}$ is finite as well. We define

$$O := U \cap \bigcap_{t \in M} \{s \in \mathrm{End}(\mathbb{A}) : sg \neq t\} \in \mathcal{T}_{\text{Zariski}}$$

and claim that $O = \{s \in \mathrm{End}(\mathbb{A}) : s \in U, s(a) = b\}$, subsequently giving $s_0 \in O \subseteq \{s \in \mathrm{End}(\mathbb{A}) : s(a) = b\}$ as desired. If $s \in U$ with $s(a) = b$, then we take $z \in A$ with $g(z) = a$ and note $sg(z) = s(a) = b \neq t(z)$ for all $t \in M$. Conversely, if $s \in U$ but $s(a) \neq b$, then $t := sg \in M$, so $s \notin O$—completing the proof. ⊣

3.3. Cores without algebraicity. Now we consider structures $\mathbb{A}$ whose model-complete cores do not have algebraicity. In our proof, we will use the following technical condition from [9]:

Lemma 3.5 [9, Lemma 5.3]. *Let X be an infinite set and let S be a subsemigroup of X^X such that for every $a \in X$ there exist $\alpha, \beta, \gamma_1, \dots, \gamma_n \in S$ for some $n \in \mathbb{N}$ such that the following hold:*

(i) $\alpha|_{X\setminus\{a\}} = \beta|_{X\setminus\{a\}}$ *and* $\alpha(a) \neq \beta(a)$;
(ii) $a \in \mathrm{Im}(\gamma_i)$ *for all* $i \in \{1, \dots, n\}$;
(iii) *For every* $s \in S$ *and every* $x \in X \setminus \{s(a)\}$, *there is* $i \in \{1, \dots, n\}$ *so that* $\mathrm{Im}(\gamma_i) \cap s^{-1}(x) = \emptyset$.

Then the Zariski topology of S is the pointwise topology.

We remark that (i) corresponds to Lemma 2.2 and that the proof proceeds by constructing the generating sets of the pointwise topology from the sets $\{s \in S : a \in \mathrm{Im}(s)\}$ exhibited in Lemma 2.3.

The fact that the model-complete core does not have algebraicity will come into play via the following observation:

Lemma 3.6. *Let $\mathbb{B}$ be a countably infinite structure without algebraicity and let $b \in \mathbb{B}$. Then there exist f, h in the $\mathcal{T}_{pw}$-closure of* $\mathrm{Aut}(\mathbb{B})$ *such that $f(b) = b = h(b)$ and $f(B) \cap h(B) = \{b\}$ (so there exist two copies of $\mathbb{B}$ within $\mathbb{B}$ which only have b in common).*

Proof. We enumerate $B = \{b_n : n \in \mathbb{N}\}$ where $b_0 = b$. First, we recursively construct automorphisms $\alpha_n, \beta_n \in \mathrm{Aut}(\mathbb{B})$, $n \in \mathbb{N}$, such that $\alpha_{n+1}|_{\{b_0,\dots,b_n\}} = \alpha_n|_{\{b_0,\dots,b_n\}}$, $\beta_{n+1}|_{\{b_0,\dots,b_n\}} = \beta_n|_{\{b_0,\dots,b_n\}}$, and $\alpha_n(\{b_0, \dots, b_n\}) \cap \beta_n(\{b_0, \dots, b_n\}) = \{b\}$ for all $n \in \mathbb{N}$. We start by setting $\alpha_0 = \beta_0 := \mathrm{id}_B$. If α_n and β_n are already defined, we put $Y := \alpha_n(\{b_0, \dots, b_n\})$ as well as $Z := \beta_n(\{b_0, \dots, b_n\})$. Since $\mathbb{B}$ has no algebraicity, the relative orbits $\mathrm{Orb}(\alpha_n(b_{n+1}); Y)$ and $\mathrm{Orb}(\beta_n(b_{n+1}); Z)$ are infinite, so we can find $c_{n+1} \in \mathrm{Orb}(\alpha_n(b_{n+1}); Y)$ which is not contained in Z and then find $d_{n+1} \in \mathrm{Orb}(\beta_n(b_{n+1}); Z)$ which is not contained in $Y \cup \{c_{n+1}\}$. Taking $\gamma \in \mathrm{Stab}(Y)$ with $\gamma(\alpha_n(b_{n+1})) = c_{n+1}$ as well as $\delta \in \mathrm{Stab}(Z)$ with $\delta(\beta_n(b_{n+1})) = d_{n+1}$, and setting $\alpha_{n+1} := \gamma\alpha_n$ as well as $\beta_{n+1} := \delta\beta_n$ completes the construction. Finally, we set $f := \lim_{n\in\mathbb{N}} \alpha_n$ and $h := \lim_{n\in\mathbb{N}} \beta_n$; these maps are contained in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{B})$ and have the desired properties. ⊣

Proposition 3.7. *Let $\mathbb{A}$ be an ω-categorical structure without algebraicity which has a mobile core. If the model-complete core of $\mathbb{A}$ is infinite and does not have algebraicity, then the Zariski topology on* $\mathrm{End}(\mathbb{A})$ *coincides with the pointwise topology.*

Proof. We check the assumptions of Lemma 3.5.

Since $\mathbb{A}$ is ω-categorical without algebraicity, property (i) follows from Lemma 2.2.

For properties (ii) and (iii), we fix $a \in A$, set $n = 2$, and construct γ_1, γ_2. Since $\mathbb{A}$ has a mobile core, there exist a copy $\mathbb{C}$ of the model-complete core of $\mathbb{A}$ and $g \in \mathrm{End}(\mathbb{A})$ such that $a \in g(A) \subseteq C$. Since $\mathbb{C}$ has no algebraicity, there exist f, h in the $\mathcal{T}_{pw}$-closure of $\mathrm{Aut}(\mathbb{C})$ such that $f(a) = a = h(a)$ and $f(C) \cap h(C) = \{a\}$

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by Lemma 3.6. Using the homomorphism $g\colon \mathbb{A} \to \mathbb{C}$, we set $\gamma_1 := fg$ and $\gamma_2 := hg$, considered as endomorphisms of $\mathbb{A}$. Then $a \in \mathrm{Im}(\gamma_i)$, i.e., (ii) holds. Suppose now that for some $s \in \mathrm{End}(\mathbb{A})$ and $x \in A$, we have $\mathrm{Im}(\gamma_i) \cap s^{-1}\{x\} \neq \emptyset$ for $i = 1, 2$. In order to prove (iii), the goal is to show $x = s(a)$. We rewrite to obtain the existence of $x_i \in A$ with $sfg(x_1) = s\gamma_1(x_1) = x = s\gamma_2(x_2) = shg(x_2)$. As a homomorphism from $\mathbb{C}$ to $\mathbb{A}$, the restriction $s|_C : \mathbb{C} \to \mathbb{A}$ is an embedding by Lemma 2.5, in particular injective. Hence,

$$fg(x_1) = hg(x_2) \in f(C) \cap h(C) = \{a\},$$

yielding $x = sfg(x_1) = s(a)$ as desired. ⊣

§4. Counterexample. In this section, we give an example of an ω-categorical (even homogeneous in a finite language) and transitive structure without algebraicity such that the Zariski topology on its endomorphism monoid does not coincide with the pointwise topology, thus answering Question 1.1. By our results in Section 3, the model-complete core of this structure must be infinite and have algebraicity. Informally speaking, we take a complete graph on countably many vertices where each point has as fine structure a complete *bipartite* graph on countably many vertices (see Figure 1).

4.1. Definitions, notation, and preliminary properties. We start by formally introducing our structure and giving some notation.

DEFINITION 4.1.

(i) Let $\mathbb{K}_{2,\omega}$ denote the *complete irreflexive bipartite graph on countably many vertices*: We write the domain as $K_{2,\omega} := A_{+1} \dot\cup A_{-1}$ where A_{+1} and A_{-1} are countably infinite sets referred to as the *parts* of $\mathbb{K}_{2,\omega}$, and define the edge relation as $E^{\mathbb{K}_{2,\omega}} := A_{-1} \times A_{+1} \cup A_{+1} \times A_{-1}$, i.e., two points are connected if and only if they are contained in different parts of $\mathbb{K}_{2,\omega}$.

(ii) Let $\mathbb{G}$ denote the following structure over the language of two binary relations: We set $G := \mathbb{N} \times K_{2,\omega}$ (countably many copies of $K_{2,\omega}$) and define the relations as follows:

$$E_1^{\mathbb{G}} := \left\{((i,x),(j,y)) \in G^2 : i \neq j\right\},$$
$$E_2^{\mathbb{G}} := \left\{((i,x),(j,y)) \in G^2 : i = j \text{ and } (x,y) \in E^{\mathbb{K}_{2,\omega}}\right\}.$$

This means that the set of copies of $K_{2,\omega}$ forms a complete graph with respect to E_1 and that each copy $\{i\} \times K_{2,\omega}$ of $K_{2,\omega}$ is indeed a copy of the graph $\mathbb{K}_{2,\omega}$ (with respect to E_2) (see Figure 1).

Note that an endomorphism s of $\mathbb{K}_{2,\omega}$ acts as a permutation on the set $\{A_{+1}, A_{-1}\}$ of parts since two ($E_2^{\mathbb{K}_{2,\omega}}$-connected) elements from different parts of $\mathbb{K}_{2,\omega}$ cannot be mapped to the same part of $\mathbb{K}_{2,\omega}$ – we either have $s(A_{+1}) \subseteq A_{+1}$ and $s(A_{-1}) \subseteq A_{-1}$ or $s(A_{+1}) \subseteq A_{-1}$ and $s(A_{-1}) \subseteq A_{+1}$.

DEFINITION 4.2. For $s \in \mathrm{End}(\mathbb{K}_{2,\omega})$, we put $\mathrm{sgn}(s) \in \{+1, -1\}$ to be the sign of the permutation induced by s on $\{A_{+1}, A_{-1}\}$. Explicitly, this means that $s(A_e) \subseteq A_{e\cdot\mathrm{sgn}(s)}$ for $e = \pm 1$. As a slight abuse of notation, we will refer to $\mathrm{sgn}(s)$ as the *sign* of s.

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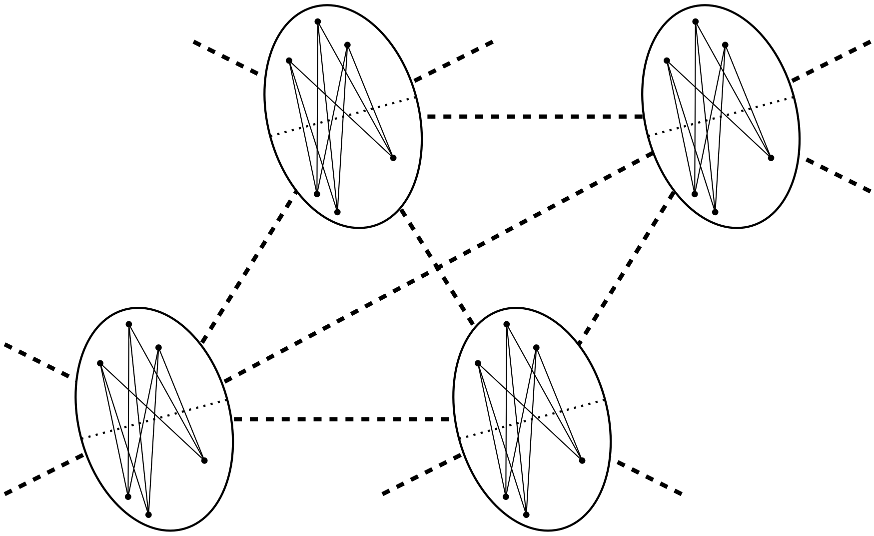

FIGURE 1. The structure $\mathbb{G}$: complete graph on countably many vertices (dashed) where each point has a complete bipartite graph on countably many vertices as fine structure (solid).

Clearly, we have $\operatorname{sgn}(st) = \operatorname{sgn}(s)\operatorname{sgn}(t)$ for $s, t \in \operatorname{End}(\mathbb{K}_{2,\omega})$. As a tool, we define two very simple endomorphisms of $\mathbb{K}_{2,\omega}$.

NOTATION 4.3.

(i) *In the sequel, $a_{+1} \in A_{+1}$ and $a_{-1} \in A_{-1}$ shall denote fixed elements.*
(ii) *We define $c_{+1} \in \operatorname{End}(\mathbb{K}_{2,\omega})$ and $c_{-1} \in \operatorname{End}(\mathbb{K}_{2,\omega})$ to be the unique endomorphisms of $\mathbb{K}_{2,\omega}$ with image $\{a_{+1}, a_{-1}\}$ and sign $+1$ and -1, respectively. So c_{+1} is constant on A_e with value a_e and c_{-1} is constant on A_e with value a_{-e} for $e = \pm 1$.*

In order to describe the automorphism group and endomorphism monoid of $\mathbb{G}$, the following notation will be useful.

NOTATION 4.4. *Let X be a set, let $\tau\colon \mathbb{N} \to \mathbb{N}$, and let $s_i\colon X \to X$ for each $i \in \mathbb{N}$. Then $\bigsqcup^{\tau}_{i\in\mathbb{N}} s_i$ shall denote the self-map of $\mathbb{N} \times X$ defined by*

$$\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i : \begin{cases} \mathbb{N} \times X, & \to \mathbb{N} \times X, \\ (i, x), & \mapsto (\tau(i), s_i(x)). \end{cases}$$

For $\tau\colon \mathbb{N} \to \mathbb{N}$ and $s\colon X \to X$, we further set $\tau \ltimes s := \bigsqcup^{\tau}_{i\in\mathbb{N}} s$.

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LEMMA 4.5.

(i) $\mathrm{End}(\mathbb{G}) = \left\{\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i : \tau \in \mathrm{Inj}(\mathbb{N}), s_i \in \mathrm{End}(\mathbb{K}_{2,\omega})\right\}$.
(ii) $\mathrm{Aut}(\mathbb{G}) = \left\{\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \alpha_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})\right\}$.

REMARK 4.6. Lemma 4.5 exactly expresses that $\mathrm{End}(\mathbb{G})$ and $\mathrm{Aut}(\mathbb{G})$ are the (unrestricted) wreath products of $\mathrm{End}(\mathbb{K}_{2,\omega})$ with $\mathrm{Inj}(\mathbb{N})$ and $\mathrm{Aut}(\mathbb{K}_{2,\omega})$ with $\mathrm{Sym}(\mathbb{N})$, respectively, by the canonical actions of $\mathrm{Inj}(\mathbb{N})$ and $\mathrm{Sym}(\mathbb{N})$ on $\mathbb{N}$.

PROOF (OF LEMMA 4.5). It is straightforward to see that the maps $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ in (i) and $\bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i$ in (ii) form endomorphisms and automorphisms, respectively. Thus, (ii) follows immediately from (i) since $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ can only be bijective if $\tau \in \mathrm{Sym}(\mathbb{N})$ and $s_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})$.

To show (i), we first note that for any $s \in \mathrm{End}(\mathbb{K}_{2,\omega})$ and any two elements $(i,x),(i,y) \in G$ in the same copy of $K_{2,\omega}$, the images $s(i,x)$ and $s(i,y)$ are also contained in the same copy of $K_{2,\omega}$: Either x, y are connected in $\mathbb{K}_{2,\omega}$ in which case $s(i,x)$ and $s(i,y)$ are $E_2^{\mathbb{G}}$-connected and therefore contained in the same copy, or x, y are both connected in $\mathbb{K}_{2,\omega}$ to a common element z in which case $s(i,x)$ and $s(i,y)$ are both $E_2^{\mathbb{G}}$-connected to $s(i,z)$ and therefore contained in the same copy. Setting $\tau(i)$ to be the index of this copy, i.e., $s(i,x), s(i,y) \in \{\tau(i)\} \times K_{2,\omega}$, we obtain that s can be written as $\bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$ for some functions $s_i\colon K_{2,\omega} \to K_{2,\omega}$. By compatibility of s with $E_1^{\mathbb{G}}$, the map τ needs to be injective. Further, the maps s_i are endomorphisms of $\mathbb{K}_{2,\omega}$ since s is compatible with $E_2^{\mathbb{G}}$. ⊣

The representation in (ii) readily yields the following properties of $\mathbb{G}$ by means of lifting from $\mathrm{Sym}(\mathbb{N})$ and $\mathrm{Aut}(\mathbb{K}_{2,\omega})$:

LEMMA 4.7. *$\mathbb{G}$ is ω-categorical, homogeneous, transitive, and has no algebraicity.*

PROOF. We start by showing that $\mathbb{G}$ is homogeneous which will also yield the ω-categoricity since $\mathbb{G}$ has a finite language. Let $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ be tuples in G and let $m\colon \bar{a} \mapsto \bar{b}$ be a finite partial isomorphism. Writing $a_k = (i_k, x_k)$ and $b_k = (j_k, y_k)$, we note that i_k and i_ℓ coincide if and only if j_k and j_ℓ coincide (for otherwise, either m or m^{-1} would not be compatible with $E_1^{\mathbb{G}}$). Hence, the map $i_k \mapsto j_k$ is a well-defined finite partial bijection and can thus easily be extended to some $\sigma \in \mathrm{Sym}(\mathbb{N})$ (in other words, the structure with domain $\mathbb{N}$ and without any relations is homogeneous). Further, if $i_{k_1} = \cdots = i_{k_N} =: i$, then $m_i\colon x_{k_1} \mapsto y_{k_1}, \dots, x_{k_N} \mapsto y_{k_N}$ is a finite partial isomorphism of $\mathbb{K}_{2,\omega}$ since m is a finite partial isomorphism with respect to $E_2^{\mathbb{G}}$. The graph $\mathbb{K}_{2,\omega}$ is homogeneous, so m_i extends to $\alpha_i \in \mathrm{Aut}(\mathbb{K}_{2,\omega})$. Setting $\alpha_i = \mathrm{id}_{K_{2,\omega}}$ for all i such that no x_k is contained in the i-th copy of $K_{2,\omega}$ and putting $\alpha := \bigsqcup_{i\in\mathbb{N}}^{\sigma} \alpha_i \in \mathrm{Aut}(\mathbb{G})$, we obtain an extension of m.

Next, observe that $\mathbb{G}$ is transitive: given $a, b \in G$, the map $a \mapsto b$ is a finite partial isomorphism since neither $E_1^{\mathbb{G}}$ nor $E_2^{\mathbb{G}}$ contain any loops. Thus, homogeneity yields $\alpha \in \mathrm{Aut}(\mathbb{G})$ with $\alpha(a) = b$.

Finally, $\mathbb{G}$ does not have algebraicity since $\mathbb{K}_{2,\omega}$ does not have algebraicity: For a finite set $Y \subseteq G$ and $a = (i_0, x_0) \in G \setminus Y$, we set $Y_{i_0} := \{y \in K_{2,\omega} : (i_0, y) \in Y\} \not\ni x_0$ and note that $\mathrm{Orb}_{\mathbb{G}}(a; Y)$ encompasses the infinite set $\{i_0\} \times \mathrm{Orb}_{\mathbb{K}_{2,\omega}}(x_0; Y_{i_0})$ as witnessed by the automorphisms $\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}} \alpha_i$ where $\alpha_{i_0} \in \mathrm{Stab}_{\mathbb{K}_{2,\omega}}(Y_{i_0})$ and $\alpha_i = \mathrm{id}_{K_{2,\omega}}$ for $i \neq i_0$. ⊣

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REMARK 4.8. An alternative construction of $\mathbb{G}$ is as a first-order reduct of the *free superposition* (see [6]; this is a type of construction to combine two structures with different signatures in a “free” way) of $\mathbb{K}_{2,\omega}$ with the irreflexive equivalence relation with countably many equivalence classes of countable size. Since both structures are transitive and have no algebraicity, the superposition structure has the same properties which are then inherited by $\mathbb{G}$ since a first-order reduct can only have additional automorphisms.

To simplify the presentation, we additionally define a few notational shorthands concerning endomorphisms of $\mathbb{G}$:

NOTATION 4.9.
(i) *For* $p = \bigsqcup_{i\in\mathbb{N}}^{\xi} p_i \in \mathrm{End}(\mathbb{G})$, *we define* $\tilde{p} := \xi \in \mathrm{Inj}(\mathbb{N})$.
(ii) *Given* $p_0, \dots, p_k \in \mathrm{End}(\mathbb{G})$ *and* $\varphi(s) := p_k s p_{k-1} s \dots s p_0$, $s \in \mathrm{End}(\mathbb{G})$, *we define* $\tilde{\varphi}(\tau) := \tilde{p}_k \tau \tilde{p}_{k-1} \tau \dots \tau \tilde{p}_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.

4.2. Proof strategy. The goal of Section 4 is to prove the following:

THEOREM 4.10. *On the endomorphism monoid of the structure* $\mathbb{G}$, *the pointwise topology is strictly finer than the Zariski topology.*

REMARK 4.11. Before we go into the details of the proof, let us remark that the structure $\mathbb{G}$ needs to have an infinite model-complete core which has algebraicity in order to have a chance of satisfying Theorem 4.10—for otherwise, Theorem 3.2 would apply.

The model-complete core of $\mathbb{K}_{2,\omega}$ is just the graph consisting of a single edge, as witnessed for instance, by the substructure induced on $\{a_{+1}, a_{-1}\}$ and the homomorphism $c_{+1}\colon \mathbb{K}_{2,\omega} \to \{a_{+1}, a_{-1}\}$. We claim that the model-complete core of $\mathbb{G}$ is the complete graph on countably many vertices where each point has as fine structure a single edge, i.e., the substructure $\mathbb{C}$ of $\mathbb{G}$ induced on $C := \mathbb{N} \times \{a_{+1}, a_{-1}\} \subseteq G$ (see Figure 2).

Similarly to the proof of Lemma 4.5, one easily checks that (here, $c_{\pm 1}$ are considered as self-maps of $\{a_{+1}, a_{-1}\}$)

$$\mathrm{End}(\mathbb{C}) = \left\{ \bigsqcup_{i\in\mathbb{N}}^{\tau} \gamma_i : \tau \in \mathrm{Inj}(\mathbb{N}), \gamma_i \in \{c_{+1}, c_{-1}\} \right\},$$

$$\mathrm{Aut}(\mathbb{C}) = \left\{ \bigsqcup_{i\in\mathbb{N}}^{\sigma} \gamma_i : \sigma \in \mathrm{Sym}(\mathbb{N}), \gamma_i \in \{c_{+1}, c_{-1}\} \right\}.$$

Thus, any endomorphism is locally interpolated by an automorphism, and $\mathbb{C}$ is indeed a model-complete core. Additionally, $\mathbb{G}$ and $\mathbb{C}$ are homomorphically equivalent—an example of a homomorphism $\mathbb{G} \to \mathbb{C}$ is given by $\bigsqcup_{i\in\mathbb{N}}^{\mathrm{id}} c_{+1}$ (where c_{+1} is considered as a map defined on $K_{2,\omega}$).

Finally, $\mathbb{C}$ has algebraicity: any automorphism of $\mathbb{C}$ which stabilises $Y := \{(0, a_{+1})\}$ also stabilises $a := (0, a_{-1})$, so the Y-related orbit of a is finite.

In order to show Theorem 4.10, we will prove that $\mathcal{T}_{\mathrm{Zariski}}$-open sets on $\mathrm{End}(\mathbb{G})$ cannot determine the sign of the components s_i of $s = \bigsqcup_{i\in\mathbb{N}}^{\tau} s_i$, in other words decide whether the functions s_i switch the two parts of $\mathbb{K}_{2,\omega}$ or not. On the other

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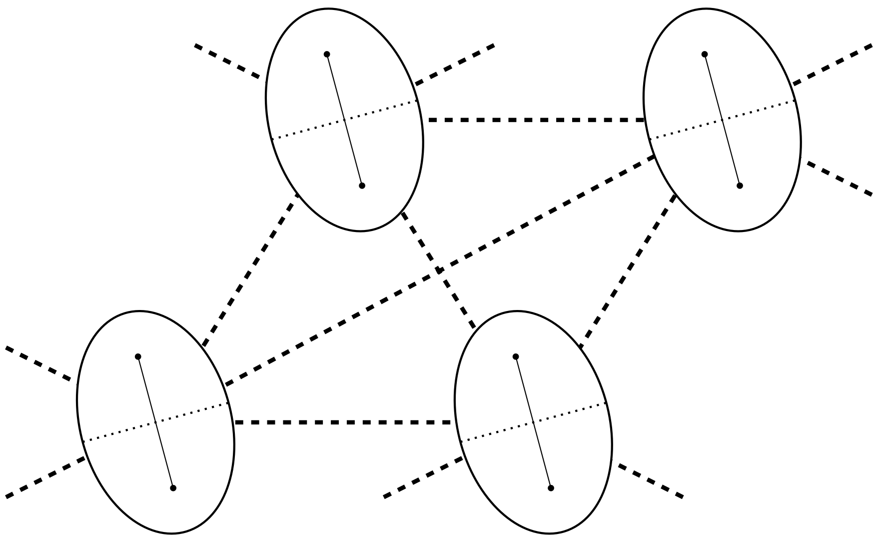

FIGURE 2. The model-complete core of $\mathbb{G}$: complete graph on countably many vertices (dashed) where each point has a single edge (solid) as fine structure.

hand, $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$-open sets can determine the sign of finitely many components, thus showing $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})} \neq \mathcal{T}_{\mathrm{Zariski}}$. More precisely, we will prove that if a $\mathcal{T}_{\mathrm{Zariski}}$-generating set $M_{\varphi,\psi}$ contains $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$, then it also contains $\tau \ltimes c_{-1}$ for all elements τ of a "big" subset of $\mathrm{Inj}(\mathbb{N})$—where "big" means either "$\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$" (if the terms φ and ψ have equal lengths; see Lemmas 4.12 and 4.13) or "$\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set" (if the terms φ and ψ have different lengths; see Lemma 4.14).

Our (almost trivial) first lemma analogously holds in a more general setting. Since we only apply it in case of terms of equal lengths, we formulate it in the present form.

LEMMA 4.12.

(i) *Let* $k \geq 1$ *and let* $\xi_0, \dots, \xi_k, \theta_0, \dots, \theta_k \in \mathrm{Inj}(\mathbb{N})$ *as well as* $\tilde{\varphi}(\tau) := \xi_k \tau \xi_{k-1} \tau \dots \tau \xi_0$ *and* $\tilde{\psi}(\tau) := \theta_k \tau \theta_{k-1} \tau \dots \tau \theta_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.

If $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) \neq \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$, *then* $M_{\tilde{\varphi},\tilde{\psi}} = \{\tau \in \mathrm{Inj}(\mathbb{N}) : \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau)\}$ *is a* $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$*-open neighbourhood of* $\mathrm{id}_{\mathbb{N}}$.

(ii) *Let* $k \geq 1$ *and let* $p_0, \dots, p_k, q_0, \dots, q_k \in \mathrm{End}(\mathbb{G})$ *as well as* $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ *and* $\psi(s) := q_k s q_{k-1} s \dots s q_0$, $s \in \mathrm{End}(\mathbb{G})$. *Assume* $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) \neq \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$ (*using the shorthand from Notation* 4.9).

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$*-open neighbourhood* U *of* $\mathrm{id}_{\mathbb{N}}$ *such that* $\tau \ltimes t \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ *for all* $\tau \in U$ *and* $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$. *In particular,* $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ *for all* $\tau \in U$.

The second lemma really requires the terms to be of equal length.

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LEMMA 4.13. *Let $k \geq 1$ and let $p_0, \dots, p_k, q_0, \dots, q_k \in \mathrm{End}(\mathbb{G})$ as well as $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_k s q_{k-1} s \dots s q_0$, $s \in \mathrm{End}(\mathbb{G})$. Assume $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}) \neq \psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})$ but $\tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) = \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$.*

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood U of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ for all $\tau \in U$.

Finally, we formulate a result for terms of different lengths.

LEMMA 4.14.

(i) *Let $\ell < k$ and let $\xi_0, \dots, \xi_k, \theta_0, \dots, \theta_\ell \in \mathrm{Inj}(\mathbb{N})$ as well as $\tilde{\varphi}(\tau) := \xi_k \tau \xi_{k-1} \tau \dots \tau \xi_0$ and $\tilde{\psi}(\tau) := \theta_\ell \tau \theta_{\ell-1} \tau \dots \tau \theta_0$, $\tau \in \mathrm{Inj}(\mathbb{N})$.*

Then $M_{\tilde{\varphi},\tilde{\psi}} = \{\tau \in \mathrm{Inj}(\mathbb{N}) : \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau)\}$ is $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open.

(ii) *Let $\ell < k$ and let $p_0, \dots, p_k, q_0, \dots, q_\ell \in \mathrm{End}(\mathbb{G})$ as well as $\varphi(s) := p_k s p_{k-1} s \dots s p_0$ and $\psi(s) := q_\ell s q_{\ell-1} s \dots s q_0$, $s \in \mathrm{End}(\mathbb{G})$.*

Then there exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set V such that $\tau \ltimes t \in M_{\varphi,\psi} = \{s \in \mathrm{End}(\mathbb{G}) : \varphi(s) \neq \psi(s)\}$ for all $\tau \in V$ and $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$. In particular, $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ for all $\tau \in V$.

We first demonstrate how these auxiliary statements are used and prove Theorem 4.10 before showing the statements themselves in Section 4.3.

PROOF (of Theorem 4.10 given Lemmas 4.12–4.14). We will show that any $\mathcal{T}_{\mathrm{Zariski}}$-open set O containing $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$ also contains $\tau \ltimes c_{-1}$ for some $\tau \in \mathrm{Inj}(\mathbb{N})$. This implies in particular that the $\mathcal{T}_{pw}$-open set $\{s \in \mathrm{End}(\mathbb{G}) : s(0, a_{+1}) = (0, a_{+1})\}$ cannot be $\mathcal{T}_{\mathrm{Zariski}}$-open—proving $\mathcal{T}_{\mathrm{Zariski}} \neq \mathcal{T}_{pw}$.

It suffices to consider $\mathcal{T}_{\mathrm{Zariski}}$-*basic* open sets O, i.e., $O = \bigcap_{h \in H} M_{\varphi_h,\psi_h} \ni \mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$ for some finite set H. If the terms φ_h and ψ_h have equal length, we apply Lemma 4.12 or 4.13 to find a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood U_h of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi_h,\psi_h}$ for all $\tau \in U_h$. If φ_h and ψ_h have different lengths, we instead apply[3] Lemma 4.14 to find a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set V_h such that $\tau \ltimes c_{-1} \in M_{\varphi_h,\psi_h}$ for all $\tau \in V_h$. Intersecting the respective sets U_h and V_h thus obtained yields a $\mathcal{T}_{pw}$-open neighbourhood U of $\mathrm{id}_{\mathbb{N}}$ and a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense and open set V such that $\tau \ltimes c_{-1} \in M_{\varphi_h,\psi_h}$ for all $\tau \in U$ whenever φ_h and ψ_h have equal length and such that $\tau \ltimes c_{-1} \in M_{\varphi_h,\psi_h}$ for all $\tau \in V$ whenever φ_h and ψ_h have different lengths. The intersection $U \cap V$ is nonempty; for any $\tau \in U \cap V$ we have $\tau \ltimes c_{-1} \in M_{\varphi_h,\psi_h}$ for all $h \in H$, i.e., $\tau \ltimes c_{-1} \in O$. This concludes the proof. $\dashv$

REMARK 4.15. A slight refinement of this proof even shows that the Zariski topology on $\mathrm{End}(\mathbb{G})$ is not Hausdorff since $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$ and $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1}$ cannot be separated by open sets: By the proof, a given basic open set around $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$ contains $\tau \ltimes c_{-1}$ provided that τ is an element of the intersection of a certain $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$ and a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense open set. The same idea similarly (but with an easier proof in the analogue of Lemma 4.13) yields that a given basic open set around $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1}$ contains $\tau' \ltimes c_{-1}$ provided that τ' is an element of the intersection of another $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$ and another $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense open set. The intersection of these four sets is nonempty, so the basic open sets around $\mathrm{id}_{\mathbb{N}} \ltimes c_{+1}$ and $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1}$ contain a common element (namely $\tau \ltimes c_{-1}$ for a certain $\tau \in \mathrm{Inj}(\mathbb{N})$).

[3] If ψ_h is longer than φ_h, we exchange these two terms.

The preceding remark suggests the following refinement of Question 1.1:

QUESTION 4.1. *Is there an* ω*-categorical* (*transitive*?) *relational structure* $\mathbb{A}$ *such that there exists a Hausdorff* (*even Polish*?) *semigroup topology on* $\mathrm{End}(\mathbb{A})$ *which is not finer than the topology of pointwise convergence*?

4.3. Proof details. In this subsection, we prove Lemmas 4.12–4.14 in sequence.

PROOF (OF LEMMA 4.12).
(i) The set $M_{\tilde{\varphi},\tilde{\psi}} \subseteq \mathrm{Inj}(\mathbb{N})$ is open with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$ since $\tilde{\varphi}$ and $\tilde{\psi}$ are continuous with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$.
(ii) Set $U := M_{\tilde{\varphi},\tilde{\psi}}$ and note that if $u := \varphi(\tau \ltimes t)$ and $v := \psi(\tau \ltimes t)$, then $\tilde{u} = \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau) = \tilde{v}$, so $u \neq v$. ⊣

The second lemma requires more work.

PROOF (OF LEMMA 4.13). We start by fixing some notation. We first write $p_j = \bigsqcup_{i\in\mathbb{N}}^{\xi_j} p_{j,i}$, $q_j = \bigsqcup_{i\in\mathbb{N}}^{\theta_j} q_{j,i}$ (so $\xi_j = \tilde{p}_j$, $\theta_j = \tilde{q}_j$), and $\delta := \tilde{\varphi}(\mathrm{id}_{\mathbb{N}}) = \tilde{\psi}(\mathrm{id}_{\mathbb{N}})$. Further, we define $\Xi_j := \xi_j\xi_{j-1}\dots\xi_0$ as well as $\Theta_j := \theta_j\theta_{j-1}\dots\theta_0$, $j = 0,\dots,k$. In particular, $\Xi_k = \Theta_k = \delta$. Let the two (distinct, by assumption) functions $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})$ and $\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})$ differ at the point $(h,x) \in G$. Further, set $e \in \{-1,+1\}$ such that $x \in A_e$ and choose any $x' \in A_{-e}$.

In the course of the proof, we will require the explicit expansions of the compositions in $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$:

$$\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1}) = \bigsqcup_{i\in\mathbb{N}}^{\delta} p_{k,\Xi_{k-1}(i)} c_{\pm 1} p_{k-1,\Xi_{k-2}(i)} \dots c_{\pm 1} p_{0,i},$$

$$\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1}) = \bigsqcup_{i\in\mathbb{N}}^{\delta} q_{k,\Theta_{k-1}(i)} c_{\pm 1} q_{k-1,\Theta_{k-2}(i)} \dots c_{\pm 1} q_{0,i}.$$

We proceed in two steps—first, we show that $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$; second, we extend this to $\tau \ltimes c_{-1}$ for all τ in an appropriately constructed $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood of $\mathrm{id}_{\mathbb{N}}$.

(1) $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$: We compare $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$ at (h,x) as well as (h,x'). In order to simplify notation, we define[4]

$$m := \mathrm{sgn}(p_{k-1,\Xi_{k-2}(h)}) \cdot \mathrm{sgn}(p_{k-2,\Xi_{k-3}(h)}) \cdot \dots \cdot \mathrm{sgn}(p_{0,h}),$$
$$n := \mathrm{sgn}(q_{k-1,\Theta_{k-2}(h)}) \cdot \mathrm{sgn}(q_{k-2,\Theta_{k-3}(h)}) \cdot \dots \cdot \mathrm{sgn}(q_{0,h}),$$
$$\widehat{p} := p_{k,\Xi_{k-1}(h)},$$
$$\widehat{q} := q_{k,\Xi_{k-1}(h)},$$

and conclude

$$[\varphi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x) = \Big(\delta(h), \widehat{p}(a_{me})\Big), \qquad [\varphi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) = \Big(\delta(h), \widehat{p}(a_{me(-1)^k})\Big),$$
$$[\psi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x) = \Big(\delta(h), \widehat{q}(a_{ne})\Big), \qquad [\psi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) = \Big(\delta(h), \widehat{q}(a_{ne(-1)^k})\Big),$$

[4] m and n count how many times the fixed functions (except for the outermost ones) involved in evaluating $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$ and $\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{\pm 1})$ switch the parts of the h-th copy of $\mathbb{K}_{2,\omega}$.

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$$[\varphi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x') = \Big(\delta(h), \widehat{p}(a_{-me})\Big), \quad [\varphi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x') = \Big(\delta(h), \widehat{p}(a_{-me(-1)^k})\Big),$$
$$[\psi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x') = \Big(\delta(h), \widehat{q}(a_{-ne})\Big), \quad [\psi\,(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x') = \Big(\delta(h), \widehat{q}(a_{-ne(-1)^k})\Big).$$

If

$$\big\{\widehat{p}(a_{+1}), \widehat{p}(a_{-1})\big\} \neq \big\{\widehat{q}(a_{+1}), \widehat{q}(a_{-1})\big\},$$

then $\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})$ and $\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})$ cannot coincide on both (h,x) and (h,x'), so $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed.

In case of

$$\big\{\widehat{p}(a_{+1}), \widehat{p}(a_{-1})\big\} = \big\{\widehat{q}(a_{+1}), \widehat{q}(a_{-1})\big\},$$

we distinguish further: If $m = n$, then $[\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x) \neq [\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{+1})](h,x)$ shows

$$\widehat{p}(a_{+1}) = \widehat{q}(a_{-1}) \quad \text{as well as} \quad \widehat{p}(a_{-1}) = \widehat{q}(a_{+1}),$$

which leads to[5] $[\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) \neq [\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x)$, so $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed. If on the other hand $m =- n$, then we analogously obtain

$$\widehat{p}(a_{+1}) = \widehat{q}(a_{+1}) \quad \text{as well as} \quad \widehat{p}(a_{-1}) = \widehat{q}(a_{-1})$$

and $[\varphi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x) \neq [\psi(\mathrm{id}_{\mathbb{N}} \ltimes c_{-1})](h,x)$, so $\mathrm{id}_{\mathbb{N}} \ltimes c_{-1} \in M_{\varphi,\psi}$ as claimed.

(2) There exists a $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open neighbourhood $U \subseteq \mathrm{Inj}(\mathbb{N})$ of $\mathrm{id}_{\mathbb{N}}$ such that $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ for all $\tau \in U$: One immediately checks that for arbitrary $t \in \mathrm{End}(\mathbb{K}_{2,\omega})$, the map $\chi_t\colon\ \mathrm{Inj}(\mathbb{N}) \to \mathrm{End}(\mathbb{G})$, $\chi_t(\tau) := \tau \ltimes t$ is continuous with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$ and[6] $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$. Since $M_{\varphi,\psi}$ is open with respect to $\mathcal{T}_{pw}|_{\mathrm{End}(\mathbb{G})}$, the preimage $U := \chi_{c_{-1}}^{-1}(M_{\varphi,\psi}) \subseteq \mathrm{Inj}(\mathbb{N})$ is open with respect to $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$. By (1), the set U contains $\mathrm{id}_{\mathbb{N}}$—completing the proof. ⊣

Finally, we show the third lemma.

Proof (of Lemma 4.14). **(i)** We have to prove that for two tuples $\bar{z}, \bar{w}$ of the same length such that $\bar{w}$ does not contain the same value twice (since we are working in $\mathrm{Inj}(\mathbb{N})$), the intersection $\{\tau \in \mathrm{Inj}(\mathbb{N}) : \tau(\bar{z}) = \bar{w}\} \cap M_{\tilde{\varphi},\tilde{\psi}}$ is nonempty. The idea behind the proof is to find an element $x_0 \in \mathbb{N}$ and inductively construct a partial injection $\widehat{\tau}$ which extends $\bar{z} \mapsto \bar{w}$ such that the values

$$[\tilde{\varphi}(\widehat{\tau})](x_0) = \xi_k \widehat{\tau} \xi_{k-1} \widehat{\tau} \ldots \widehat{\tau} \xi_0(x_0) \quad \text{and} \quad [\tilde{\psi}(\widehat{\tau})](x_0) = \theta_\ell \widehat{\tau} \theta_{\ell-1} \widehat{\tau} \ldots \widehat{\tau} \theta_0(x_0)$$

are welldefined (i.e., $\xi_0(x_0) \in \mathrm{Dom}(\widehat{\tau})$, $\xi_1\widehat{\tau}\xi_0(x_0) \in \mathrm{Dom}(\widehat{\tau})$, etc.) and $[\tilde{\varphi}(\widehat{\tau})](x_0) \neq [\tilde{\psi}(\widehat{\tau})](x_0)$. This gives $\tau(\bar{z}) = \bar{w}$ and $\tau \in M_{\tilde{\varphi},\tilde{\psi}}$ for any $\tau \in \mathrm{Inj}(\mathbb{N})$ extending $\widehat{\tau}$.

More precisely, we will define (not necessarily distinct) elements $x_0, \ldots, x_k$, $x_0', \ldots, x_k' \in \mathbb{N}$ and $y_0 \ldots, y_\ell, y_0', \ldots, y_\ell' \in \mathbb{N}$ such that:

(1) $x_0 = y_0$.
(2) $x_j' = \xi_j(x_j)$ for all $j = 0, \ldots, k$.
(3) $y_j' = \theta_j(y_j)$ for all $j = 0, \ldots, \ell$.

---

[5] Here we use that φ and ψ have equal lengths (or more precisely: lengths of equal parity).

[6] Caution! We briefly consider the pointwise topology on $\mathrm{End}(\mathbb{G})$ instead of the Zariski topology.

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(4) $\widehat{\tau}$ defined by $\bar{z} \mapsto \bar{w}$, $(x'_0, \dots, x'_{k-1}) \mapsto (x_1, \dots, x_k)$, $(y'_0, \dots, y'_{\ell-1}) \mapsto (y_1, \dots, y_\ell)$ is a welldefined[7] partial injection.

(5) $x'_k \neq y'_\ell$. (This will crucially depend on the assumption $\ell < k$.)

We first pick $x_0 = y_0 \in \mathbb{N}$ such that $x'_0 := \xi_0(x_0) \notin \bar{z}$ and $y'_0 := \theta_0(y_0) \notin \bar{z}$; this is possible since the set $\xi_0^{-1}(\bar{z}) \cup \theta_0^{-1}(\bar{w})$ of forbidden points is finite by injectivity of ξ_0 and θ_0. Note that x'_0 and y'_0 are not necessarily different (in particular, $\xi_0 = \theta_0$ is possible).

Suppose that $1 \leq i \leq \ell$ and that $x_0, \dots, x_{i-1}, x'_0, \dots, x'_{i-1}$ as well as $y_0, \dots, y_{i-1}$, $y'_0, \dots, y'_{i-1}$ are already defined such that (1)–(4) hold (with $i-1$ in place of both k and ℓ). We abbreviate $X_{i-1} := \{x_0, \dots, x_{i-1}\}$, $X'_{i-1} := \{x'_0, \dots, x'_{i-1}\}$ and $Y_{i-1} := \{y_0, \dots, y_{i-1}\}$, $Y'_{i-1} := \{y'_0, \dots, y'_{i-1}\}$. Pick $x_i, y_i \notin \bar{w} \cup X_{i-1} \cup Y_{i-1}$ such that $x'_i := \xi_i(x_i) \notin \bar{z} \cup X'_{i-1} \cup Y'_{i-1}$ and $y'_i := \theta_i(y_i) \notin \bar{z} \cup X'_{i-1} \cup Y'_{i-1}$ with the additional property that[8] x_i and y_i are chosen to be distinct if and only if x'_{i-1} and y'_{i-1} are distinct (to obtain a welldefined partial injection in (4)). As with the construction of x_0 above, this is possible by finiteness of the forbidden sets.

If $\ell + 1 \leq i \leq k$ and if $x_0, \dots, x_{i-1}, x'_0, \dots, x'_{i-1}$ as well as $y_0, \dots, y_\ell, y'_0, \dots, y'_\ell$ are already defined such that (1)–(4) hold (with $i-1$ in place of k), then we again abbreviate $X_{i-1} := \{x_0, \dots, x_{i-1}\}$, $X'_{i-1} := \{x'_0, \dots, x'_{i-1}\}$ and $Y_\ell := \{y_0, \dots, y_\ell\}$, $Y'_\ell := \{y'_0, \dots, y'_\ell\}$. Analogously to the previous step, we pick $x_i \notin \bar{w} \cup X_{i-1} \cup Y_\ell$ such that $x'_i := \xi_i(x_i) \notin \bar{z} \cup X'_{i-1} \cup Y'_\ell$. Note that in the final step $i = k$, we are picking x_k such that[9] $x'_k \notin \bar{z} \cup X'_{k-1} \cup Y'_\ell$. In particular, we require $x'_k \neq y'_\ell$, i.e., (5).

(ii) The set $V := M_{\tilde{\varphi},\tilde{\psi}} \subseteq \mathrm{Inj}(\mathbb{N})$ is $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-dense by the first statement and clearly $\mathcal{T}_{pw}|_{\mathrm{Inj}(\mathbb{N})}$-open. For $\tau \in V$, we set $u := \varphi(\tau \ltimes c_{-1})$ as well as $v := \psi(\tau \ltimes c_{-1})$ and note that $\tilde{u} = \tilde{\varphi}(\tau) \neq \tilde{\psi}(\tau) = \tilde{v}$. This yields $\tau \ltimes c_{-1} \in M_{\varphi,\psi}$ as desired. ⊣

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[7] This means that if, e.g., $x'_0 = y'_0$, then $x_1 = y_1$.

[8] This ensures that $\widehat{\tau}$ is welldefined and injective.

[9] At this point, it is crucial that $\ell < k$ since we would never enter the second phase $\ell + 1 \leq i \leq k$ of the construction otherwise (more precisely, if ℓ were equal to k, we would have to determine x'_k at the same time as y'_ℓ and could not make sure that they are different).

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