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Toeplitz determinants with a one-cut regular potential and Fisher–Hartwig singularities I. Equilibrium measure supported on the unit circle

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We consider Toeplitz determinants whose symbol has: (i) a one-cut regular potential V, (ii) Fisher–Hartwig singularities and (iii) a smooth function in the background. The potential V is associated with an equilibrium measure that is assumed to be supported on the whole unit circle. For constant potentials V, the equilibrium measure is the uniform measure on the unit circle and our formulas reduce to well-known results for Toeplitz determinants with Fisher–Hartwig singularities. For non-constant V, our results appear to be new even in the case of no Fisher–Hartwig singularities. As applications of our results, we derive various statistical properties of a determinantal point process which generalizes the circular unitary ensemble.

Keywords: asymptotics; Fisher–Hartwig singularities; Riemann–Hilbert problems; Toeplitz determinants

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1. Introduction

In this work, we obtain large n asymptotics of the Toeplitz determinant

$$D_n(\vec{\alpha},\vec{\beta},V,W) := \det(f_{j-k})_{j,k=0,\dots,n-1}, \quad f_k := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}\mathrm{d}\theta, \tag{1.1}$$

where f is supported on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and is of the form

$$f(z) = e^{-nV(z)}e^{W(z)}\omega(z), \quad z \in \mathbb{T}. \tag{1.2}$$

We assume that V and W are analytic in a neighbourhood of $\mathbb{T}$ and that the potential V is real-valued on $\mathbb{T}$. The function $\omega(z) = \omega(z; \vec{\alpha}, \vec{\beta})$ in (1.2) contains Fisher–Hartwig singularities and is defined in (1.8) below. Since the functions V and W are analytic on $\mathbb{T}$, there exists an open annulus U containing $\mathbb{T}$ on which they admit Laurent series representations of the form

$$V(z) = V_0 + V_+(z) + V_-(z), \quad V_+(z) = \sum_{k=1}^{+\infty} V_k z^k, \quad V_-(z) = \sum_{k=-\infty}^{-1} V_k z^k, \tag{1.3}$$

$$W(z) = W_0 + W_+(z) + W_-(z), \quad W_+(z) = \sum_{k=1}^{+\infty} W_k z^k, \quad W_-(z) = \sum_{k=-\infty}^{-1} W_k z^k, \tag{1.4}$$

where $V_k, W_k \in \mathbb{C}$ are the Fourier coefficients of V and W, i.e. $V_k = 1/2\pi \int_0^{2\pi} V(e^{i\theta}) e^{-ik\theta} \mathrm{d}\theta$ and similarly for W_k. Associated to V there is an equilibrium measure μ_V, which is the unique minimizer of the functional

$$\mu \mapsto \iint \log \frac{1}{|z-s|} \mathrm{d}\mu(z)\mathrm{d}\mu(s) + \int V(z)\mathrm{d}\mu(z) \tag{1.5}$$

among all Borel probability measures μ on $\mathbb{T}$. In this paper, we make the assumption that μ is supported on the whole unit circle. We further assume that V is regular, i.e. that the function ψ given by

$$\psi(z) = \frac{1}{2\pi} - \frac{1}{2\pi} \sum_{\ell=1}^{+\infty} \ell (V_\ell z^\ell + \overline{V_\ell} z^{-\ell}), \quad z \in U, \tag{1.6}$$

is strictly positive on $\mathbb{T}$. Under these assumptions, we show in appendix A that

$$\mathrm{d}\mu_V(e^{i\theta}) = \psi(e^{i\theta})\mathrm{d}\theta, \qquad \theta \in [0, 2\pi). \tag{1.7}$$

The function ω appearing in (1.2) is defined by

$$\omega(z) = \prod_{k=0}^{m} \omega_{\alpha_k}(z)\omega_{\beta_k}(z), \tag{1.8}$$

where $\omega_{\alpha_k}(z)$ and $\omega_{\beta_k}(z)$ are defined for $z = e^{i\theta}$ by

$$\omega_{\alpha_k}(z) = |z - t_k|^{\alpha_k}, \;\; \omega_{\beta_k}(z) = e^{i(\theta-\theta_k)\beta_k} \times \begin{cases} e^{i\pi\beta_k}, & \text{if } 0 \leqslant \theta < \theta_k, \\ e^{-i\pi\beta_k}, & \text{if } \theta_k \leqslant \theta < 2\pi, \end{cases} \quad \theta \in [0, 2\pi), \tag{1.9}$$

and

$$t_k := e^{i\theta_k}, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi. \tag{1.10}$$

At $t_k = e^{i\theta_k}$, the functions ω_{α_k} and ω_{β_k} have root- and jump-type singularities, respectively. Note that ω_{β_k} is continuous at $z = 1$ if $k \neq 0$. We allow the parameters $\theta_1, \ldots, \theta_m$ to vary with n, but we require them to lie in a compact subset of $(0, 2\pi)^m_{\mathrm{ord}} := \{(\theta_1, \ldots, \theta_m) : 0 < \theta_1 < \cdots < \theta_m < 2\pi\}$.

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To summarize, the $n \times n$ Toeplitz determinant (1.1) depends on n, m, V, W, $\vec{t} = (t_1, \ldots, t_m)$, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\vec{\beta} = (\beta_1, \ldots, \beta_m)$, but for convenience the dependence on m and $\vec{t}$ is omitted in the notation $D_n(\vec{\alpha}, \vec{\beta}, V, W)$. We now state our main result.

THEOREM 1.1 Large n asymptotics of $D_n(\vec{\alpha}, \vec{\beta}, V, W)$. *Let $m \in \mathbb{N} := \{0, 1, \ldots\}$, and let $t_k = e^{i\theta_k}$, $\alpha_k \in \mathbb{C}$ and $\beta_k \in \mathbb{C}$ be such that*

$$0 = \theta_0 < \theta_1 < \ldots < \theta_m < 2\pi, \qquad \textit{and}$$
$$\operatorname{Re}\alpha_k > -1, \quad \operatorname{Re}\beta_k \in (-\tfrac{1}{2}, \tfrac{1}{2}) \qquad \textit{for } k = 0, \ldots, m.$$

Let $V : \mathbb{T} \to \mathbb{R}$ and $W : \mathbb{T} \to \mathbb{C}$, and suppose V and W can be extended to analytic functions in a neighbourhood of $\mathbb{T}$. Suppose that the equilibrium measure $\mathrm{d}\mu_V(e^{i\theta}) = \psi(e^{i\theta})\mathrm{d}\theta$ associated to V is supported on $\mathbb{T}$ and that $\psi > 0$ on $\mathbb{T}$. Then, as $n \to \infty$,

$$D_n(\vec{\alpha}, \vec{\beta}, V, W) = \exp(C_1 n^2 + C_2 n + C_3 \log n + C_4 + \mathcal{O}(n^{-1+2\beta_{\max}})), \tag{1.11}$$

with $\beta_{\max} = \max\{|\operatorname{Re}\beta_1|, \ldots, |\operatorname{Re}\beta_m|\}$ and

$$C_1 = -\frac{V_0}{2} - \frac{1}{2}\int_0^{2\pi} V(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta}),$$

$$C_2 = \sum_{k=0}^{m} \frac{\alpha_k}{2}(V(t_k) - V_0) - \sum_{k=0}^{m} 2i\beta_k \operatorname{Im}(V_+(t_k)) + \int_0^{2\pi} W(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta}),$$

$$C_3 = \sum_{k=0}^{m}\left(\frac{\alpha_k^2}{4} - \beta_k^2\right),$$

$$\begin{aligned} C_4 = {} & \sum_{\ell=1}^{+\infty} \ell W_\ell W_{-\ell} - \sum_{k=0}^{m}\frac{\alpha_k}{2}(W(t_k) - W_0) + \sum_{k=0}^{m}\beta_k\left(W_+(t_k) - W_-(t_k)\right) \\ & + \sum_{0 \leqslant j < k \leqslant m}\left\{\frac{\alpha_j i\beta_k - \alpha_k i\beta_j}{2}(\theta_k - \theta_j - \pi) + \left(2\beta_j\beta_k - \frac{\alpha_j\alpha_k}{2}\right)\log|t_j - t_k|\right\} \\ & + \sum_{k=0}^{m}\log\frac{G(1 + \frac{\alpha_k}{2} + \beta_k)G(1 + \frac{\alpha_k}{2} - \beta_k)}{G(1 + \alpha_k)} + \sum_{k=0}^{m}\frac{\beta_k^2 - \frac{\alpha_k^2}{4}}{\psi(t_k)}\left(\frac{1}{2\pi} - \psi(t_k)\right), \end{aligned}$$

where G is Barnes' G-function. Furthermore, the above asymptotics are uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, for all β_k in compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z \in (-\frac{1}{2}, \frac{1}{2})\}$ and for all $(\theta_1, \ldots, \theta_m)$ in compact subsets of $(0, 2\pi)^m_{\mathrm{ord}}$. The above asymptotics can also be differentiated with respect to $\alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_m$ as follows: if $k_0, \ldots, k_{2m+1} \in \mathbb{N}$, $k_0 + \ldots + k_{2m+1} \geqslant 1$ and $\partial^{\vec{k}} := \partial_{\alpha_0}^{k_0} \ldots \partial_{\alpha_m}^{k_m}\partial_{\beta_0}^{k_{m+1}} \ldots \partial_{\beta_m}^{k_{2m+1}}$, then

$$\partial^{\vec{k}}\left(\log D_n(\vec{\alpha}, \vec{\beta}, V, W) - \log \widehat{D}_n\right) = \mathcal{O}\left(\frac{(\log n)^{k_{m+1} + \ldots + k_{2m+1}}}{n^{1-2\beta_{\max}}}\right), \qquad \textit{as } n \to +\infty, \tag{1.12}$$

where $\widehat{D}_n$ denotes the right-hand side of (1.11) without the error term.

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1.1. History and related work

In the case when the potential $V(z)$ in (1.2) vanishes identically, the asymptotic evaluation of Toeplitz determinants of the form (1.1) has a long and distinguished history. The first important result was obtained by Szegő in 1915 who determined the leading behaviour of $D_n(\vec{\alpha},\vec{\beta},V,W)$ in the case when $\vec{\alpha}=\vec{\beta}=\vec{0}$ and $V=0$, that is, when the symbol $f(z)$ is given by $f(z)=e^{W(z)}$. In our notation, this result, known as the first Szegő limit theorem [**45**], can be expressed as

$$D_n(\vec{0},\vec{0},0,W)=\exp\left(\frac{n}{2\pi}\int_0^{2\pi}W(e^{i\theta})\mathrm{d}\theta+o(n)\right)\qquad \text{as } n\to\infty. \tag{1.13}$$

Later, in the 1940s, it became clear from the pioneering work of Kaufmann and Onsager that a more detailed understanding of the error term in (1.13) could be used to compute two-point correlation functions in the two-dimensional Ising model in the thermodynamic limit [**39**]. This inspired Szegő to seek for a stronger version of (1.13). The outcome was the so-called strong Szegő limit theorem [**46**], which in our notation states that

$$D_n(\vec{0},\vec{0},0,W)=\exp\left(\frac{n}{2\pi}\int_0^{2\pi}W(e^{i\theta})\mathrm{d}\theta+\sum_{\ell=1}^{+\infty}\ell W_\ell W_{-\ell}+o(1)\right)\qquad \text{as } n\to\infty. \tag{1.14}$$

We observe that if $V=0$, then $\mathrm{d}\mu_V(e^{i\theta})=\frac{\mathrm{d}\theta}{2\pi}$; thus, Szegő's theorems are consistent with our main result, theorem 1.11, in the special case when $\vec{\alpha}=\vec{\beta}=\vec{0}$ and $V=0$. (The strong Szegő theorem actually holds under much weaker assumptions on W than what is assumed in this paper, see e.g. the survey [**7**].)

In a groundbreaking paper from 1968, Fisher and Hartwig introduced a class of singular symbols $f(z)$ for which they convincingly conjectured a detailed asymptotic formula for the associated Toeplitz determinant [**32**]. The Fisher–Hartwig class consists of symbols $f(z)$ of form (1.2) with $V=0$. In our notation, the Fisher–Hartwig conjecture can be formulated as

$$D_n(\vec{\alpha},\vec{\beta},0,W)\sim\exp\left(\frac{n}{2\pi}\int_0^{2\pi}W(e^{i\theta})\mathrm{d}\theta+\sum_{k=0}^{m}\left(\frac{\alpha_k^2}{4}-\beta_k^2\right)\log n+C_4\right)\qquad \text{as } n\to\infty, \tag{1.15}$$

where C_4 is a constant to be determined, and the Fisher–Hartwig singularities are encoded in the vectors $\vec{\alpha}$ and $\vec{\beta}$. Symbols with Fisher–Hartwig singularities arise in many applications. For example, in the 1960s, Lenard proved [**41**] that no Bose–Einstein condensation exists in the ground state for a one-dimensional system of impenetrable bosons by considering Toeplitz determinants with symbols of the form $f(z)=|z-e^{i\theta_1}||z-e^{-i\theta_1}|$ with $\theta_1\in\mathbb{R}$. Lenard's proof hinges on an inequality whose proof was provided by Szegő, see [**41**, Theorem 2]. We observe that (1.15) is consistent with theorem 1.11 in the special case when $V=0$.

There are too many works devoted to proofs and generalizations of the Fisher–Hartwig conjecture (1.15) for us to cite them all, but we refer to [**4**, **11**,

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47] for some early works, and to [5, 6, 10, 25] for four reviews. The current state-of-the-art for non-merging singularities and for $\vec{\alpha}$, $\vec{\beta}$ in compact subsets was set by Ehrhardt in his 1997 Ph.D. thesis (see [29]) and by Deift, Its and Krasovsky in [24, 26]. Since our proof builds on the results for the case of $V = 0$, we have included a version of the asymptotic formulas of [24, 26, 29] in theorem 4.1. We also refer to [21, 31] for studies of merging Fisher–Hartwig singularities with $V = 0$, and to [17] for the case of large discontinuities with $V = 0$.

Note that if $V = V_0$ is a constant, then $D_n(\vec{\alpha}, \vec{\beta}, V_0, W) = e^{-n^2 V_0} D_n(\vec{\alpha}, \vec{\beta}, 0, W)$.

The novelty of the present work is that we consider symbols that include a non-constant potential V; we are not aware of any previous works on the unit circle including such potentials. Our main result is formulated under the assumption that $\operatorname{Re}\beta_k \in (-\frac{1}{2}, \frac{1}{2})$ for all k. The general case where $\operatorname{Re}\beta_k \in \mathbb{R}$ was treated in the case of $V = 0$ in [24]. Asymptotic formulas for Hankel determinants with a one-cut regular potential V and Fisher–Hartwig singularities were obtained in [8, 14, 19], and the corresponding multi-cut case was considered in [18]. Our proofs draw on some of the techniques developed in these papers.

1.2. Application: a determinantal point process on the unit circle

The Toeplitz determinant (1.1) admits the Heine representation

$$D_n(\vec{\alpha}, \vec{\beta}, V, W) = \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} \prod_{1 \leqslant j < k \leqslant n} |e^{i\phi_k} - e^{i\phi_j}|^2 \prod_{j=1}^{n} f(e^{i\phi_j}) \mathrm{d}\phi_j. \tag{1.16}$$

This suggests that the results of theorem 1.11 can be applied to obtain information about the point process on $\mathbb{T}$ defined by the probability measure

$$\frac{1}{n!(2\pi)^n Z_n} \prod_{1 \leqslant j < k \leqslant n} |e^{i\phi_k} - e^{i\phi_j}|^2 \prod_{j=1}^{n} e^{-nV(e^{i\phi_j})} \mathrm{d}\phi_j, \qquad \phi_1, \ldots, \phi_n \in [0, 2\pi), \tag{1.17}$$

where $Z_n = D_n(\vec{0}, \vec{0}, V, 0)$ is the normalization constant (also called the partition function). In what follows, we use theorem 1.11 to obtain smooth statistics, log statistics, counting statistics and rigidity bounds for the point process (1.17). In the case of constant V, the point process (1.17) describes the distribution of eigenvalues of matrices drawn from the circular unitary ensemble and has already been widely studied. We are not aware of any earlier work where the process (1.17) is considered explicitly for non-constant V. However, the point process (1.17), but with $nV(e^{i\phi})$ replaced by the highly oscillatory potential $V(e^{in\phi})$, is studied in [2, 34]. We also refer to [12, 13] for other determinantal generalizations of the circular unitary ensemble.

Let $\mathsf{p}_n(z) := \prod_{j=1}^{n}(e^{i\phi_j} - z)$ be the characteristic polynomial associated to (1.17), and define $\log \mathsf{p}_n(z)$ for $z \in \mathbb{T} \setminus \{e^{i\phi_1}, \ldots, e^{i\phi_n}\}$ by

$$\log \mathsf{p}_n(z) := \sum_{j=1}^{n} \log(e^{i\phi_j} - z), \qquad \mathrm{Im}\, \log(e^{i\phi_j} - z)$$
$$:= \frac{\phi_j + \arg_0 z}{2} + \begin{cases} \frac{3\pi}{2}, & \text{if } 0 \leqslant \phi_j < \arg_0 z, \\ \frac{\pi}{2}, & \text{if } \arg_0 z < \phi_j < 2\pi, \end{cases}$$

where $\arg_0 z \in [0, 2\pi)$. In particular, if $\theta_k \notin \{\phi_1, \ldots, \phi_n\}$,

$$e^{2i\beta_k(\mathrm{Im}\, \log \mathsf{p}_n(t_k) - n\theta_k - n\pi)} = \prod_{j=1}^{n} \omega_{\beta_k}(e^{i\phi_j}) = e^{-i\beta_k(\pi+\theta_k)n} e^{2\pi i\beta_k N_n(\theta_k)} \prod_{j=1}^{n} e^{i\beta_k\phi_j}, \tag{1.18}$$

where $N_n(\theta) := \#\{\phi_j \in [0, \theta]\} \in \{0, 1, \ldots, n\}$. Using the first identity in (1.18) and the fact that $\{\theta_0, \ldots, \theta_m\} \cap \{\phi_1, \ldots, \phi_n\} = \emptyset$ with probability one, it is straightforward to see that

$$\mathbb{E}\left[\prod_{j=1}^{n} e^{W(e^{i\phi_j})} \prod_{k=0}^{m} e^{\alpha_k \mathrm{Re}\, \log \mathsf{p}_n(t_k)} e^{2i\beta_k(\mathrm{Im}\, \log \mathsf{p}_n(t_k) - n\theta_k - n\pi)}\right] = \frac{D_n(\vec{\alpha}, \vec{\beta}, V, W)}{D_n(\vec{0}, \vec{0}, V, 0)}. \tag{1.19}$$

Furthermore, if $\beta_0 = -\beta_1 - \ldots - \beta_m$, then the second identity in (1.18) together with (1.19) implies

$$\frac{D_n(\vec{\alpha}, \vec{\beta}, V, W)}{D_n(\vec{0}, \vec{0}, V, 0)} = \prod_{k=1}^{m} e^{-i\beta_k\theta_k n} \times \mathbb{E}\left[\prod_{j=1}^{n} e^{W(e^{i\phi_j})} \prod_{k=0}^{m} |\mathsf{p}_n(t_k)|^{\alpha_k} e^{2\pi i\beta_k N_n(\theta_k)}\right]. \tag{1.20}$$

Lemma 1.2. *For any $z \in \mathbb{T}$, we have*

$$\frac{V(z) - V_0}{2} = \int_0^{2\pi} \log|e^{i\theta} - z| \mathrm{d}\mu_V(e^{i\theta}), \tag{1.21}$$

$$\frac{\arg_0 z}{2\pi} - \frac{\mathrm{Im}\, V_+(z) - \mathrm{Im}\, V_+(1)}{\pi} = \int_0^{\arg_0 z} \mathrm{d}\mu_V(e^{i\theta}). \tag{1.22}$$

Proof. The equilibrium measure μ_V is uniquely characterized by the Euler–Lagrange variational equality

$$2\int_0^{2\pi} \log|z - e^{i\theta}| \mathrm{d}\mu_V(e^{i\theta}) = V(z) - \ell, \qquad \text{for } z \in \mathbb{T}, \tag{1.23}$$

where $\ell \in \mathbb{R}$ is a constant, see e.g. [**42**]. In particular, the identity (1.21) is equivalent to the statement that $\ell = V_0$. The equality $\ell = V_0$ can be established by integrating

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(1.23) over $z = e^{i\phi} \in \mathbb{T}$ and dividing by 2π:

$$\ell = \int_0^{2\pi} \ell \frac{\mathrm{d}\phi}{2\pi} = \int_0^{2\pi} \left(V(z) - 2\int_0^{2\pi} \log|e^{i\phi} - e^{i\theta}| \mathrm{d}\mu_V(e^{i\theta}) \right) \frac{\mathrm{d}\phi}{2\pi} = V_0,$$

where we have used the well-known (see e.g. [**42**, Example 0.5.7]) identity $\int_0^{2\pi} \log|e^{i\phi} - e^{i\theta}| \frac{\mathrm{d}\phi}{2\pi} = 0$ for $\theta \in [0, 2\pi)$. This proves (1.21). Identity (1.22) follows from (1.6) and (1.3). □

Combining (1.20), theorem 1.1 and lemma 1.2, we get the following.

THEOREM 1.3. *Let $m \in \mathbb{N}$, and let $t_k = e^{i\theta_k}$, $\alpha_0, \ldots, \alpha_m \in \mathbb{C}$ and $u_1, \ldots, u_m \in \mathbb{C}$ be such that*

$$0 = \theta_0 < \theta_1 < \ldots < \theta_m < 2\pi, \quad \text{and} \quad \operatorname{Re}\alpha_k > -1, \quad \operatorname{Im} u_k \in (-\pi, \pi) \quad \text{for all } k.$$

Let $V : \mathbb{T} \to \mathbb{R}$, $W : \mathbb{T} \to \mathbb{C}$ and suppose V, W can be extended to analytic functions in a neighbourhood of $\mathbb{T}$. Suppose that the equilibrium measure $\mathrm{d}\mu_V(e^{i\theta}) = \psi(e^{i\theta})\mathrm{d}\theta$ associated to V is supported on $\mathbb{T}$ and that $\psi > 0$ on $\mathbb{T}$. Then, as $n \to \infty$, we have

$$\mathbb{E}\left[\prod_{j=1}^{n} e^{W(e^{i\phi_j})} \prod_{k=0}^{m} |\mathsf{p}_n(t_k)|^{\alpha_k} \prod_{k=1}^{m} e^{u_k N_n(\theta_k)} \right]$$
$$= \exp\left(\tilde{C}_1 n + \tilde{C}_2 \log n + \tilde{C}_3 + \mathcal{O}\left(n^{-1+\frac{u_{\max}}{\pi}} \right) \right), \tag{1.24}$$

with $u_{\max} = \max\{|\operatorname{Im} u_1|, \ldots, |\operatorname{Im} u_m|\}$ and

$$\tilde{C}_1 = \sum_{k=0}^{m} \alpha_k \int_0^{2\pi} \log|e^{i\phi} - t_k| \mathrm{d}\mu_V(e^{i\phi}) + \sum_{k=1}^{m} u_k \int_0^{\theta_k} \mathrm{d}\mu_V(e^{i\phi})$$
$$+ \int_0^{2\pi} W(e^{i\phi}) \mathrm{d}\mu_V(e^{i\phi}), \tag{1.25}$$

$$\tilde{C}_2 = \sum_{k=0}^{m} \left(\frac{\alpha_k^2}{4} + \frac{u_k^2}{4\pi^2} \right), \tag{1.26}$$

$$\tilde{C}_3 = \sum_{\ell=1}^{+\infty} \ell W_\ell W_{-\ell} - \sum_{k=0}^{m} \alpha_k \frac{W_+(t_k) + W_-(t_k)}{2} + \sum_{k=0}^{m} \frac{u_k}{\pi} \frac{W_+(t_k) - W_-(t_k)}{2i} \tag{1.27}$$

$$+ \sum_{0 \leqslant j < k \leqslant m} \left\{ \frac{\alpha_j u_k - \alpha_k u_j}{4\pi} (\theta_k - \theta_j - \pi) - \left(\frac{u_j u_k}{2\pi^2} + \frac{\alpha_j \alpha_k}{2} \right) \log|t_j - t_k| \right\} \tag{1.28}$$

$$+ \sum_{k=0}^{m} \log \frac{G(1 + \frac{\alpha_k}{2} + \frac{u_k}{2\pi i}) G(1 + \frac{\alpha_k}{2} - \frac{u_k}{2\pi i})}{G(1 + \alpha_k)} - \sum_{k=0}^{m} \frac{\frac{u_k^2}{\pi^2} + \alpha_k^2}{4\psi(t_k)} \left(\frac{1}{2\pi} - \psi(t_k) \right), \tag{1.29}$$

where G is Barnes' G-function and $u_0 := -u_1 - \ldots - u_m$. Furthermore, the above asymptotics are uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, for

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all u_k *in compact subsets of* $\{z \in \mathbb{C} : \mathrm{Im}\, z \in (-\pi, \pi)\}$ *and for all* $(\theta_1, \ldots, \theta_m)$ *in compact subsets of* $(0, 2\pi)^m_{\mathrm{ord}}$*. The above asymptotics can also be differentiated with respect to* $\alpha_0, \ldots, \alpha_m, u_1, \ldots, u_m$ *as follows: if* $k_0, \ldots, k_{2m} \in \mathbb{N}$, $k_0 + \ldots + k_{2m} \geqslant 1$ *and* $\partial^{\vec{k}} := \partial_{\alpha_0}^{k_0} \ldots \partial_{\alpha_m}^{k_m} \partial_{u_1}^{k_{m+1}} \ldots \partial_{u_m}^{k_{2m}}$*, then as* $n \to +\infty$

$$\partial^{\vec{k}} \left(\log \mathbb{E}\left[\prod_{j=1}^{n} e^{W(e^{i\phi_j})} \prod_{k=0}^{m} |\mathsf{p}_n(t_k)|^{\alpha_k} \prod_{k=1}^{m} e^{u_k N_n(\theta_k)} \right] - \log \widehat{E}_n \right)$$
$$= \mathcal{O}\left(\frac{(\log n)^{k_{m+1}+\ldots+k_{2m}}}{n^{1-\frac{u_{\max}}{\pi}}} \right),$$

where $\widehat{E}_n$ *denotes the right-hand side of (1.24) without the error term.*

Our first corollary is concerned with the smooth linear statistics of (1.17). For $V = 0$, the central limit theorem stated in corollary 1.4 was already obtained in [**38**].

Corollary 1.4 Smooth statistics. *Let* V *and* W *be as in theorem 1.3, and assume furthermore that* $W : \mathbb{T} \to \mathbb{R}$*. Let* $\{\kappa_j\}_{j=1}^{+\infty}$ *be the cumulants of* $\sum_{j=1}^n W(e^{i\phi_j})$*, i.e.*

$$\kappa_j := \partial_t^j \log \mathbb{E}[e^{t\sum_{j=1}^n W(e^{i\phi_j})}]\big|_{t=0}. \tag{1.30}$$

As $n \to +\infty$*, we have*

$$\mathbb{E}\left[\sum_{j=1}^n W(e^{i\phi_j})\right] = n\int_0^{2\pi} W(e^{i\phi}) \mathrm{d}\mu_V(e^{i\phi}) + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\mathrm{Var}\left[\sum_{j=1}^n W(e^{i\phi_j})\right] = 2\sum_{\ell=1}^{+\infty} \ell W_\ell W_{-\ell} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$\kappa_j = \mathcal{O}\left(\frac{1}{n}\right), \qquad j \geqslant 3.$$

Moreover, if W *is non-constant, then*

$$\frac{\sum_{j=1}^n W(e^{i\phi_j}) - n\int_0^{2\pi} W(e^{i\phi}) \mathrm{d}\mu_V(e^{i\phi})}{(2\sum_{k=1}^{+\infty} k W_k W_{-k})^{1/2}}$$

converges in distribution to a standard normal random variable.

Our second corollary considers linear statistics for a test function with a log-singularity at t. We let $\gamma_{\mathrm{E}} \approx 0.5772$ denote Euler's constant.

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Corollary 1.5 $\log|\cdot|$-statistics. *Let $t = e^{i\theta} \in \mathbb{T}$ with $\theta \in [0, 2\pi)$, and let $\{\kappa_j\}_{j=1}^{+\infty}$ be the cumulants of $\log|\mathsf{p}_n(t)|$, i.e.*

$$\kappa_j := \partial_\alpha^j \log \mathbb{E}[e^{\alpha \log|\mathsf{p}_n(t)|}]\big|_{\alpha=0}. \tag{1.31}$$

As $n \to +\infty$, we have

$$\mathbb{E}[\log|\mathsf{p}_n(t)|] = n\int_0^{2\pi} \log|e^{i\phi} - t| \mathrm{d}\mu_V(e^{i\phi}) + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\mathrm{Var}[\log|\mathsf{p}_n(t)|] = \frac{\log n}{2} + \frac{1+\gamma_{\mathrm{E}}}{2} - \frac{\frac{1}{2\pi} - \psi(t)}{2\psi(t)} + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\kappa_j = (-1 + 2^{1-j})\,(\log G)^{(j)}(1) + \mathcal{O}\left(\frac{1}{n}\right), \qquad j \geqslant 3,$$

and

$$\frac{\log|\mathsf{p}_n(t)| - n\int_0^{2\pi} \log|e^{i\phi} - t| \mathrm{d}\mu_V(e^{i\phi})}{\sqrt{\log n}/\sqrt{2}}$$

converges in distribution to a standard normal random variable.

Counting statistics of determinantal point processes have been widely studied over the years [**22**, **44**] and is still a subject of active research, see e.g. the recent works [**16**, **23**, **43**]. Our third corollary established various results on the counting statistics of (1.17).

Corollary 1.6 Counting statistics. *Let $t = e^{i\theta} \in \mathbb{T}$ be bounded away from $t_0 := 1$, with $\theta \in (0, 2\pi)$, and let $\{\kappa_j\}_{j=1}^{+\infty}$ be the cumulants of $N_n(\theta)$, i.e.*

$$\kappa_j := \partial_u^j \log \mathbb{E}[e^{uN_n(\theta)}]\big|_{u=0}. \tag{1.32}$$

As $n \to +\infty$, we have

$$\mathbb{E}[N_n(\theta)] = n\int_0^{\theta} \mathrm{d}\mu_V(e^{i\phi}) + \mathcal{O}\left(\frac{\log n}{n}\right),$$

$$\mathrm{Var}[N_n(\theta)] = \frac{\log n}{\pi^2} + \frac{1+\gamma_{\mathrm{E}} + \log|t-1|}{\pi^2} - \frac{\frac{1}{2\pi} - \psi(1)}{2\pi^2\psi(1)} - \frac{\frac{1}{2\pi} - \psi(t)}{2\pi^2\psi(t)} + \mathcal{O}\left(\frac{(\log n)^2}{n}\right),$$

$$\kappa_{2j+1} = \mathcal{O}\left(\frac{(\log n)^{2j+1}}{n}\right), \qquad j \geqslant 1,$$

$$\kappa_{2j+2} = \frac{(-1)^{j+1}}{2^{2j}\pi^{2j+2}}\,(\log G)^{(2j+2)}(1) + \mathcal{O}\left(\frac{(\log n)^{2j+2}}{n}\right), \qquad j \geqslant 1,$$

and $\frac{N_n(\theta) - n\int_0^\theta \mathrm{d}\mu_V(e^{i\phi})}{\sqrt{\log n}/\pi}$ converges in distribution to a standard normal random variable.

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REMARK 1.7. There are several differences between smooth, log- and counting statistics that are worth pointing out:

- The variance of the smooth statistics is of order 1, while the variances of the log- and counting statistics are of order $\log n$.
- The third and higher order cumulants of the smooth statistics are all $\mathcal{O}(n^{-1})$, while for the log-statistics the corresponding cumulants are all of order 1. On the other hand, the third and higher order cumulants of the counting statistics are as follows: the odd cumulants are $o(1)$, while the even cumulants are of order 1. This phenomenon for the counting statistics was already noticed in [**43**, eq (29)] for a class of determinantal point processes.

Another consequence of theorem 1.3 is the following result about the individual fluctuations of the ordered angles. Corollary 1.8 is an analogue for (1.17) of Gustavsson's well-known result [**36**, Theorem 1.2] for the Gaussian unitary ensemble.

COROLLARY 1.8 Ordered statistics. *Let $\xi_1 \leqslant \xi_2 \leqslant \ldots \leqslant \xi_n$ denote the ordered angles,*

$$\xi_1 = \min\{\phi_1, \ldots, \phi_n\}, \quad \xi_j = \inf_{\theta \in [0,2\pi)} \{\theta : N_n(\theta) = j\}, \quad j = 1, \ldots, n, \tag{1.33}$$

and let η_k be the classical location of the k-th smallest angle ξ_k,

$$\int_0^{\eta_k} \mathrm{d}\mu_V(e^{i\phi}) = \frac{k}{n}, \qquad k = 1, \ldots, n. \tag{1.34}$$

Let $t = e^{i\theta} \in \mathbb{T}$ with $\theta \in (0, 2\pi)$. Let $k_\theta = [n \int_0^\theta \mathrm{d}\mu_V(e^{i\phi})]$, where $[x] := \lfloor x + \frac{1}{2} \rfloor$ is the closest integer to x. As $n \to +\infty$, $\frac{n\psi(e^{i\eta_{k_\theta}})}{\sqrt{\log n}/\pi}(\xi_{k_\theta} - \eta_{k_\theta})$ converges in distribution to a standard normal random variable.

There has been a lot of progress in recent years towards understanding the global rigidity of various point processes, see e.g. [**1**, **20**, **30**]. Our next corollary is a contribution in this direction: it establishes global rigidity upper bounds for (i) the counting statistics of (1.17) and (ii) the ordered statistics of (1.17).

COROLLARY 1.9 Rigidity. *For each $\epsilon > 0$ sufficiently small, there exist $c > 0$ and $n_0 > 0$ such that*

$$\mathbb{P}\left(\sup_{0 \leqslant \theta < 2\pi} \left| N_n(\theta) - n \int_0^\theta \mathrm{d}\mu_V(e^{i\phi}) \right| \leqslant (1+\epsilon)\frac{1}{\pi} \log n \right) \geqslant 1 - \frac{c}{\log n}, \tag{1.35}$$

$$\mathbb{P}\left(\max_{1 \leqslant k \leqslant n} \psi(e^{i\eta_k}) |\xi_k - \eta_k| \leqslant (1+\epsilon)\frac{1}{\pi}\frac{\log n}{n} \right) \geqslant 1 - \frac{c}{\log n}, \tag{1.36}$$

for all $n \geqslant n_0$.

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REMARK 1.10. It follows from (1.36) that $\lim_{n\to\infty} \mathbb{P}(\max_{1\leqslant k\leqslant n} \psi(e^{i\eta_k})|\xi_k - \eta_k| \leqslant (1+\epsilon)\frac{1}{\pi}\frac{\log n}{n}) = 1$. We believe that the upper bound $(1+\epsilon)\frac{1}{\pi}$ is sharp, in the sense that we expect the following to hold true:

$$\lim_{n\to+\infty} \mathbb{P}\left((1-\epsilon)\frac{1}{\pi}\frac{\log n}{n} \leqslant \max_{1\leqslant k\leqslant n} \psi(e^{i\eta_k})|\xi_k - \eta_k| \leqslant (1+\epsilon)\frac{1}{\pi}\frac{\log n}{n}\right) = 1. \quad (1.37)$$

Our belief is supported by the fact that (1.37) was proved in [**1**, Theorem 1.5] for $V = 0$, $\psi(e^{i\theta}) = \frac{1}{2\pi}$.

2. Differential identity for D_n

Our general strategy to prove theorem 1.1 is inspired by the earlier works [**8**, **14**, **24**, **40**]. The first step consists of establishing a differential identity which expresses derivatives of $\log D_n(\vec{\alpha}, \vec{\beta}, V, W)$ in terms of the solution Y to a Riemann–Hilbert (RH) problem (see proposition 2.2). Throughout the paper, $\mathbb{T}$ is oriented in the counterclockwise direction. We first state the RH problem for Y.

RH problem for $Y(\cdot) = Y_n(\cdot; \vec{\alpha}, \vec{\beta}, V, W)$

(a) $Y : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For each $z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}$, the boundary values $\lim_{z'\to z} Y(z')$ from the interior and exterior of $\mathbb{T}$ exist, and are denoted by $Y_+(z)$ and $Y_-(z)$ respectively. Furthermore, Y_+ and Y_- are continuous on $\mathbb{T} \setminus \{t_0, \ldots, t_m\}$, and are related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n} f(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}, \qquad (2.1)$$

where f is given by (1.2).

(c) Y has the following asymptotic behaviour at infinity:

$$Y(z) = (1 + \mathcal{O}(z^{-1}))z^{n\sigma_3}, \qquad \text{as } z \to \infty,$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) As $z \to t_k$, $k = 0, \ldots, m$, $z \in \mathbb{C} \setminus \mathbb{T}$,

$$Y(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z-t_k|^{\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z-t_k|^{\alpha_k}) \end{pmatrix}, & \text{if } \operatorname{Re}\alpha_k \neq 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-t_k|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-t_k|) \end{pmatrix}, & \text{if } \operatorname{Re}\alpha_k = 0. \end{cases}$$

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Suppose $\{p_k(z) = \kappa_k z^k + \ldots\}_{k\geqslant 0}$ and $\{\hat{p}_k(z) = \kappa_k z^k + \ldots\}_{k\geqslant 0}$ are two families of polynomials satisfying the orthogonality conditions

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} p_k(z)z^{-j}f(z)\mathrm{d}\theta = \kappa_k^{-1}\delta_{jk}, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \hat{p}_k(z^{-1})z^{j}f(z)\mathrm{d}\theta = \kappa_k^{-1}\delta_{jk}, \end{cases} \qquad z = e^{i\theta}, \quad j = 0, \ldots, k. \tag{2.2}$$

Then the function $Y(z)$ defined by

$$Y(z) = \begin{pmatrix} \kappa_n^{-1}p_n(z) & \kappa_n^{-1}\displaystyle\int_{\mathbb{T}} \frac{p_n(s)f(s)}{2\pi i s^n(s-z)}\mathrm{d}s \\ -\kappa_{n-1}z^{n-1}\hat{p}_{n-1}(z^{-1}) & -\kappa_{n-1}\displaystyle\int_{\mathbb{T}} \frac{\hat{p}_{n-1}(s^{-1})f(s)}{2\pi i s(s-z)}\mathrm{d}s \end{pmatrix} \tag{2.3}$$

solves the RH problem for Y. It was first noticed by Fokas, Its and Kitaev [33] that orthogonal polynomials can be characterized by RH problems (for a contour on the real line). The above RH problem for Y, whose jumps lie on the unit circle, was already considered in e.g. [3, eq. (1.26)] and [24, eq. (3.1)] for more specific f.

The monic orthogonal polynomials $\kappa_n^{-1}p_n, \kappa_n^{-1}\hat{p}_n$, and also Y, are unique (if they exist). The orthogonal polynomials exist if f is strictly positive almost everywhere on $\mathbb{T}$ (this is the case if W is real-valued, $\alpha_k > -1$ and $i\beta_k \in (-\frac{1}{2}, \frac{1}{2})$). More generally, a sufficient condition to ensure existence of $p_n, \hat{p}_n$ (and therefore of Y) is that $D_n^{(n)} \neq 0 \neq D_{n+1}^{(n)}$, where $D_l^{(n)} =: \det(f_{j-k})_{j,k=0,\ldots,l-1}$, $l \geqslant 1$ (note that $D_n^{(n)} = D_n(\vec{\alpha}, \vec{\beta}, V, W)$), see e.g. [21, Section 2.1]. In fact,

$$p_k(z) = \frac{\begin{vmatrix} f_0 & f_{-1} & \cdots & f_{-k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{k-1} & f_{k-2} & \cdots & f_{-1} \\ 1 & z & \cdots & z^k \end{vmatrix}}{\sqrt{D_k^{(n)}}\sqrt{D_{k+1}^{(n)}}}, \quad \hat{p}_k(z) = \frac{\begin{vmatrix} f_0 & f_{-1} & \cdots & f_{-k+1} & 1 \\ f_1 & f_0 & \cdots & f_{-k+2} & z \\ \vdots & \vdots & & \vdots & \vdots \\ f_k & f_{k-1} & \cdots & f_1 & z^k \end{vmatrix}}{\sqrt{D_k^{(n)}}\sqrt{D_{k+1}^{(n)}}}, \tag{2.4}$$

and $\kappa_k = (D_k^{(n)})^{1/2}/(D_{k+1}^{(n)})^{1/2}$. (Note that p_k, $\hat{p}_k$ and κ_k are unique only up to multiplicative factors of -1. This can be fixed with a choice of the branch for the above roots. However, since Y only involves $\kappa_n^{-1}p_n$ and $\kappa_{n-1}\hat{p}_{n-1}$, which are unique, this choice for the branch is unimportant for us.) If $D_k^{(n)} \neq 0$ for $k = 0, 1, \ldots, n+1$, it follows that

$$D_n(\vec{\alpha}, \vec{\beta}, V, W) = \prod_{j=0}^{n-1} \kappa_j^{-2}. \tag{2.5}$$

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LEMMA 2.1. *Let $n \in \mathbb{N}$ be fixed, and assume that $D_k^{(n)}(f) \neq 0$, $k = 0, 1, \ldots, n+1$. For any $z \neq 0$, we have*

$$[Y^{-1}(z)Y'(z)]_{21} z^{-n+1} = \sum_{k=0}^{n-1} \hat{p}_k(z^{-1}) p_k(z), \tag{2.6}$$

where $Y(\cdot) = Y_n(\cdot; \vec{\alpha}, \vec{\beta}, V, W)$.

Proof. The assumptions imply that $\kappa_k = (D_k^{(n)})^{1/2}/(D_{k+1}^{(n)})^{1/2}$ is finite and nonzero and that $p_k, \hat{p}_k$ exist for all $k \in \{0, \ldots, n\}$. Note that (a) $\det Y : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}$ is analytic, (b) $(\det Y)_+(z) = (\det Y)_-(z)$ for $z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}$, (c) $\det Y(z) = o(|z - t_k|^{-1})$ as $z \to t_k$ and (d) $\det Y(z) = 1 + o(1)$ as $z \to \infty$. Hence, using successively Morera's theorem, Riemann's removable singularities theorem and Liouville's theorem, we conclude that $\det Y \equiv 1$. Using (2.3) and the fact that $\det Y \equiv 1$, we obtain

$$\left[Y^{-1}(z)Y'(z)\right]_{21} = \frac{z^n}{\kappa_n} \cdot \frac{\kappa_{n-1}}{z} \hat{p}_{n-1}(z^{-1}) \frac{d}{dz} p_n(z) - \kappa_n^{-1} p_n(z) \frac{d}{dz}\left[z^n \cdot \frac{\kappa_{n-1}}{z} \hat{p}_{n-1}(z^{-1})\right].$$

Using the recurrence relation (see [**24**, Lemma 2.2])

$$\frac{\kappa_{n-1}}{z} \hat{p}_{n-1}(z^{-1}) = \kappa_n \hat{p}_n(z^{-1}) - \hat{p}_n(0) z^{-n} p_n(z),$$

we then find

$$\left[Y^{-1}(z)Y'(z)\right]_{21} = z^{n-1}\left(-n p_n(z)\hat{p}_n(z^{-1}) + z\left(\hat{p}_n(z^{-1})\frac{d}{dz}p_n(z) - p_n(z)\frac{d}{dz}\hat{p}_n(z^{-1})\right)\right).$$

The claim now directly follows from the Christoffel–Darboux formula [**24**, Lemma 2.3]. □

PROPOSITION 2.2. *Let $n \in \mathbb{N}_{\geqslant 1} := \{1, 2, \ldots\}$ be fixed and suppose that f depends smoothly on a parameter γ. If $D_k^{(n)}(f) \neq 0$ for $k = n-1, n, n+1$, then the following differential identity holds*

$$\partial_\gamma \log D_n(\vec{\alpha}, \vec{\beta}, V, W) = \frac{1}{2\pi} \int_0^{2\pi} [Y^{-1}(z)Y'(z)]_{21} z^{-n+1} \partial_\gamma f(z) \mathrm{d}\theta, \qquad z = e^{i\theta}. \tag{2.7}$$

REMARK 2.3. Identity (2.7) will be used (with a particular choice of γ) in the proof of proposition 4.4 to deform the potential, see (4.8).

Proof. We first prove the claim under the stronger assumption that $D_k^{(n)}(f) \neq 0$ for $k = 0, 1, \ldots, n+1$. In this case, $\kappa_k = (D_k^{(n)})^{1/2}/(D_{k+1}^{(n)})^{1/2}$ is finite and nonzero

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and $p_k, \hat{p}_k$ exist for all $k = 0, 1, \ldots, n$. Replacing z^{-j} with $\hat{p}_j(z^{-1})\kappa_j^{-1}$ in the first orthogonality condition in (2.2) (with $k = j$), and differentiating with respect to γ, we obtain, for $j = 0, \ldots, n-1$,

$$-\frac{\partial_\gamma[\kappa_j]}{\kappa_j} = \frac{\kappa_j}{2\pi}\partial_\gamma\left[\int_0^{2\pi} p_j(z)\hat{p}_j(z^{-1})\kappa_j^{-1}f(z)\mathrm{d}\theta\right]$$
$$= \frac{1}{2\pi}\int_0^{2\pi} p_j(z)\hat{p}_j(z^{-1})\partial_\gamma[f(z)]\mathrm{d}\theta + \frac{\kappa_j}{2\pi}\int_0^{2\pi}\partial_\gamma\left[p_j(z)\hat{p}_j(z^{-1})\kappa_j^{-1}\right]f(z)\mathrm{d}\theta. \tag{2.8}$$

The second term on the right-hand side can be simplified as follows:

$$\frac{\kappa_j}{2\pi}\int_0^{2\pi}\partial_\gamma\left[p_j(z)\hat{p}_j(z^{-1})\kappa_j^{-1}\right]f(z)\mathrm{d}\theta$$
$$= \frac{\kappa_j}{2\pi}\int_0^{2\pi}\partial_\gamma[p_j(z)]\hat{p}_j(z^{-1})\kappa_j^{-1}f(z)\mathrm{d}\theta = \frac{\partial_\gamma[\kappa_j]}{\kappa_j}, \tag{2.9}$$

where the first and second equalities use the first and second relations in (2.2), respectively. Combining (2.8) and (2.9), we find

$$-2\frac{\partial_\gamma[\kappa_j]}{\kappa_j} = \frac{1}{2\pi}\int_0^{2\pi} p_j(z)\hat{p}_j(z^{-1})\partial_\gamma[f(z)]\mathrm{d}\theta. \tag{2.10}$$

Taking the log of both sides of (2.5) and differentiating with respect to γ, we get

$$\partial_\gamma \log D_n(\vec{\alpha}, \vec{\beta}, V, W) = -2\sum_{j=0}^{n-1}\frac{\partial_\gamma[\kappa_j]}{\kappa_j} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{j=0}^{n-1} p_j(z)\hat{p}_j(z^{-1})\right)\partial_\gamma[f(z)]\mathrm{d}\theta. \tag{2.11}$$

An application of lemma 2.1 completes the proof under the assumption that $D_k^{(n)}(f) \neq 0$, $k = 0, 1, \ldots, n+1$. Since the existence of Y only relies on the weaker assumption $D_k^{(n)}(f) \neq 0$, $k = n-1, n, n+1$, the claim follows from a simple continuity argument. □

3. Steepest descent analysis

In this section, we use the Deift-Zhou [**28**] steepest descent method to obtain large n asymptotics for Y.

3.1. Equilibrium measure and g-function

The first step of the method is to normalize the RH problem at ∞ by means of a so-called g-function built in terms of the equilibrium measure (1.7). Recall from (1.3), (1.4) and (1.6) that U is an open annulus containing $\mathbb{T}$ in which V, W and ψ are analytic.

Define the function $g : \mathbb{C} \setminus ((-\infty,-1] \cup \mathbb{T}) \to \mathbb{C}$ by

$$g(z) = \int_{\mathbb{T}} \log_s(z-s)\psi(s)\frac{\mathrm{d}s}{is}, \tag{3.1}$$

where for $s = e^{i\theta} \in \mathbb{T}$ and $\theta \in [-\pi,\pi)$, the function $z \mapsto \log_s(z-s)$ is analytic in $\mathbb{C} \setminus ((-\infty,-1] \cup \{e^{i\theta'} : -\pi \leqslant \theta' \leqslant \theta\})$ and such that $\log_s(2) = \log|2|$.

Lemma 3.1. *The function g defined in (3.1) is analytic in $\mathbb{C} \setminus ((-\infty,-1] \cup \mathbb{T})$, satisfies $g(z) = \log z + \mathcal{O}(z^{-1})$ as $z \to \infty$ and possesses the following properties:*

$$g_+(z) + g_-(z) = 2\int_{\mathbb{T}} \log|z-s|\psi(s)\frac{\mathrm{d}s}{is} + i(\pi + \hat{c} + \arg z), \quad z \in \mathbb{T}, \tag{3.2}$$

$$g_+(z) - g_-(z) = 2\pi i \int_{\arg z}^{\pi} \psi(e^{i\theta})\mathrm{d}\theta, \quad z \in \mathbb{T}, \tag{3.3}$$

$$g_+(z) - g_-(z) = 2\pi i, \quad z \in (-\infty,-1), \tag{3.4}$$

where $\hat{c} = \int_{-\pi}^{\pi} \theta\psi(e^{i\theta})\mathrm{d}\theta$ and $\arg z \in (-\pi,\pi)$.

Proof. In the case where the equilibrium measure satisfies the symmetry $\psi(e^{i\theta}) = \psi(e^{-i\theta})$, we have $\hat{c} = 0$ and in this case (3.2)–(3.4) follow from [**3**, Lemma 4.2]. In the more general setting of a non-symmetric equilibrium measure, (3.2)–(3.4) can be proved along the same lines as [**3**, proof of Lemma 4.2] (the main difference is that $F(\pi) = \pi$ in [**3**, proof of Lemma 4.2] should here be replaced by $F(\pi) = \pi + \hat{c}$). □

It follows from (3.3) that

$$g'_+(z) - g'_-(z) = -\frac{2\pi}{z}\psi(z), \quad z \in \mathbb{T}. \tag{3.5}$$

Substituting (3.2) into the Euler–Lagrange equality (1.23) and recalling that $\mathrm{d}\mu_V(s) = \psi(s)\frac{\mathrm{d}s}{is}$, we get

$$V(z) = g_+(z) + g_-(z) + \ell - \log z - i(\pi + \hat{c}), \qquad z \in \mathbb{T}, \tag{3.6}$$

where the principal branch is taken for the logarithm. Consider the function

$$\xi(z) = \begin{cases} -i\pi \displaystyle\int_{-1}^{z} \psi(s)\frac{\mathrm{d}s}{is}, & \text{if } |z| < 1,\ z \in U, \\ i\pi \displaystyle\int_{-1}^{z} \psi(s)\frac{\mathrm{d}s}{is}, & \text{if } |z| > 1,\ z \in U, \end{cases} \tag{3.7}$$

where the contour of integration (except for the starting point -1) lies in $U \setminus ((-\infty,0] \cup \mathbb{T})$ and the first part of the contour lies in $\{z : \operatorname{Im} z \geqslant 0\}$. Since ψ is real-valued on $\mathbb{T}$, we have $\operatorname{Re}\xi(z) = 0$ for $z \in \mathbb{T}$. Using the Cauchy–Riemann equations in polar coordinates and the compactness of the unit circle, we verify that there exists an open annulus $U' \subseteq U$ containing $\mathbb{T}$ such that $\operatorname{Re}\xi(z) > 0$ for $z \in U' \setminus \mathbb{T}$.

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Redefining U if necessary, we can (and do) assume that $U' = U$. Furthermore, for $z = e^{i\theta} \in \mathbb{T}$, $\theta \in (-\pi, \pi)$, we have

$$\xi_+(z) - \xi_-(z) = 2\xi_+(z) = -2\pi i \int_{-1}^{z} \psi(s) \frac{\mathrm{d}s}{is} = 2\pi i \int_{\theta}^{\pi} \psi(e^{i\theta'})\mathrm{d}\theta' = g_+(z) - g_-(z), \tag{3.8}$$

$$2\xi_\pm(z) - 2g_\pm(z) = \ell - V(z) - \log z - i\pi - i\hat{c}. \tag{3.9}$$

Analytically continuing $\xi(z) - g(z)$ in (3.9), we obtain

$$\xi(z) = g(z) + \frac{1}{2}\left(\ell - V(z) - \log z - i\pi - i\hat{c}\right), \quad \text{for all } z \in U \setminus ((-\infty, 0] \cup \mathbb{T}). \tag{3.10}$$

Note also that

$$\xi_+(x) - \xi_-(x) = \pi i, \qquad x \in U \cap (-\infty, -1), \tag{3.11}$$

$$\xi_+(x) - \xi_-(x) = -\pi i, \qquad x \in U \cap (-1, 0), \tag{3.12}$$

where $\xi_\pm(x) := \lim_{\epsilon \to 0^+} \xi(z \pm i\epsilon)$ for $x \in U \cap ((-\infty, -1) \cup (-1, 0))$.

3.2. Transformations $Y \to T \to S$

The first transformation $Y \to T$ is defined by

$$T(z) = e^{-\frac{n(\pi+\hat{c})i}{2}\sigma_3} e^{\frac{n\ell}{2}\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{-\frac{n\ell}{2}\sigma_3} e^{\frac{n(\pi+\hat{c})i}{2}\sigma_3}. \tag{3.13}$$

For $z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}$, the function T satisfies the jump relation $T_+ = T_- J_T$ where the jump matrix J_T is given by

$$J_T(z) = \begin{pmatrix} e^{-n(g_+(z) - g_-(z))} & z^{-n} e^{-n[V(z) - g_+(z) - g_-(z) - \ell + i\pi + i\hat{c}]} e^{W(z)} \omega(z) \\ 0 & e^{n(g_+(z) - g_-(z))} \end{pmatrix}.$$

Combining the above with (3.4), (3.6) and (3.8), we conclude that T satisfies the following RH problem.

RH problem for T

(a) $T : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The boundary values T_+ and T_- are continuous on $\mathbb{T} \setminus \{t_0, \ldots, t_m\}$ and are related by

$$T_+(z) = T_-(z) \begin{pmatrix} e^{-2n\xi_+(z)} & e^{W(z)}\omega(z) \\ 0 & e^{-2n\xi_-(z)} \end{pmatrix}, \qquad z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}.$$

(c) As $z \to \infty$, $T(z) = I + \mathcal{O}(z^{-1})$.

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(d) As $z \to t_k$, $k = 0, \dots, m$, $z \in \mathbb{C} \setminus \mathbb{T}$,

$$T(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z-t_k|^{\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z-t_k|^{\alpha_k}) \end{pmatrix}, & \text{if } \operatorname{Re}\alpha_k \neq 0, \\ \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z-t_k|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z-t_k|) \end{pmatrix}, & \text{if } \operatorname{Re}\alpha_k = 0. \end{cases}$$

The jumps of T for $z \in \mathbb{T} \setminus \{t_0, \dots, t_m\}$ can be factorized as

$$\begin{pmatrix} e^{-2n\xi_+(z)} & e^{W(z)}\omega(z) \\ 0 & e^{-2n\xi_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi_-(z)} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & e^{W(z)}\omega(z) - e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi_+(z)} & 1 \end{pmatrix}.$$

Before proceeding to the second transformation, we first describe the analytic continuations of the functions appearing in the above factorization. The functions ω_{β_k}, $k = 0, \dots, m$, have a straightforward analytic continuation from $\mathbb{T} \setminus \{t_k\}$ to $\mathbb{C} \setminus \{\lambda t_k : \lambda \geqslant 0\}$, which is given by

$$\omega_{\beta_k}(z) = z^{\beta_k} t_k^{-\beta_k} \times \begin{cases} e^{i\pi\beta_k}, & 0 \leqslant \arg_0 z < \theta_k, \\ e^{-i\pi\beta_k}, & \theta_k \leqslant \arg_0 z < 2\pi, \end{cases} \quad z \in \mathbb{C} \setminus \{\lambda t_k : \lambda \geqslant 0\}, \quad k = 0, \dots, m, \tag{3.14}$$

where $\arg_0 z \in [0, 2\pi)$, $t_k^{-\beta_k} := e^{-i\beta_k\theta_k}$ and $z^{\beta_k} := |z|^{\beta_k} e^{i\beta_k \arg_0 z}$. For the root-type singularities, we follow [**24**] and analytically continue ω_{α_k} from $\mathbb{T} \setminus \{t_k\}$ to $\mathbb{C} \setminus \{\lambda t_k : \lambda \geqslant 0\}$ as follows

$$\omega_{\alpha_k}(z) = \frac{(z-t_k)^{\alpha_k}}{(zt_k e^{i\ell_k(z)})^{\alpha_k/2}} := \frac{e^{\alpha_k(\log|z-t_k| + i\hat{\arg}_k(z-t_k))}}{e^{\frac{\alpha_k}{2}(\log|z| + i\arg_0(z) + i\theta_k + i\ell_k(z))}}, \quad z \in \mathbb{C} \setminus \{\lambda t_k : \lambda \geqslant 0\}, \quad k = 0, \dots, m,$$

where $\hat{\arg}_k z \in (\theta_k, \theta_k + 2\pi)$, and

$$\ell_k(z) = \begin{cases} 3\pi, & 0 \leqslant \arg_0 z < \theta_k, \\ \pi, & \theta_k \leqslant \arg_0 z < 2\pi. \end{cases}$$

Now, we open lenses around $\mathbb{T} \setminus \{t_0, \dots, t_m\}$ as shown in figure 1. The part of the lens-shaped contour lying in $\{|z| < 1\}$ is denoted γ_+, and the part lying in $\{|z| > 1\}$ is denoted γ_-. We require that $\gamma_+, \gamma_- \subset U$. The transformation $T \mapsto S$ is defined

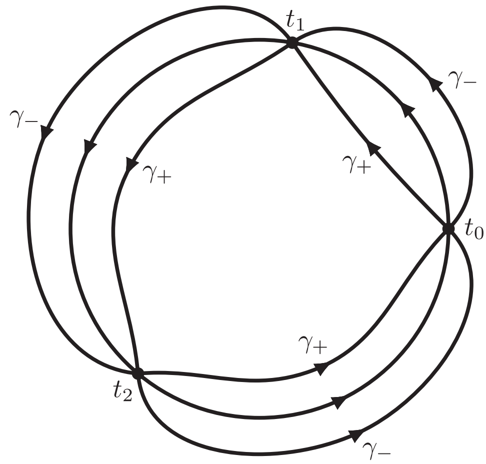

Figure 1. The jump contour for S with $m = 2$.

by

$$S(z) = T(z) \times \begin{cases} I, & \text{if } z \text{ is outside the lenses,} \\ \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } |z| > 1 \text{ and inside the lenses,} \\ \begin{pmatrix} 1 & 0 - e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, & \text{if } |z| < 1 \text{ and inside the lenses.} \end{cases} \tag{3.15}$$

Note from (3.11)–(3.12) that $e^{-2n\xi(z)}$ is analytic in $U \cap ((-\infty, -1) \cup (-1, 0))$. It can be verified using the RH problem for T and (3.15) that S satisfies the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus (\gamma_+ \cup \gamma_- \cup \mathbb{T}) \to \mathbb{C}^{2\times 2}$ is analytic, where γ_+, γ_- are the contours in figure 1 lying inside and outside $\mathbb{T}$, respectively.

(b) The jumps for S are as follows.

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & e^{W(z)}\omega(z) - e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \quad z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\},$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix}, \quad z \in \gamma_+ \cup \gamma_-.$$

(c) As $z \to \infty$, $S(z) = I + \mathcal{O}(z^{-1})$.

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(d) As $z \to t_k$, $k = 0, \dots, m$, we have

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z-t_k)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z-t_k)) \end{pmatrix}, & \text{if } z \text{ is outside the lenses,} \\ \begin{pmatrix} \mathcal{O}(\log(z-t_k)) & \mathcal{O}(\log(z-t_k)) \\ \mathcal{O}(\log(z-t_k)) & \mathcal{O}(\log(z-t_k)) \end{pmatrix}, & \text{if } z \text{ is inside the lenses,} \end{cases}$$

$$\text{if } \operatorname{Re}\alpha_k = 0,$$

$$S(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, & \text{if } z \text{ is outside the lenses,} \\ \begin{pmatrix} \mathcal{O}((z-t_k)^{-\alpha_k}) & \mathcal{O}(1) \\ \mathcal{O}((z-t_k)^{-\alpha_k}) & \mathcal{O}(1) \end{pmatrix}, & \text{if } z \text{ is inside the lenses,} \end{cases}$$

$$\text{if } \operatorname{Re}\alpha_k > 0,$$

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}((z-t_k)^{\alpha_k}) \\ \mathcal{O}(1) & \mathcal{O}((z-t_k)^{\alpha_k}) \end{pmatrix},$$

$$\text{if } \operatorname{Re}\alpha_k < 0.$$

Since $\gamma_+, \gamma_- \subset U$ and $\operatorname{Re}\xi(z) > 0$ for $z \in U \setminus \mathbb{T}$ (recall the discussion below (3.7)), the jump matrices $S_-(z)^{-1}S_+(z)$ on $\gamma_+ \cup \gamma_-$ are exponentially close to I as $n \to +\infty$, and this convergence is uniform outside fixed neighbourhoods of $t_0, \dots, t_m$.

Our next task is to find suitable approximations (called 'parametrices') for S in different regions of the complex plane.

3.3. Global parametrix $P^{(\infty)}$

In this subsection, we will construct a global parametrix $P^{(\infty)}$ that is defined as the solution to the following RH problem. We will show in subsection 3.5 below that $P^{(\infty)}$ is a good approximation of S outside fixed neighbourhoods of $t_0, \dots, t_m$.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The jumps are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & e^{W(z)}\omega(z) - e^{-W(z)}\omega(z)^{-1} & 0 \end{pmatrix}, \qquad z \in \mathbb{T} \setminus \{t_0, \dots, t_m\}. \tag{3.16}$$

(c) As $z \to \infty$, we have $P^{(\infty)}(z) = I + \mathcal{O}(z^{-1})$.

(d) As $z \to t_k$ from $|z| \lessgtr 1$, $k \in \{0, \dots, m\}$, we have $P^{(\infty)}(z) = \mathcal{O}(1)(z - t_k)^{-(\frac{\alpha_k}{2} \pm \beta_k)\sigma_3}$.

The unique solution to the above RH problem is given by

$$P^{(\infty)}(z) = \begin{cases} D(z)^{\sigma_3} \begin{pmatrix} 0 & 1-1 & 0 \end{pmatrix}, & \text{if } |z| < 1, \\ D(z)^{\sigma_3}, & \text{if } |z| > 1, \end{cases} \tag{3.17}$$

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where $D(z)$ is the Szegő function defined by

$$D(z) = D_W(z) \prod_{k=0}^{m} D_{\alpha_k}(z) D_{\beta_k}(z), \qquad D_W(z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{W(s)}{s-z}\mathrm{d}s\right), \tag{3.18}$$

$$D_{\alpha_k}(z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \omega_{\alpha_k}(s)}{s-z}\mathrm{d}s\right), \quad D_{\beta_k}(z) = \exp\left(\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \omega_{\beta_k}(s)}{s-z}\mathrm{d}s\right). \tag{3.19}$$

The branches of the logarithms in (3.19) can be arbitrarily chosen as long as $\log \omega_{\alpha_k}(s)$ and $\log \omega_{\beta_k}(s)$ are continuous on $\mathbb{T} \setminus t_k$. The function D is analytic on $\mathbb{C} \setminus \mathbb{T}$ and satisfies the jump condition $D_+(z) = D_-(z) e^{W(z)} \omega(z)$ on $\mathbb{T} \setminus \{t_0, \ldots, t_m\}$. The expressions for D_{α_k} and D_{β_k} can be simplified as in [**24**, eqs. (4.9)–(4.10)]; we have

$$D_{\alpha_k}(z) D_{\beta_k}(z) = \begin{cases} \left(\dfrac{z-t_k}{t_k e^{i\pi}}\right)^{\frac{\alpha_k}{2}+\beta_k} = \dfrac{e^{(\frac{\alpha_k}{2}+\beta_k)(\log|z-t_k| + i\hat{\arg}_k(z-t_k))}}{e^{(\frac{\alpha_k}{2}+\beta_k)(i\theta_k + i\pi)}}, & \text{if } |z|<1, \\ \left(\dfrac{z-t_k}{z}\right)^{-\frac{\alpha_k}{2}+\beta_k} = \dfrac{e^{(\beta_k-\frac{\alpha_k}{2})(\log|z-t_k| + i\hat{\arg}_k(z-t_k))}}{e^{(\beta_k-\frac{\alpha_k}{2})(\log|z| + i\hat{\arg}_k z)}}, & \text{if } |z|>1, \end{cases} \tag{3.20}$$

where $\hat{\arg}_k$ was defined below (3.14). Using (1.4), we can also simplify D_W as

$$D_W(z) = \begin{cases} e^{W_0 + W_+(z)}, & |z|<1, \\ e^{-W_-(z)}, & |z|>1. \end{cases} \tag{3.21}$$

3.4. Local parametrices $P^{(t_k)}$

In this subsection, we build parametrices $P^{(t_k)}(z)$ in small open disks $\mathcal{D}_{t_k}$ of t_k, $k = 0, \ldots, m$. The disks $\mathcal{D}_{t_k}$ are taken sufficiently small such that $\mathcal{D}_{t_k} \subset U$ and $\mathcal{D}_{t_k} \cap \mathcal{D}_{t_j} = \emptyset$ for $j \neq k$. Since we assume that the t_k's remain bounded away from each other, we can (and do) choose the radii of the disks to be fixed. The parametrices $P^{(t_k)}(z)$ are defined as the solution to the following RH problem. We will show in subsection 3.5 below that $P^{(t_k)}$ is a good approximation for S in $\mathcal{D}_{t_k}$.

RH problem for $P^{(t_k)}$

(a) $P^{(t_k)} : \mathcal{D}_{t_k} \setminus (\mathbb{T} \cup \gamma_+ \cup \gamma_-) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $z \in (\mathbb{T} \cup \gamma_+ \cup \gamma_-) \cap \mathcal{D}_{t_k}$, $P_-^{(t_k)}(z)^{-1} P_+^{(t_k)}(z) = S_-(z)^{-1} S_+(z)$.

(c) As $n \to +\infty$, $P^{(t_k)}(z) = (I + \mathcal{O}(n^{-1+2|\mathrm{Re}\,\beta_k|}))P^{(\infty)}(z)$ uniformly for $z \in \partial\mathcal{D}_{t_k}$.

(d) As $z \to t_k$, $S(z) P^{(t_k)}(z)^{-1} = \mathcal{O}(1)$.

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A solution to the above RH problem can be constructed using hypergeometric functions as in [**24**, **35**]. Consider the function

$$f_{t_k}(z) := 2\pi i \int_{t_k}^{z} \psi(s) \frac{\mathrm{d}s}{is}, \qquad z \in \mathcal{D}_{t_k},$$

where the path is a straight line segment from t_k to z. This is a conformal map from $\mathcal{D}_{t_k}$ to a neighbourhood of 0, which satisfies

$$f_{t_k}(z) = 2\pi t_k^{-1} \psi(t_k)(z - t_k)\left(1 + \mathcal{O}(z - t_k)\right), \qquad \text{as } z \to t_k. \tag{3.22}$$

If $\mathcal{D}_{t_k} \cap (-\infty, 0] = \emptyset$, f_{t_k} can also be expressed as

$$f_{t_k}(z) = -2 \times \begin{cases} \xi(z) - \xi_+(t_k), & |z| < 1, \\ -(\xi(z) - \xi_-(t_k)), & |z| > 1. \end{cases}$$

If $\mathcal{D}_{t_k} \cap (-\infty, 0] \neq \emptyset$, then instead we have

$$f_{t_k}(z) = -2 \times \begin{cases} \xi(z) - \xi_+(t_k), & |z| < 1, \ \mathrm{Im}\, z > 0, \\ \xi(z) - \xi_+(t_k) - \pi i, & |z| < 1, \ \mathrm{Im}\, z < 0, \\ -(\xi(z) - \xi_-(t_k)), & |z| > 1, \ \mathrm{Im}\, z > 0, \\ -(\xi(z) - \xi_-(t_k) + \pi i), & |z| > 1, \ \mathrm{Im}\, z < 0, \end{cases} \quad \text{if } \mathrm{Im}\, t_k > 0, \tag{3.23}$$

$$f_{t_k}(z) = -2 \times \begin{cases} \xi(z) - \xi_+(t_k) + \pi i, & |z| < 1, \ \mathrm{Im}\, z > 0, \\ \xi(z) - \xi_+(t_k), & |z| < 1, \ \mathrm{Im}\, z < 0, \\ -(\xi(z) - \xi_-(t_k) - \pi i), & |z| > 1, \ \mathrm{Im}\, z > 0, \\ -(\xi(z) - \xi_-(t_k)), & |z| > 1, \ \mathrm{Im}\, z < 0, \end{cases} \quad \text{if } \mathrm{Im}\, t_k < 0.$$

If $t_k = -1$, (3.23) also holds with $\xi_\pm(t_k) := \lim_{\epsilon \to 0^+} \xi_\pm(e^{(\pi - \epsilon)i}) = 0$. We define ω_k and $\widetilde{W}_k$ by

$$\omega_k(z) = e^{-2\pi i \beta_k \hat{\theta}(z;k)} z^{\beta_k} t_k^{-\beta_k} \prod_{j \neq k} \omega_{\alpha_j}(z) \omega_{\beta_j}(z),$$

$$\widetilde{W}_k(z) = \breve{\omega}_{\alpha_k}(z)^{\frac{1}{2}} \times \begin{cases} e^{-\frac{i\pi\alpha_k}{2}}, & z \in Q_{+,k}^R \cup Q_{-,k}^L, \\ e^{\frac{i\pi\alpha_k}{2}}, & z \in Q_{-,k}^R \cup Q_{+,k}^L, \end{cases}$$

where $\hat{\theta}(z;k) = 1$ if $\mathrm{Im}\, z < 0$ and $k = 0$ and $\hat{\theta}(z;k) = 0$ otherwise, $z^{\beta_k} := |z|^{\beta_k} e^{i\beta_k \arg_0 z}$,

$$\breve{\omega}_{\alpha_k}(z)^{1/2} := \frac{(z - t_k)^{\frac{\alpha_k}{2}}}{(z t_k e^{i\ell_k(z)})^{\alpha_k/4}} := \frac{e^{\frac{\alpha_k}{2}(\log|z - t_k| + i\breve{\arg}_k(z - t_k))}}{e^{\frac{\alpha_k}{4}(\log|z| + i\arg_0(z) + i\theta_k + i\ell_k(z))}}, \tag{3.24}$$

and (see figure 2)

$$Q_{\pm,k}^R = \{z \in \mathcal{D}_{t_k} : \mp \mathrm{Re}\, f_{t_k}(z) > 0, \ \mathrm{Im}\, f_{t_k}(z) > 0\},$$

$$Q_{\pm,k}^L = \{z \in \mathcal{D}_{t_k} : \mp \mathrm{Re}\, f_{t_k}(z) > 0, \ \mathrm{Im}\, f_{t_k}(z) < 0\}.$$

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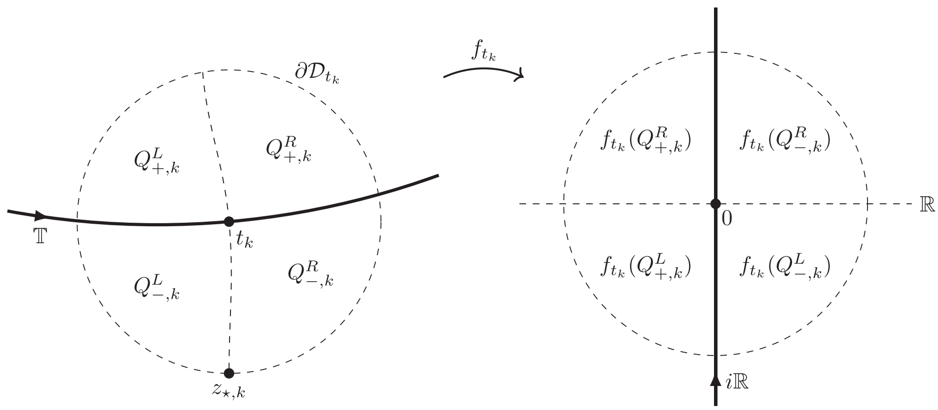

Figure 2. The four quadrants $Q^R_{\pm,k}$, $Q^L_{\pm,k}$ near t_k and their images under the map f_{t_k}.

The argument $\check{\arg}_k(z-t_k)$ in (3.24) is defined to have a discontinuity for $z \in (\overline{Q^L_{-,k}} \cap \overline{Q^R_{-,k}}) \cup [z_{\star,k}, t_k\infty)$, $z_{\star,k} := \overline{Q^L_{-,k}} \cap \overline{Q^R_{-,k}} \cap \partial\mathcal{D}_{t_k}$, and such that $\check{\arg}_k((1-0_+)t_k - t_k) = \theta_k + \pi$. Note that $\check{\arg}_k(z-t_k)$ is merely a small deformation of the argument $\hat{\arg}_k(z-t_k)$ defined below (3.14). This small deformation is needed to ensure that E_{t_k} in (3.26) below is analytic in $\mathcal{D}_{t_k}$.

Note that ω_k is analytic in $\mathcal{D}_{t_k}$. We now use the confluent hypergeometric model RH problem, whose solution is denoted $\Phi_{\mathrm{HG}}(z;\alpha_k,\beta_k)$ (see appendix B for the definition and properties of Φ_{HG}). If $k \neq 0$ and $\mathcal{D}_{t_k} \cap (-\infty, 0] = \emptyset$, we define

$$P^{(t_k)}(z) = E_{t_k}(z)\Phi_{\mathrm{HG}}(nf_{t_k}(z);\alpha_k,\beta_k)\widetilde{W}_k(z)^{-\sigma_3}e^{-n\xi(z)\sigma_3}e^{-\frac{W(z)}{2}\sigma_3}\omega_k(z)^{-\frac{\sigma_3}{2}}, \tag{3.25}$$

where E_{t_k} is given by

$$\begin{aligned} E_{t_k}(z) &= P^{(\infty)}(z)\omega_k(z)^{\frac{\sigma_3}{2}}e^{\frac{W(z)}{2}\sigma_3}\widetilde{W}_k(z)^{\sigma_3} \\ &\times \begin{Bmatrix} e^{\dfrac{i\pi\alpha_k}{4}\sigma_3}e^{-i\pi\beta_k\sigma_3}, & z \in Q^R_{+,k} \\ e^{-\dfrac{i\pi\alpha_k}{4}\sigma_3}e^{-i\pi\beta_k\sigma_3}, & z \in Q^L_{+,k} \\ e^{\dfrac{i\pi\alpha_k}{4}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q^L_{-,k} \\ e^{-\dfrac{i\pi\alpha_k}{4}\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in Q^R_{-,k} \end{Bmatrix} \quad e^{n\xi_+(t_k)\sigma_3}(nf_{t_k}(z))^{\beta_k\sigma_3}. \end{aligned} \tag{3.26}$$

Here the branch of $f_{t_k}(z)^{\beta_k}$ is such that $f_{t_k}(z)^{\beta_k} = |f_{t_k}(z)|^{\beta_k}e^{\beta_k i \arg f_{t_k}(z)}$ with $\arg f_{t_k}(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2})$, and the branch for the square root of $\omega_k(z)$ can be chosen arbitrarily as long as $\omega_k(z)^{1/2}$ is analytic in $\mathcal{D}_{t_k}$ (note that $P^{(t_k)}(z)$ is invariant under a sign change of $\omega_k(z)^{1/2}$). If $k \neq 0$, $\mathcal{D}_{t_k} \cap (-\infty, 0] \neq \emptyset$ and $\operatorname{Im} t_k \geqslant 0$ (resp. $\operatorname{Im} t_k < 0$), then we define $P^{(t_k)}(z)$ as in (3.25) but with $\xi(z)$ replaced by

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$\xi(z)+\pi i\theta_-(z)$ (resp. $\xi(z)+\pi i\theta_+(z)$), where

$$\theta_-(z) := \begin{cases} 1, & \text{if } \operatorname{Im} z<0,\ |z|>1, \\ -1, & \text{if } \operatorname{Im} z<0,\ |z|<1, \\ 0, & \text{otherwise}, \end{cases} \quad \theta_+(z) := \begin{cases} -1, & \text{if } \operatorname{Im} z>0,\ |z|>1, \\ 1, & \text{if } \operatorname{Im} z>0,\ |z|<1, \\ 0, & \text{otherwise}. \end{cases}$$

Using the definition of $\widetilde{W}_k$ and the jumps (3.16) of $P^{(\infty)}$, we verify that E_{t_k} has no jumps in $\mathcal{D}_{t_k}$. Moreover, since $P^{(\infty)}(z)=\mathcal{O}(1)(z-t_k)^{-(\frac{\alpha_k}{2}\pm\beta_k)\sigma_3}$ as $z\to t_k$, $\pm(1-|z|)>0$, we infer from (3.26) that $E_{t_k}(z)=\mathcal{O}(1)$ as $z\to t_k$, and therefore E_{t_k} is analytic in $\mathcal{D}_{t_k}$. Using (3.26), we see that $E_{t_k}(z)=\mathcal{O}(1)n^{\beta_k\sigma_3}$ as $n\to\infty$, uniformly for $z\in\mathcal{D}_{t_k}$. Since $P^{(t_k)}$ and S have the same jumps on $(\mathbb{T}\cup\gamma_+\cup\gamma_-)\cap\mathcal{D}_{t_k}$, $S(z)P^{(t_k)}(z)^{-1}$ is analytic in $\mathcal{D}_{t_k}\setminus\{t_k\}$. Furthermore, by (B.5) and condition (d) in the RH problem for S, as $z\to t_k$ from outside the lenses we have that $S(z)P^{(t_k)}(z)^{-1}$ is $\mathcal{O}(\log(z-t_k))$ if $\operatorname{Re}\alpha_k=0$, is $\mathcal{O}(1)$ if $\operatorname{Re}\alpha_k>0$, and is $\mathcal{O}((z-t_k)^{\alpha_k})$ if $\operatorname{Re}\alpha_k<0$. In all cases, the singularity of $S(z)P^{(t_k)}(z)^{-1}$ at $z=t_k$ is removable and therefore $P^{(t_k)}$ in (3.25) satisfies condition (d) of the RH problem for $P^{(t_k)}$.

The value of $E_{t_k}(t_k)$ can be obtained by taking the limit $z\to t_k$ in (3.26) (e.g. from the quadrant $Q^R_{+,k}$). Using (3.17), (3.18), (3.20), (3.22) and (3.26), we obtain

$$E_{t_k}(t_k)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Lambda_k^{\sigma_3}, \tag{3.27}$$

where

$$\Lambda_k = e^{\frac{W(t_k)}{2}}D_{W,+}(t_k)^{-1} \times\left[\prod_{j\neq k}D_{\alpha_j,+}(t_k)^{-1}D_{\beta_j,+}(t_k)^{-1}\omega_{\alpha_j}^{\frac{1}{2}}(t_k)\omega_{\beta_j}^{\frac{1}{2}}(t_k)\right](2\pi\psi(t_k)n)^{\beta_k}e^{n\xi_+(t_k)}. \tag{3.28}$$

In (3.28), the branch of $\omega_{\alpha_j}^{\frac{1}{2}}(t_k)$ is as in (3.24) and $\omega_{\beta_j}^{\frac{1}{2}}(t_k)$ is defined by

$$\omega_{\beta_j}^{\frac{1}{2}}(t_k) := e^{i\frac{\beta_j}{2}(\theta_k-\theta_j)}\times\begin{cases} e^{\frac{i\pi}{2}\beta_j}, & \text{if } 0\leqslant\theta_k<\theta_j, \\ e^{-\frac{i\pi}{2}\beta_j}, & \text{if } \theta_j\leqslant\theta_k<2\pi. \end{cases}$$

The expression for Λ_k can be further simplified as follows. A simple computation shows that

$$\begin{aligned} D_{\alpha_j,+}(z) &= |z-t_j|^{\frac{\alpha_j}{2}}\exp\left(\frac{i\alpha_j}{2}\left[\hat{\arg}_j(z-t_j)-\theta_j-\pi\right]\right) \\ &= |z-t_j|^{\frac{\alpha_j}{2}}\exp\left(\frac{i\alpha_j}{2}\left[\frac{\ell_j(z)}{2}+\frac{\arg_0 z-\theta_j}{2}-\pi\right]\right), \\ D_{\beta_j,+}(z) &= |z-t_j|^{\beta_j}\exp\left(i\beta_j\left[\hat{\arg}_j(z-t_j)-\theta_j-\pi\right]\right) \\ &= |z-t_j|^{\beta_j}\exp\left(i\beta_j\left[\frac{\ell_j(z)}{2}+\frac{\arg_0 z-\theta_j}{2}-\pi\right]\right) \end{aligned}$$

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for $z \in \mathbb{T}$. Therefore, the product in brackets in (3.28) can be rewritten as

$$\prod_{j\neq k} |t_k - t_j|^{-\beta_j} \exp\left(-\frac{i\alpha_j}{2}\frac{\theta_k-\theta_j}{2}\right)\prod_{j=0}^{k-1}\exp\left(\frac{\pi i\alpha_j}{4}\right)\prod_{j=k+1}^{m}\exp\left(-\frac{\pi i\alpha_j}{4}\right),$$

and thus

$$\Lambda_k = e^{\frac{W(t_k)}{2}} D_{W,+}(t_k)^{-1} e^{\frac{i\lambda_k}{2}} (2\pi\psi(t_k)n)^{\beta_k} \prod_{j\neq k} |t_k - t_j|^{-\beta_j},$$

where

$$\lambda_k = \sum_{j=0}^{k-1}\frac{\pi\alpha_j}{2} - \sum_{j=k+1}^{m}\frac{\pi\alpha_j}{2} - \sum_{j\neq k}\frac{\alpha_j(\theta_k-\theta_j)}{2} + 2\pi n\int_{t_k}^{-1}\psi(s)\frac{\mathrm{d}s}{is}. \tag{3.29}$$

Using (3.25) and (B.2), we obtain

$$\begin{aligned} &P^{(t_k)}(z)P^{(\infty)}(z)^{-1} \\ &\quad = I + \frac{\beta_k^2 - \frac{\alpha_k^2}{4}}{nf_{t_k}(z)} E_{t_k}(z)\begin{pmatrix} -1 & \tau(\alpha_k,\beta_k) - \tau(\alpha_k,-\beta_k) & 1 \end{pmatrix} E_{t_k}(z)^{-1} \\ &\qquad + \mathcal{O}(n^{-2+2|\operatorname{Re}\beta_k|}), \end{aligned} \tag{3.30}$$

as $n \to \infty$ uniformly for $z \in \partial\mathcal{D}_{t_k}$, where $\tau(\alpha_k, \beta_k)$ is defined in (B.3).

3.5. Small norm RH problem

We consider the function R defined by

$$R(z) = \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C}\setminus(\cup_{k=0}^m \overline{\mathcal{D}_{t_k}} \cup \mathbb{T} \cup \gamma_+ \cup \gamma_-), \\ S(z)P^{(t_k)}(z)^{-1}, & z \in \mathcal{D}_{t_k} \setminus (\mathbb{T}\cup\gamma_+\cup\gamma_-),\ k = 0,\ldots,m. \end{cases} \tag{3.31}$$

We have shown in the previous section that $P^{(t_k)}$ and S have the same jumps on $\mathbb{T}\cup\gamma_+\cup\gamma_-$ and that $S(z)P^{(t_k)}(z)^{-1} = \mathcal{O}(1)$ as $z \to t_k$. Hence, R is analytic in $\cup_{k=0}^m \mathcal{D}_{t_k}$. Using also the RH problems for S, $P^{(\infty)}$ and $P^{(t_k)}$, we conclude that R satisfies the following RH problem.

RH problem for R

(a) $R : \mathbb{C}\setminus\Gamma_R \to \mathbb{C}^{2\times 2}$ is analytic, where $\Gamma_R = \cup_{k=0}^m \partial\mathcal{D}_{t_k} \cup ((\gamma_+\cup\gamma_-)\setminus \cup_{k=0}^m \mathcal{D}_{t_k})$ and the circles $\partial\mathcal{D}_{t_k}$ are oriented in the clockwise direction.

(b) The jumps are given by

$$\begin{aligned} R_+(z) &= R_-(z)P^{(\infty)}(z) \\ &\quad\times\begin{pmatrix} 1 & 0 \\ e^{-W(z)}\omega(z)^{-1}e^{-2n\xi(z)} & 1 \end{pmatrix} P^{(\infty)}(z)^{-1}, \quad z \in (\gamma_+\cup\gamma_-)\setminus \cup_{k=0}^m \overline{\mathcal{D}_{t_k}}, \\ R_+(z) &= R_-(z)P^{(t_k)}(z)P^{(\infty)}(z)^{-1}, \qquad z \in \partial\mathcal{D}_{t_k},\ k = 0,\ldots,m. \end{aligned}$$

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(c) As $z \to \infty$, $R(z) = I + \mathcal{O}(z^{-1})$.

(d) As $z \to z^* \in \Gamma_R^*$, where Γ_R^* is the set of self-intersecting points of Γ_R, we have $R(z) = \mathcal{O}(1)$.

Recall that $\operatorname{Re}\xi(z) \geqslant c > 0$ for $z \in (\gamma_+ \cup \gamma_-) \setminus \cup_{k=0}^m \mathcal{D}_{t_k}$. Moreover, we see from (3.17) that $P^{(\infty)}(z)$ is bounded for z away from the points $t_0, \ldots, t_m$. Using also (3.30), we conclude that as $n \to +\infty$

$$J_R(z) = I + \mathcal{O}(e^{-cn}), \quad \text{uniformly for } z \in (\gamma_+ \cup \gamma_-) \setminus \cup_{k=0}^m \overline{\mathcal{D}_{t_k}}, \tag{3.32}$$

$$J_R(z) = I + J_R^{(1)}(z) n^{-1} + \mathcal{O}(n^{-2+2\beta_{\max}}), \quad \text{uniformly for } z \in \cup_{k=0}^m \partial\mathcal{D}_{t_k}, \tag{3.33}$$

where $J_R(z) := R_-^{-1}(z) R_+(z)$ and

$$J_R^{(1)}(z) = \frac{\beta_k^2 - \frac{\alpha_k^2}{4}}{f_{t_k}(z)} E_{t_k}(z) \begin{pmatrix} -1 & \tau(\alpha_k,\beta_k) - \tau(\alpha_k,-\beta_k) & 1 \end{pmatrix} E_{t_k}(z)^{-1}, \quad z \in \partial\mathcal{D}_{t_k}.$$

Furthermore, it is easy to see that the $\mathcal{O}$-terms in (3.32)–(3.33) are uniform for $(\theta_1, \ldots, \theta_m)$ in any given compact subset $\Theta \subset (0, 2\pi)^m_{\mathrm{ord}}$, for $\alpha_0, \ldots, \alpha_m$ in any given compact subset $\mathfrak{A} \subset \{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, and for $\beta_0, \ldots, \beta_m$ in any given compact subset $\mathfrak{B} \subset \{z \in \mathbb{C} : \operatorname{Re} z \in (-\frac{1}{2}, \frac{1}{2})\}$. Therefore, R satisfies a small norm RH problem, and the existence of R for all sufficiently large n can be proved using standard theory [**27**, **28**] as follows. Define the operator $\mathcal{C} : L^2(\Gamma_R) \to L^2(\Gamma_R)$ by $\mathcal{C}f(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(s)}{s-z} dz$, and denote $\mathcal{C}_+ f$ and $\mathcal{C}_- f$ for the left and right non-tangential limits of $\mathcal{C}f$. Since Γ_R is a compact set, by (3.32)–(3.33) we have $J_R - I \in L^2(\Gamma_R) \cap L^\infty(\Gamma_R)$, and we can define

$$\begin{aligned} \mathcal{C}_{J_R} : L^2(\Gamma_R) + L^\infty(\Gamma_R) &\to L^2(\Gamma_R), \\ \mathcal{C}_{J_R} f &= \mathcal{C}_-(f(J_R - I)), \\ f &\in L^2(\Gamma_R) + L^\infty(\Gamma_R). \end{aligned}$$

Using $\|\mathcal{C}_{J_R}\|_{L^2(\Gamma_R) \to L^2(\Gamma_R)} \leqslant C \|J_R - I\|_{L^\infty(\Gamma_R)}$ and (3.32)–(3.33), we infer that there exists $n_0 = n_0(\Theta, \mathfrak{A}, \mathfrak{B})$ such that $\|\mathcal{C}_{J_R}\|_{L^2(\Gamma_R) \to L^2(\Gamma_R)} < 1$ for all $n \geqslant n_0$, all $(\theta_1, \ldots, \theta_m) \in \Theta$, all $\alpha_0, \ldots, \alpha_m \in \mathfrak{A}$ and all $\beta_0, \ldots, \beta_m \in \mathfrak{B}$. Hence, for $n \geqslant n_0$, $I - \mathcal{C}_{J_R} : L^2(\Gamma_R) \to L^2(\Gamma_R)$ can be inverted as a Neumann series and thus R exists and is given by

$$R = I + \mathcal{C}(\mu_R(J_R - I)), \qquad \text{where} \quad \mu_R := I + (I - \mathcal{C}_{J_R})^{-1} \mathcal{C}_{J_R}(I). \tag{3.34}$$

Using (3.34), (3.32) and (3.33), we obtain

$$R(z) = I + R^{(1)}(z) n^{-1} + \mathcal{O}(n^{-2+2\beta_{\max}}), \qquad \text{as } n \to +\infty, \tag{3.35}$$

uniformly for $(\theta_1, \ldots, \theta_m) \in \Theta$, $\alpha_0, \ldots, \alpha_m \in \mathfrak{A}$ and $\beta_0, \ldots, \beta_m \in \mathfrak{B}$, where $R^{(1)}$ is given by

$$R^{(1)}(z) = \sum_{k=0}^m \frac{1}{2\pi i} \int_{\partial\mathcal{D}_{t_k}} \frac{J_R^{(1)}(s)}{s-z} \mathrm{d}s.$$

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Since the jumps J_R are analytic in a neighbourhood of Γ_R, expansion (3.35) holds uniformly for $z \in \mathbb{C} \setminus \Gamma_R$. It also follows from (3.34) that (3.35) can be differentiated with respect to z without increasing the error term. For $z \in \mathbb{C} \setminus \cup_{k=0}^{m} \mathcal{D}_{t_k}$, a residue calculation using (3.22), (3.27) and (3.30) shows that (recall that $\partial\mathcal{D}_{t_k}$ is oriented in the clockwise direction)

$$R^{(1)}(z) = \sum_{k=0}^{m} \frac{1}{z-t_k} \frac{(\beta_k^2 - \frac{\alpha_k^2}{4})t_k}{2\pi\psi(t_k)} \begin{pmatrix} 1 & \Lambda_k^{-2}\tau(\alpha_k,-\beta_k) \\ -\Lambda_k^{2}\tau(\alpha_k,\beta_k) & -1 \end{pmatrix}. \tag{3.36}$$

Remark 3.2. Above, we have discussed the uniformity of (3.32)–(3.33) and (3.35) in the parameters $\theta_k, \alpha_k, \beta_k$. In § 4, we will also need the following fact, which can be proved via a direct analysis (we omit the details here, see e.g. [**8**, Lemma 4.35] for a similar situation): If V is replaced by sV, then (3.32)–(3.33) and (3.35) also hold uniformly for $s \in [0,1]$.

Remark 3.3. If $k_0, \dots, k_{2m+1} \in \mathbb{N}$, $k_0 + \dots + k_{2m+1} \geqslant 1$ and $\partial^{\vec{k}} := \partial_{\alpha_0}^{k_0} \dots \partial_{\alpha_m}^{k_m} \partial_{\beta_0}^{k_{m+1}} \dots \partial_{\beta_m}^{k_{2m+1}}$, then by (3.17) we have

$$\partial^{\vec{k}} J_R(z) = \mathcal{O}(e^{-cn}), \quad \text{uniformly for } z \in (\gamma_+ \cup \gamma_-) \setminus \cup_{k=0}^{m} \overline{\mathcal{D}_{t_k}},$$

and by the same type of arguments that led to (3.30) we have

$$\partial^{\vec{k}} J_R(z) = \partial^{\vec{k}} (J_R^{(1)}(z)) n^{-1} + \mathcal{O}\left(\frac{(\log n)^{k_{m+1}+\dots+k_{2m+1}}}{n^{2-2\beta_{\max}}} \right), \quad \text{uniformly for } z \in \cup_{k=0}^{m} \partial\mathcal{D}_{t_k}.$$

It follows that

$$\partial^{\vec{k}} R(z) = \partial^{\vec{k}} (R^{(1)}(z)) n^{-1} + \mathcal{O}\left(\frac{(\log n)^{k_{m+1}+\dots+k_{2m+1}}}{n^{2-2\beta_{\max}}} \right), \qquad \text{as } n \to +\infty.$$

If W is replaced by tW, $t \in [0,1]$, then the asymptotics (3.32), (3.33) and (3.35) are uniform with respect to t and can also be differentiated any number of times with respect to t without worsening the error term.

4. Integration in V

Our strategy is inspired by [**8**] and considers a linear deformation in the potential (in [**8**] the authors study Hankel determinants related to point processes on the real line, see also [**14**, **18**, **19**] for subsequent works using similar deformation techniques).

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Consider the potential $\hat{V}_s := sV$, where $s \in [0,1]$. It is immediate to verify that

$$2\int_0^{2\pi} \log|z - e^{i\theta}| \mathrm{d}\mu_{\hat{V}_0}(e^{i\theta}) = \hat{V}_0(z) - \ell_0, \text{ for } z \in \mathbb{T}, \tag{4.1}$$

with $\mathrm{d}\mu_{\hat{V}_0}(e^{i\theta}) := \frac{1}{2\pi}\mathrm{d}\theta$ and $\ell_0 = 0$. Using a linear combination of (4.1) and (1.23) (writing $\hat{V}_s = (1-s)\hat{V}_0 + sV$), we infer that

$$2\int_0^{2\pi} \log|z - e^{i\theta}| \mathrm{d}\mu_{\hat{V}_s}(e^{i\theta}) = \hat{V}_s(z) - \ell_s, \text{ for } z \in \mathbb{T}, \tag{4.2}$$

holds for each $s \in [0,1]$ with $\ell_s := s\ell$ and $\mathrm{d}\mu_{\hat{V}_s}(e^{i\theta}) = \psi_s(e^{i\theta})\mathrm{d}\theta$, $\psi_s(e^{i\theta}) := \frac{1-s}{2\pi} + s\psi(e^{i\theta})$. In particular, this shows that $\psi_s(e^{i\theta}) > 0$ for all $s \in [0,1]$ and all $\theta \in [0, 2\pi)$. Hence, we can (and will) use the analysis of § 3 with V replaced by $\hat{V}_s$.

We first recall the following result, which will be used for our proof.

Theorem 4.1 Taken from [**24**, **29**]. *Let $m \in \mathbb{N}$, and let $t_k = e^{i\theta_k}$, α_k and β_k be such that*

$$0 = \theta_0 < \theta_1 < \ldots < \theta_m < 2\pi, \qquad \text{and} \qquad \operatorname{Re}\alpha_k > -1, \quad \operatorname{Re}\beta_k \in (-\tfrac{1}{2}, \tfrac{1}{2}) \quad \text{for } k = 0, \ldots, m.$$

Let $W : \mathbb{T} \to \mathbb{R}$ be analytic, and define W_+ and W_- as in (1.4). As $n \to +\infty$, we have

$$D_n(\vec{\alpha}, \vec{\beta}, 0, W) = \exp\left(D_2 n + D_3 \log n + D_4 + \mathcal{O}\left(\frac{1}{n^{1-2\beta_{\max}}}\right)\right), \tag{4.3}$$

where

$$\begin{aligned}
D_2 &= W_0, \\
D_3 &= \sum_{k=0}^{m}\left(\frac{\alpha_k^2}{4} - \beta_k^2\right), \\
D_4 &= \sum_{\ell=1}^{+\infty} \ell W_\ell W_{-\ell} + \sum_{k=0}^{m}\left(\beta_k - \frac{\alpha_k}{2}\right) W_+(t_k) - \sum_{k=0}^{m}\left(\beta_k + \frac{\alpha_k}{2}\right) W_-(t_k) \\
&\quad + \sum_{0 \leqslant j < k \leqslant m}\left\{\frac{\alpha_j i\beta_k - \alpha_k i\beta_j}{2}(\theta_k - \theta_j - \pi) + \left(2\beta_j\beta_k - \frac{\alpha_j\alpha_k}{2}\right)\log|t_j - t_k|\right\} \\
&\quad + \sum_{k=0}^{m} \log\frac{G(1 + \frac{\alpha_k}{2} + \beta_k)G(1 + \frac{\alpha_k}{2} - \beta_k)}{G(1+\alpha_k)},
\end{aligned}$$

where G is Barnes' G-function. Furthermore, the above asymptotics are uniform for all α_k in compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, for all β_k in compact subsets of $\{z \in \mathbb{C} : \operatorname{Re} z \in (-\frac{1}{2}, \frac{1}{2})\}$ and for all $(\theta_1, \ldots, \theta_m)$ in compact subsets of $(0, 2\pi)^m_{\mathrm{ord}}$.

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REMARK 4.2. The above theorem, but with the $\mathcal{O}$-term replaced by $o(1)$, was proved by Ehrhardt in [**29**]. The stronger estimate $\mathcal{O}(n^{-1+2\beta_{\max}})$ was obtained in [**26**, Remark 1.4]. (In fact the results [**26**, **29**] are valid for more general values of the β_k's, but this will not be needed for us.)

LEMMA 4.3. *For $z \in \mathbb{T}$, we have*

$$\frac{1}{i\pi}\fint_{\mathbb{T}} \frac{V'(w)}{w-z}dw = \frac{1}{z}\left(1 - 2\pi\psi(z)\right), \tag{4.4}$$

$$\frac{1}{i\pi}\fint_{\mathbb{T}} \frac{V(w)}{w-z}dw = V_0 + V_+(z) - V_-(z) = V_0 + 2i\,\mathrm{Im}\,(V_+(z)), \tag{4.5}$$

where $\fint$ stands for principal value integral.

Proof. The first identity (4.4) can be proved by a direct residue calculation using (1.3) and (1.6). We give here another proof, more in the spirit of [**8**, Lemma 5.8] and [**19**, Lemma 8.1]. Let $H, \varphi : \mathbb{C}\setminus\mathbb{T} \to \mathbb{C}$ be functions given by

$$H(z) = \varphi(z)\left(g'(z) - \frac{1}{2z}\right) - \frac{1}{2z} + \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{V'(w)}{w-z}dw, \quad \varphi(z) = \begin{cases} -1, & |z| < 1, \\ 1, & |z| > 1. \end{cases} \tag{4.6}$$

Clearly, $H(\infty) = 0$, and for $z \in \mathbb{T}$ we have

$$H_+(z) - H_-(z) = -\left(g'_+(z) + g'_-(z) - \frac{1}{z}\right) + V'(z) = 0,$$

where for the last equality we have used (3.6). So $H(z) \equiv 0$ by Liouville's theorem. Identity (4.4) now follows from relations (3.5) and

$$0 = H_+(z) + H_-(z) = -(g'_+(z) - g'_-(z)) - \frac{1}{z} + \frac{1}{i\pi}\fint_{\mathbb{T}} \frac{V'(w)}{w-z}dz, \qquad z \in \mathbb{T}.$$

The second identity (4.5) follows from a direct residue computation, using (1.3). $\square$

PROPOSITION 4.4. *As $n \to +\infty$,*

$$\log \frac{D_n(\vec{\alpha}, \vec{\beta}, V, W)}{D_n(\vec{\alpha}, \vec{\beta}, 0, W)} = c_1 n^2 + c_2 n + c_3 + \mathcal{O}(n^{-1+2\beta_{\max}}), \tag{4.7}$$

where

$$c_1 = -\frac{V_0}{2} - \frac{1}{2}\int_0^{2\pi} V(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta}),$$

$$c_2 = \sum_{k=0}^{m} \frac{\alpha_k}{2}(V(t_k) - V_0) - \sum_{k=0}^{m} 2i\beta_k \mathrm{Im}\,(V_+(t_k)) + \int_0^{2\pi} W(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta}) - W_0,$$

$$c_3 = \sum_{k=0}^{m} \frac{\beta_k^2 - \frac{\alpha_k^2}{4}}{\psi(t_k)}\left(\frac{1}{2\pi} - \psi(t_k)\right).$$

Proof. We will use (2.7) with $V = \hat{V}_s$ and $\gamma = s$, i.e.

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, \hat{V}_s, W) = \frac{1}{2\pi} \int_0^{2\pi} [Y^{-1}(z)Y'(z)]_{21} z^{-n+1} \partial_s f(z) \mathrm{d}\theta, \tag{4.8}$$

where $f(z) = e^{-n\hat{V}_s(z)}\omega(z)$ and $Y(\cdot) = Y_n(\cdot; \vec{\alpha}, \vec{\beta}, \hat{V}_s, W)$. Recall from proposition 2.2 that (4.8) is valid only when $D_k^{(n)}(f) \neq 0$, $k = n-1, n, n+1$. However, it follows from the analysis of subsection 3.5 (see also remark 3.2) that the right-hand side of (4.8) exists for all n sufficiently large, for all $(\theta_1, \ldots, \theta_m) \in \Theta$, all $\alpha_0, \ldots, \alpha_m \in \mathfrak{A}$, all $\beta_0, \ldots, \beta_m \in \mathfrak{B}$ and all $s \in [0,1]$. Hence, we can extend (4.8) by continuity (see also [**14**, **19**, **26**, **37**, **40**] for similar situations with more details provided). By (2.1), for $z \in \mathbb{T} \setminus \{t_0, \ldots, t_m\}$ we have

$$[Y(z)^{-1}Y'(z)]_{21,+} = [Y(z)^{-1}Y'(z)]_{21,-}, \tag{4.9}$$

$$[Y(z)^{-1}Y'(z)]_{21} = -\frac{z^n}{f(z)} \left([Y(z)^{-1}Y'(z)]_{11,+} - [Y(z)^{-1}Y'(z)]_{11,-} \right), \tag{4.10}$$

and thus, using that $\partial_s \log f(z) = -nV(z)$ is analytic in a neighbourhood of $\mathbb{T}$,

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, \hat{V}_s, W) = \frac{-1}{2\pi i} \int_{\mathcal{C}_e \cup \mathcal{C}_i} \left[Y^{-1}(z)Y'(z)\right]_{11} \partial_s \log f(z) dz, \tag{4.11}$$

where $\mathcal{C}_i \subset \{z : |z| < 1\} \cap U$ is a closed curve oriented counterclockwise and surrounding 0, and $\mathcal{C}_e \subset \{z : |z| > 1\} \cap U$ is a closed curve oriented clockwise and surrounding 0. We choose $\mathcal{C}_i$ and $\mathcal{C}_e$ such that they do not intersect $\mathbb{T} \cup \gamma_+ \cup \gamma_- \cup \mathcal{D}_{t_0} \cup \cdots \cup \mathcal{D}_{t_m}$.

Inverting the transformations $Y \mapsto T \mapsto S \mapsto R$ of § 3 using (3.13), (3.15) and (3.31), for $z \in \mathcal{C}_e \cup \mathcal{C}_i$ we find

$$\begin{aligned}\left[Y^{-1}(z)Y'(z)\right]_{11} = ng'(z) &+ \left[P^{(\infty)}(z)^{-1}P^{(\infty)\prime}(z)\right]_{11} \\ &+ \left[P^{(\infty)}(z)^{-1}R(z)^{-1}R'(z)P^{(\infty)}(z)\right]_{11}.\end{aligned}$$

Substituting the above in (4.11), we find the following exact identity:

$$\partial_s \log D_n(\vec{\alpha}, \vec{\beta}, \hat{V}_s, W) = I_{1,s} + I_{2,s} + I_{3,s},$$

where

$$I_{1,s} = \frac{-n}{2\pi i} \int_{\mathcal{C}_e \cup \mathcal{C}_i} g'(z) \partial_s \log f(z) dz, \tag{4.12}$$

$$I_{2,s} = \frac{-1}{2\pi i} \int_{\mathcal{C}_e \cup \mathcal{C}_i} \left[P^{(\infty)}(z)^{-1}P^{(\infty)\prime}(z)\right]_{11} \partial_s \log f(z) dz, \tag{4.13}$$

$$I_{3,s} = \frac{-1}{2\pi i} \int_{\mathcal{C}_e \cup \mathcal{C}_i} \left[P^{(\infty)}(z)^{-1}R(z)^{-1}R'(z)P^{(\infty)}(z)\right]_{11} \partial_s \log f(z) dz. \tag{4.14}$$

Using $\partial_s \log f(z) = -nV(z)$ and (3.5) (with ψ replaced by ψ_s), we find

$$I_{1,s} = \frac{n^2}{2\pi i} \int_{\mathbb{T}} (g'_+(z) - g'_-(z)) V(z) dz = -n^2 \int_{\mathbb{T}} V(z) \psi_s(z) \frac{dz}{iz},$$

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and since $\psi_s = \frac{1-s}{2\pi} + s\psi$,

$$\int_0^1 I_{1,s}\mathrm{d}s = -\frac{n^2}{2}\int_0^{2\pi} V(e^{i\theta})\left(\frac{1}{2\pi} + \psi(e^{i\theta})\right)\mathrm{d}\theta$$
$$= -\frac{n^2}{2}\left(V_0 + \int_0^{2\pi} V(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta})\right) = c_1 n^2. \tag{4.15}$$

Now we turn to the analysis of $I_{2,s}$. Using (3.17), we obtain

$$\left[P^{(\infty)}(z)^{-1}P^{(\infty)\prime}(z)\right]_{11} = \varphi(z)\partial_z[\log D(z)] \tag{4.16}$$

where φ is defined in (4.6). Also, by (3.18), (3.20) and (3.21), we have

$$\partial_z \log D(z) = \begin{cases} W_+'(z) + \sum_{k=0}^m \left(\beta_k + \frac{\alpha_k}{2}\right)\frac{1}{z-t_k}, & |z|<1, \\ -W_-'(z) + \sum_{k=0}^m \left(\beta_k - \frac{\alpha_k}{2}\right)\left(\frac{1}{z-t_k} - \frac{1}{z}\right), & |z|>1, \end{cases} \tag{4.17}$$

where $W_\pm$ are defined in (1.4), and by (1.6), we have

$$-\sum_{k=-\infty}^{+\infty} |k| W_k V_{-k} = \int_0^{2\pi} W(e^{i\theta})\mathrm{d}\mu_V(e^{i\theta}) - W_0. \tag{4.18}$$

Substituting (4.16) and (4.17) in (4.13), and doing a residue computation, we obtain

$$I_{2,s} = -n\sum_{k=-\infty}^{+\infty}|k|W_k V_{-k} + n\sum_{k=0}^m \frac{\alpha_k}{2}(V(t_k) - V_0)$$
$$- n\sum_{k=0}^m \beta_k\left(\frac{1}{\pi i}\int_{\mathbb{T}} \frac{V(z)}{z-t_k}dz - V_0\right) = c_2 n,$$

where for the last equality we have used (4.5) and (4.18). Clearly, $I_{2,s}$ is independent of s, and therefore $\int_0^1 I_{2,s}\mathrm{d}s = c_2 n$. We now analyse $I_{3,s}$ as $n\to+\infty$. From (3.35), we have

$$R^{-1}(z)R'(z) = n^{-1}R^{(1)\prime}(z) + \mathcal{O}(n^{-2+2\beta_{\max}}),$$

and, using first (3.17) and then (3.36),

$$\left[P^{(\infty)}(z)^{-1}n^{-1}R^{(1)\prime}(z)P^{(\infty)}(z)\right]_{11}$$
$$= \frac{1}{n}\times\begin{cases}\left[R^{(1)\prime}(z)\right]_{22}, & |z|<1 \\ \left[R^{(1)\prime}(z)\right]_{11}, & |z|>1\end{cases} = \frac{-\varphi(z)}{2\pi n}\sum_{k=0}^m \frac{(\beta_k^2 - \frac{\alpha_k^2}{4})t_k}{\psi(t_k)(z-t_k)^2}.$$

Therefore, as $n\to+\infty$

$$I_{3,s} = \frac{1}{2\pi}\sum_{k=0}^m \frac{(\beta_k^2 - \frac{\alpha_k^2}{4})t_k}{\psi(t_k)}\frac{1}{2\pi i}\left(\int_{C_i} - \int_{C_e}\right)\frac{V(z)}{(z-t_k)^2}dz + \mathcal{O}(n^{-1+2\beta_{\max}}).$$

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Partial integration yields

$$\frac{1}{2\pi i}\left(\int_{C_i}-\int_{C_e}\right)\frac{V(z)}{(z-t_k)^2}dz=\frac{1}{2\pi i}\left(\int_{C_i}-\int_{C_e}\right)\frac{V'(z)}{z-t_k}dz=\frac{1}{\pi i}\fint_{\mathbb{T}}\frac{V'(z)}{z-t_k}dz,$$

and thus, by (4.4), we have

$$I_{3,s}=\frac{1}{2\pi}\sum_{k=0}^{m}\frac{(\beta_k^2-\frac{\alpha_k^2}{4})t_k}{\psi(t_k)}\frac{1}{t_k}\left(1-2\pi\psi(t_k)\right)+\mathcal{O}(n^{-1+2\beta_{\max}}),\qquad \text{as } n\to+\infty.$$

Since the above asymptotics are uniform for $s\in[0,1]$ (see remark 3.2), the claim follows. □

Theorem 1.1 now directly follows by combining proposition 4.4 with theorem 4.1. (The estimate (1.12) follows from remark 3.3.)

5. Proofs of corollaries 1.4, 1.5, 1.6, 1.8, 1.9

Let $e^{\phi_1},\ldots,e^{i\phi_n}$ be distributed according to (1.17) with $\phi_1,\ldots,\phi_n\in[0,2\pi)$. Recall that $N_n(\theta)=\#\{\phi_j\in[0,\theta)\}$ and that the angles $\phi_1,\ldots,\phi_n$ arranged in increasing order are denoted by $0\leqslant\xi_1\leqslant\xi_2\leqslant\ldots\leqslant\xi_n<2\pi$.

Proof of corollary 1.4. The asymptotics for the cumulants $\{\kappa_j\}_{j=1}^{+\infty}$ follow directly from (1.30), theorem 1.3 (with $m=0$, $\alpha_0=0$ and with W replaced by tW) and the fact that (1.24) can be differentiated any number of time with respect to t without worsening the error term (see remark 3.3). Furthermore, if W is non-constant, then $\sum_{k=1}^{+\infty}kW_kW_{-k}=\sum_{k=1}^{+\infty}k|W_k|^2>0$ (because W is assumed to be real-valued) and from theorem 1.3 (with $m=0$, $\alpha_0=0$ and with W replaced by $\frac{tW}{(2\sum_{k=1}^{+\infty}kW_kW_{-k})^{1/2}}$, $t\in\mathbb{R}$) we also have

$$\mathbb{E}\left[\exp\left(t\frac{\sum_{j=1}^{n}W(e^{i\phi_j})-n\int_0^{2\pi}W(e^{i\phi})\mathrm{d}\mu_V(e^{i\phi})}{(2\sum_{k=1}^{+\infty}kW_kW_{-k})^{1/2}}\right)\right]=e^{\frac{t^2}{2}+\mathcal{O}(n^{-1})},$$

as $n\to+\infty$ with $t\in\mathbb{R}$ arbitrary but fixed. The convergence in distribution stated in corollary 1.4 now follows from standard theorems (see e.g. [**9**, top of page 415]).

Proof of corollary 1.5. The proof is similar to the proof of corollary 1.4. The main difference is that (i) for the asymptotics of the cumulants, one needs to use theorem 1.3 with $W=0$, $m=0$ if $t=1$, and with $W=0$, $m=1$, $\alpha_0=0$, $u_1=0$ if $t\in\mathbb{T}\setminus\{1\}$, and (ii) for the convergence in distribution, one needs to use theorem 1.3 with $W=0$, $m=0$ and α_0 replaced by $\alpha\sqrt{2}/\sqrt{\log n}$, $\alpha\in\mathbb{R}$ fixed, if $t=1$, and with $W=0$, $m=1$, $\alpha_0=0$, $u_1=0$ and α_1 replaced by $\alpha\sqrt{2}/\sqrt{\log n}$, $\alpha\in\mathbb{R}$ fixed, if $t\in\mathbb{T}\setminus\{1\}$.

Proof of corollary 1.6. This proof is also similar to the proof of corollary 1.4. For the asymptotics of the cumulants, one needs to use theorem 1.3 with $W=0$, $m=1$, $\alpha_0=\alpha_1=0$ and for the convergence in distribution, one needs to use theorem 1.3 with $W=0$, $m=1$, $\alpha_0=\alpha_1=0$, and with u_1 replaced by $\pi u/\sqrt{\log n}$, $u\in\mathbb{R}$ fixed.

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Proof of corollary 1.8. The proof is inspired by Gustavsson [**36**, Theorem 1.2]. Let $\theta \in (0, 2\pi)$ and $k_\theta = [n\int_0^\theta \mathrm{d}\mu_V(e^{i\phi})]$, where $[x] := \lfloor x + \frac{1}{2} \rfloor$, and consider the random variable

$$Y_n := \frac{n\int_0^{\xi_{k_\theta}} \mathrm{d}\mu_V(e^{i\phi}) - k_\theta}{\sqrt{\log n}/\pi} = \frac{\mu_n(\xi_{k_\theta}) - k_\theta}{\sigma_n}, \tag{5.1}$$

where $\mu_n(\xi) := n\int_0^\xi \mathrm{d}\mu_V(e^{i\phi})$ and $\sigma_n := \frac{1}{\pi}\sqrt{\log n}$. For $y \in \mathbb{R}$, we have

$$\mathbb{P}\big[Y_n \leqslant y\big] = \mathbb{P}\Big[\xi_{k_\theta} \leqslant \mu_n^{-1}\left(k_\theta + y\sigma_n\right)\Big] = \mathbb{P}\Big[N_n\left(\mu_n^{-1}\left(k_\theta + y\sigma_n\right)\right) \geqslant k_\theta\Big]. \tag{5.2}$$

Letting $\tilde{\theta} := \mu_n^{-1}(k_\theta + y\sigma_n)$, we can rewrite (5.2) as

$$\mathbb{P}\big[Y_n \leqslant y\big] = \mathbb{P}\left[\frac{N_n(\tilde{\theta}) - \mu_n(\tilde{\theta})}{\sqrt{\sigma_n^2}} \geqslant \frac{k_\theta - \mu_n(\tilde{\theta})}{\sigma_n}\right] = \mathbb{P}\left[\frac{\mu_n(\tilde{\theta}) - N_n(\tilde{\theta})}{\sigma_n} \leqslant y\right]. \tag{5.3}$$

As $n \to +\infty$, we have

$$k_\theta = [\mu_n(\theta)] = \mathcal{O}(n), \qquad \tilde{\theta} = \theta\left(1 + \mathcal{O}\left(\tfrac{\sqrt{\log n}}{n}\right)\right). \tag{5.4}$$

Since theorem 1.3 also holds in the case where θ depends on n but remains bounded away from 0, the same is true for the convergence in distribution in corollary 1.6. By (5.4), $\tilde{\theta}$ remains bounded away from 0, and therefore corollary 1.6 together with (5.3) implies that Y_n converges in distribution to a standard normal random variable. Since

$$\begin{aligned}
&\mathbb{P}\left[\frac{n\psi(e^{i\eta_{k_\theta}})}{\sqrt{\log n}/\pi}(\xi_{k_\theta} - \eta_{k_\theta}) \leqslant y\right] \\
&\quad = \mathbb{P}\left[Y_n \leqslant \frac{\mu_n(\eta_{k_\theta} + y\frac{\sigma_n}{n\psi(e^{i\eta_{k_\theta}})}) - \mu_n(\eta_{k_\theta})}{\sigma_n}\right] \\
&\quad = \mathbb{P}\left[Y_n \leqslant \int_{\eta_{k_\theta}}^{\eta_{k_\theta} + \frac{y\sigma_n}{n\psi(e^{i\eta_{k_\theta}})}} \frac{n\psi(e^{i\phi})}{\sigma_n}\mathrm{d}\phi\right] = \mathbb{P}\big[Y_n \leqslant y + o(1)\big]
\end{aligned}$$

as $n \to +\infty$, this implies the convergence in distribution in the statement of corollary 1.8.

Proof of corollary 1.9. Let $\mu_n(\xi) := n\int_0^\xi \mathrm{d}\mu_V(e^{i\phi})$, $\sigma_n := \frac{1}{\pi}\sqrt{\log n}$, and for $\theta \in [0, 2\pi)$, let $\overline{N}_n(\theta) := N_n(\theta) - \mu_n(\theta)$. Using theorem 1.3 with $W = 0$, $m \in \mathbb{N}_{>0}$, $\alpha_0 = \ldots = \alpha_m = 0$ and $u_1, \ldots, u_m \in \mathbb{R}$, we infer that for any $\delta \in (0, \pi)$ and $M > 0$, there exists $n_0' = n_0'(\delta, M) \in \mathbb{N}$ and $\mathrm{C} = \mathrm{C}(\delta, M) > 0$ such that

$$\mathbb{E}\left(e^{\sum_{k=1}^m u_k \overline{N}_n(\theta_k)}\right) \leqslant \mathrm{C}\exp\left(\frac{\sum_{k=0}^m u_k^2}{2}\frac{\sigma_n^2}{2}\right), \tag{5.5}$$

for all $n \geqslant n_0'$, $(\theta_1, \ldots, \theta_m)$ in compact subsets of $(0, 2\pi)^m_{\mathrm{ord}} \cap (\delta, 2\pi - \delta)^m$ and $u_1, \ldots, u_m \in [-M, M]$, and where $u_0 = -u_1 - \ldots - u_m$.

LEMMA 5.1. *For any $\delta \in (0, \pi)$, there exists $c > 0$ such that for all large enough n and small enough $\epsilon > 0$,*

$$\mathbb{P}\left(\sup_{\delta \leqslant \theta \leqslant 2\pi-\delta}\left|\frac{N_n(\theta)-\mu_n(\theta)}{\sigma_n^2}\right| \leqslant \pi(1+\epsilon)\right) \geqslant 1-\frac{c}{\log n}. \tag{5.6}$$

Proof. A naive adaptation of [**15**, Lemma 8.1] (an important difference between [**15**] and our situation is that $\sigma_n = \frac{\sqrt{\log n}}{\sqrt{2}\pi}$ in [**15**] while here we have $\sigma_n = \frac{\sqrt{\log n}}{\pi}$) yields

$$\mathbb{P}\left(\sup_{\delta \leqslant \theta \leqslant 2\pi-\delta}\left|\frac{N_n(\theta)-\mu_n(\theta)}{\sigma_n^2}\right| \leqslant \sqrt{2}\pi(1+\epsilon)\right) \geqslant 1-o(1).$$

Inequality (5.5) can in fact be used to obtain the stronger statement (5.6).[1] Recall that $\eta_k = \mu_n^{-1}(k)$ is the classical location of the k-th smallest point ξ_k and is defined in (1.34). Since μ_n and N_n are increasing functions, for $x \in [\eta_{k-1}, \eta_k]$ with $k \in \{1, \ldots, n\}$, we have

$$N_n(x) - \mu_n(x) \leqslant N_n(\eta_k) - \mu_n(\eta_{k-1}) = N_n(\eta_k) - \mu_n(\eta_k) + 1, \tag{5.7}$$

which implies

$$\sup_{\delta \leqslant x \leqslant 2\pi-\delta} \frac{N_n(x)-\mu_n(x)}{\sigma_n^2} \leqslant \sup_{k \in \mathcal{K}_n} \frac{N_n(\eta_k)-\mu_n(\eta_k)+1}{\sigma_n^2},$$

where $\mathcal{K}_n = \{k : \eta_k > \delta \text{ and } \eta_{k-1} < 2\pi - \delta\}$. Hence, for any $v > 0$,

$$\mathbb{P}\left(\sup_{\delta \leqslant x \leqslant 2\pi-\delta} \frac{N_n(x)-\mu_n(x)}{\sigma_n^2} > v\right) \leqslant \mathbb{P}\left(\sup_{k \in \mathcal{K}_n} \frac{N_n(\kappa_k)-\mu_n(\kappa_k)}{\sigma_n^2} > v - \frac{1}{\sigma_n^2}\right).$$

Let $\epsilon_0 > 0$ be small and fixed, and let $\mathcal{I}$ be an arbitrary but fixed subset of $(0, \epsilon_0]$. Claim (5.6) will follow if we can prove for any $\epsilon \in \mathcal{I}$ that

$$\mathbb{P}\left(\sup_{k \in \mathcal{K}_n} \frac{N_n(\eta_k)-\mu_n(\eta_k)}{\sigma_n^2} > \pi(1+\epsilon)\right) \leqslant \frac{c_1}{\log n}, \tag{5.8}$$

for some $c_1 = c_1(\mathcal{I}) > 0$. Let $m \in \mathbb{N}$ be fixed and S_m and S'_m be the following two collections of points of size m

$$S_m = \left\{\delta + (2\pi - 2\delta)\frac{4j+1}{4m} : \qquad j = 0, \ldots, m-1\right\},$$

$$S'_m = \left\{\delta + (2\pi - 2\delta)\frac{4j+2}{4m} : \qquad j = 0, \ldots, m-1\right\}.$$

Let $X_n(\theta) := (N_n(\theta) - \mu_n(\theta))/\sigma_n$. For any $\theta \in [\delta, 2\pi - \delta]$, we have by corollary 1.6 that $\mathbb{E}[X_n(\theta)] = \mathcal{O}(\frac{\sqrt{\log n}}{n})$ and $\mathrm{Var}[X_n(\theta)] \leqslant 2$ for all large enough n. Hence, by

[1] We are very grateful to a referee for pointing this out.

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Chebyshev's inequality, for any fixed $\ell > 0$, $\mathbb{P}(\frac{|X_n|}{\sigma_n} \geqslant \ell) \leqslant \frac{3}{\ell^2 \sigma_n^2}$ for all large enough n. Using this inequality with $\ell = \frac{\pi\epsilon}{2}$ together with a union bound, we get

$$\mathbb{P}\left(\sup_{\hat{\theta}\in S_m \cup S'_m} \left|\frac{N_n(\hat{\theta}) - \mu_n(\hat{\theta})}{\sigma_n^2}\right| > \frac{\pi\epsilon}{2}\right) = \mathbb{P}\left(\sup_{\hat{\theta}\in S_m \cup S'_m} \left|\frac{X_n(\hat{\theta})}{\sigma_n}\right| > \frac{\pi\epsilon}{2}\right) \leqslant \frac{3\times 2m}{(\frac{\pi\epsilon}{2})^2 \frac{\log n}{\pi^2}} = \frac{24m}{\epsilon^2 \log n},$$

and then

$$\begin{aligned} \mathbb{P}\left(\sup_{k\in\mathcal{K}_n} \left|\frac{N_n(\eta_k) - \mu_n(\eta_k)}{\sigma_n^2}\right| > \pi(1+\epsilon)\right) &\leqslant \frac{24m}{\epsilon^2 \log n} \\ &+ \sum_{k\in\mathcal{K}_n} \mathbb{P}\left(\left|\frac{N_n(\eta_k) - \mu_n(\eta_k)}{\sigma_n^2}\right| > \pi(1+\epsilon) \quad \text{and} \quad \sup_{\hat{\theta}\in S_m\cup S'_m}\left|\frac{X_n(\hat{\theta})}{\sigma_n}\right| \leqslant \frac{\pi\epsilon}{2}\right). \end{aligned} \tag{5.9}$$

The reason for introducing two subsets S_m, S'_m is the following: for any $k \in \mathcal{K}_n$, one must have that η_k remains bounded away from at least one of S_m, S'_m (so that (5.5) can be applied). Indeed, suppose for example that θ is bounded away from S_m, then by (5.5) (with m replaced by $m+1$ and with $u_1 = u$ and $u_2 = \ldots = u_{m+1} = -\frac{u}{m}$) we have

$$\mathbb{E}\left[\exp\left(u\overline{N}_n(\eta_k) - \frac{u}{m}\sum_{\hat{\theta}\in S_m} \overline{N}_n(\hat{\theta})\right)\right] \leqslant \mathrm{C}\exp\left\{\frac{u^2\sigma_n^2}{4}\left(1+\frac{1}{m}\right)\right\}$$

and similarly,

$$\mathbb{E}\left[\exp\left(\frac{u}{m}\sum_{\hat{\theta}\in S_m} \overline{N}_n(\hat{\theta}) - u\overline{N}_n(\eta_k)\right)\right] \leqslant \mathrm{C}\exp\left\{\frac{u^2\sigma_n^2}{4}\left(1+\frac{1}{m}\right)\right\}.$$

Hence, if η_k remains bounded away from S_m, we have (with $\gamma := \pi(1+\epsilon/2)$ and $\alpha := \frac{1}{2}(1+\frac{1}{m})$)

$$\begin{aligned} &\mathbb{P}\left(\left|\frac{\overline{N}_n(\eta_k)}{\sigma_n^2}\right| > \pi(1+\epsilon) \quad \text{and} \quad \sup_{\hat{\theta}\in S_m\cup S'_m}\left|\frac{\overline{N}_n(\hat{\theta})}{\sigma_n^2}\right| \leqslant \frac{\pi\epsilon}{2}\right) \\ &\leqslant \mathbb{P}\left(\frac{\overline{N}_n(\eta_k) - \frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta})}{\sigma_n^2} > \gamma\right) + \mathbb{P}\left(\frac{\frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta}) - \overline{N}_n(\eta_k)}{\sigma_n^2} > \gamma\right) \\ &= \mathbb{P}\left(e^{\frac{\gamma}{\alpha}(\overline{N}_n(\eta_k) - \frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta}))} > e^{\frac{\gamma^2}{\alpha}\sigma_n^2}\right) \\ &\quad + \mathbb{P}\left(e^{\frac{\gamma}{\alpha}(\frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta}) - \overline{N}_n(\eta_k))} > e^{\frac{\gamma^2}{\alpha}\sigma_n^2}\right) \end{aligned} \tag{5.10}$$

$$\leqslant \mathbb{E}\left(e^{\frac{\gamma}{\alpha}(\overline{N}_n(\eta_k)-\frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta}))}\right)e^{-\frac{\gamma^2}{\alpha}\sigma_n^2}$$
$$+\mathbb{E}\left(e^{\frac{\gamma}{\alpha}(\frac{1}{m}\sum_{\hat{\theta}\in S_m}\overline{N}_n(\hat{\theta})-\overline{N}_n(\eta_k))}\right)e^{-\frac{\gamma^2}{\alpha}\sigma_n^2}$$
$$\leqslant 2\mathrm{C}\exp\left(\frac{\gamma^2\sigma_n^2}{4\alpha^2}\left(1+\frac{1}{m}\right)-\frac{\gamma^2}{\alpha}\sigma_n^2\right)=2\mathrm{C}\exp\left(-\frac{\pi^2(1+\frac{\epsilon}{2})^2\sigma_n^2}{1+\frac{1}{m}}\right)$$
$$=2\mathrm{C}n^{-\frac{(1+\frac{\epsilon}{2})^2}{1+\frac{1}{m}}}. \tag{5.11}$$

We obtain the same bound (5.11) if η_k in (5.10) is instead bounded away from S_m'. The above exponent is less than -1 provided that m is sufficiently large relative to ϵ. Since the number of points in $\mathcal{K}_n$ is proportional to n, claim (5.6) now directly follows from (5.9) (recall also (5.8)). □

LEMMA 5.2. *Let $\delta\in(0,\frac{\pi}{2})$ and $\epsilon>0$. For all sufficiently large n, if the event*

$$\sup_{\delta\leqslant\theta\leqslant 2\pi-\delta}\left|\frac{N_n(\theta)-\mu_n(\theta)}{\sigma_n^2}\right|\leqslant\pi(1+\epsilon) \tag{5.12}$$

holds true, then we have

$$\sup_{k\in(\mu_n(2\delta),\mu_n(2\pi-2\delta))}\left|\frac{\mu_n(\xi_k)-k}{\sigma_n^2}\right|\leqslant\pi(1+\epsilon)+\frac{1}{\sigma_n^2}, \tag{5.13}$$

Proof. The proof is almost identical to the proof of [**15**, Lemma 8.2] so we omit it. □

By combining lemmas 5.1 and 5.2, we arrive at the following result (the proof is very similar to [**15**, Proof of (1.38)], so we omit it).

LEMMA 5.3. *For any $\delta\in(0,\pi)$, there exists $c>0$ such that for all large enough n and small enough $\epsilon>0$,*

$$\mathbb{P}\left(\max_{\delta n\leqslant k\leqslant(1-\delta)n}\psi(e^{i\eta_k})|\xi_k-\eta_k|\leqslant\frac{1+\epsilon}{\pi}\frac{\log n}{n}\right)\geqslant 1-\frac{c}{\log n}. \tag{5.14}$$

Extending lemmas 5.1 and 5.3 to $\delta=0$. In this paper, the support of μ_V is $\mathbb{T}$. Therefore, the point $1\in\mathbb{T}$ should play no special role in the study of the global rigidity of the points, which suggests that (5.6) and (5.14) should still hold with $\delta=0$. The next lemma shows that this is indeed the case.

LEMMA 5.4 Proof of (1.35). *For each small enough $\epsilon>0$, there exists $c>0$ such that*

$$\mathbb{P}\left(\sup_{0\leqslant\theta<2\pi}\left|\frac{N_n(\theta)-\mu_n(\theta)}{\sigma_n^2}\right|\leqslant\pi(1+\epsilon)\right)\geqslant 1-\frac{c}{\log n}$$

for all large enough n.

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Proof. For $-\pi \leqslant \theta < 0$, let $\tilde{N}_n(\theta) := \#\{\phi_j - 2\pi \in (-\pi, \theta]\}$, and for $0 \leqslant \theta < \pi$, let $\tilde{N}_n(\theta) := \#(\{\phi_j - 2\pi \in (-\pi, 0]\} \cup \{\phi_j \in [0, \theta]\})$. For $-\pi \leqslant \theta < \pi$, define also $\tilde{\mu}_n(\theta) := n \int_{-\pi}^{\theta} \mathrm{d}\mu_V(e^{i\phi})$. In the same way as for lemma 5.1, the following holds: for any $\delta \in (0, \pi)$, there exists $c_1 > 0$ such that for all large enough n and small enough $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{-\pi+\delta \leqslant \theta \leqslant \pi-\delta} \left| \frac{\tilde{N}_n(\theta) - \tilde{\mu}_n(\theta)}{\sigma_n^2} \right| \leqslant \pi(1+\epsilon) \right) \geqslant 1 - \frac{c_1}{\log n}.$$

Clearly,

$$\tilde{N}_n(\theta) = \begin{cases} N_n(\theta + 2\pi) - N_n(\pi), & \text{if } \theta \in (-\pi, 0), \\ N_n(\theta) + n - N_n(\pi), & \text{if } \theta \in (0, \pi), \end{cases}$$

$$\tilde{\mu}_n(\theta) = \begin{cases} \mu_n(\theta + 2\pi) - \mu_n(\pi), & \text{if } \theta \in (-\pi, 0), \\ \mu_n(\theta) + n - \mu_n(\pi), & \text{if } \theta \in (0, \pi), \end{cases}$$

and therefore

$$\frac{\tilde{N}_n(\theta) - \tilde{\mu}_n(\theta)}{\sigma_n^2} = -\frac{N_n(\pi) - \mu_n(\pi)}{\sigma_n^2} + \begin{cases} \frac{N_n(\theta+2\pi) - \mu_n(\theta+2\pi)}{\sigma_n^2}, & \text{if } \theta \in (-\pi, 0), \\ [0.1\,cm] \frac{N_n(\theta) - \mu_n(\theta)}{\sigma_n^2}, & \text{if } \theta \in (0, \pi). \end{cases}$$

Thus, for all large enough n,

$$\mathbb{P}\left(\sup_{\theta \in [0, 2\pi) \setminus (\pi-\delta, \pi+\delta)} \left| \frac{N_n(\theta) - \mu_n(\theta)}{\sigma_n^2} \right| \leqslant \pi(1+\epsilon) + \left| \frac{N_n(\pi) - \mu_n(\pi)}{\sigma_n^2} \right| \right) \geqslant 1 - \frac{c_1}{\log n}.$$

Combining the above with (5.6) (with c replaced by c_2), we obtain

$$\mathbb{P}\left(\sup_{\theta \in [0, 2\pi)} \left| \frac{N_n(\theta) - \mu_n(\theta)}{\sigma_n^2} \right| \leqslant \pi(1+\epsilon) + \left| \frac{N_n(\pi) - \mu_n(\pi)}{\sigma_n^2} \right| \right) \geqslant 1 - \frac{c_1 + c_2}{\log n}. \tag{5.15}$$

Let $X_n := (N_n(\pi) - \mu_n(\pi))/\sigma_n$. By corollary 1.6, $\mathbb{E}[X_n] = \mathcal{O}(\frac{\sqrt{\log n}}{n})$ and $\mathrm{Var}[X_n] \leqslant 2$ for all large enough n. Hence, by Chebyshev's inequality, for any fixed $\ell > 0$, $\mathbb{P}(\frac{|X_n|}{\sigma_n} \geqslant \ell) \leqslant \frac{3}{\ell^2 \sigma_n^2}$ for all large enough n. Applying this inequality with $\ell = \pi(1 + (1+\epsilon)\epsilon) - \pi(1+\epsilon) = \pi\epsilon^2$ we see that if $\mathbb{P}(A)$ denotes the left-hand side of (5.15), then

$$\mathbb{P}(A) \leqslant \mathbb{P}\left(A \cap \left\{ \frac{|X_n|}{\sigma_n} < \ell \right\} \right) + \mathbb{P}\left(\frac{|X_n|}{\sigma_n} \geqslant \ell \right) \leqslant \mathbb{P}\left(A \cap \left\{ \frac{|X_n|}{\sigma_n} < \ell \right\} \right) + \frac{3}{\ell^2 \sigma_n^2}.$$

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Together with (5.15), this gives

$$\mathbb{P}\left(\sup_{\theta\in[0,2\pi)}\left|\frac{N_n(\theta)-\mu_n(\theta)}{\sigma_n^2}\right|\leqslant \pi\left(1+(1+\epsilon)\epsilon\right)\right)$$
$$\geqslant \mathbb{P}\left(A\cap\left\{\tfrac{|X_n|}{\sigma_n}<\ell\right\}\right)\geqslant \mathbb{P}(A)-\frac{3}{\ell^2\sigma_n^2}$$
$$\geqslant 1-\frac{c_1+c_2}{\log n}-\frac{3}{\ell^2\sigma_n^2}\geqslant 1-\frac{c_3}{\log n},$$

for some $c_3=c_3(\epsilon)>0$, which proves the claim. □

The upper bound (1.36) can be proved using the same idea as in the proof of lemma 5.4.

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Appendix A. Equilibrium measure

Assume that μ_V is supported on $\mathbb{T}$. We make the ansatz that μ_V is of form (1.7) for some ψ. Let g be as in (3.1). Substituting (3.2) in (1.23) and differentiating, we obtain

$$g'_+(z) + g'_-(z) = V'(z) + \frac{1}{z}, \qquad z \in \mathbb{T}.$$

Since $g'(z) = \frac{1}{z} + \mathcal{O}(z^{-2})$ as $z \to \infty$, we deduce that

$$g'(z) = -\frac{\varphi(z)}{2\pi i} \int_{\mathbb{T}} \frac{\frac{1}{s} + V'(s)}{s - z} \mathrm{d}s, \qquad z \in \mathbb{C} \setminus \mathbb{T}, \tag{A.1}$$

where $\varphi(z) := +1$ if $|z| > 1$ and $\varphi(z) := -1$ if $|z| < 1$. Using (A.1) in (3.5), it follows that

$$-\frac{2\pi}{z}\psi(z) = \frac{1}{\pi i} \fint_{\mathbb{T}} \frac{\frac{1}{s} + V'(s)}{s - z} \mathrm{d}s, \qquad z \in \mathbb{T}. \tag{A.2}$$

Recall from (1.3) that V is analytic in the open annulus U and real-valued on $\mathbb{T}$, and therefore

$$V(z) = V_0 + \sum_{k \geqslant 1} (V_k z^k + \overline{V_k} z^{-k}), \qquad V'(z) = \sum_{k \geqslant 1} (k V_k z^{k-1} - k\overline{V_k} z^{-k-1}), \quad z \in U.$$

(It is straightforward to check that the series $\sum_{k \geqslant 1} k V_k z^{k-1}$ and $\sum_{k \geqslant 1} k\overline{V_k} z^{-k-1}$ are convergent in U.) Direct computation gives

$$\fint_{\mathbb{T}} \frac{\frac{1}{s} + V'(s)}{s - z} \mathrm{d}s = -\frac{\pi i}{z} + \pi i \sum_{k \geqslant 1} (k V_k z^{k-1} + k\overline{V_k} z^{-k-1}), \qquad z \in \mathbb{T},$$

which, by (A.2), proves that ψ is given by (1.6). Since the right-hand side of (1.6) is positive on $\mathbb{T}$ (by our assumption that V is regular), we conclude that $\psi(e^{i\theta})\mathrm{d}\theta$

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is a probability measure satisfying the Euler–Lagrange condition (1.23). Therefore, $\psi(e^{i\theta})\mathrm{d}\theta$ minimizes (1.5), i.e. $\psi(e^{i\theta})\mathrm{d}\theta$ is the equilibrium measure associated to V. Since the equilibrium measure is unique [**42**], this proves (1.7).

Appendix B. Confluent hypergeometric model RH problem

(a) $\Phi_{\mathrm{HG}} : \mathbb{C} \setminus \Sigma_{\mathrm{HG}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{HG} is shown in figure 3.

(b) For $z \in \Gamma_k$ (see figure 3), $k = 1, \ldots, 8$, Φ_{HG} obeys the jump relations

$$\Phi_{\mathrm{HG},+}(z) = \Phi_{\mathrm{HG},-}(z) J_k, \tag{B.1}$$

where

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} - e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 0 & e^{i\pi\beta} - e^{-i\pi\beta} & 0 \end{pmatrix},$$

$$J_3 = J_7 = \begin{pmatrix} e^{\frac{i\pi\alpha}{2}} & 0 \\ 0 & e^{-\frac{i\pi\alpha}{2}} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}, J_4 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix}, J_6 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\alpha}e^{-i\pi\beta} & 1 \end{pmatrix},$$

$$J_8 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\alpha}e^{i\pi\beta} & 1 \end{pmatrix}.$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{HG}}$, we have

$$\Phi_{\mathrm{HG}}(z) = \left(I + \sum_{k=1}^{\infty} \frac{\Phi_{\mathrm{HG},k}}{z^k} \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} M^{-1}(z), \tag{B.2}$$

where

$$\Phi_{\mathrm{HG},1} = \left(\beta^2 - \frac{\alpha^2}{4} \right) \begin{pmatrix} -1 & \tau(\alpha,\beta) - \tau(\alpha,-\beta) & 1 \end{pmatrix},$$
$$\tau(\alpha,\beta) = -\frac{\Gamma\left(\frac{\alpha}{2} - \beta\right)}{\Gamma\left(\frac{\alpha}{2} + \beta + 1\right)}, \tag{B.3}$$

and

$$M(z) = \begin{cases} e^{\dfrac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \dfrac{\pi}{2} < \arg z < \pi, \\ e^{-\dfrac{i\pi\alpha}{4}\sigma_3} e^{-i\pi\beta\sigma_3}, & \pi < \arg z < \dfrac{3\pi}{2}, \\ e^{\dfrac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 - 1 & 0 \end{pmatrix}, & -\dfrac{\pi}{2} < \arg z < 0, \\ e^{-\dfrac{i\pi\alpha}{4}\sigma_3} \begin{pmatrix} 0 & 1 - 1 & 0 \end{pmatrix}, & 0 < \arg z < \dfrac{\pi}{2}. \end{cases} \tag{B.4}$$

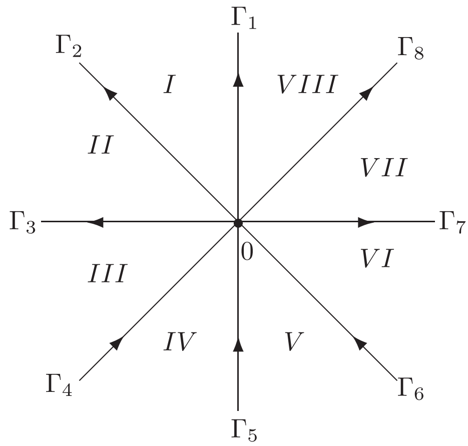

Figure B3. The jump contour $\Sigma_{\rm HG}$ for $\Phi_{\rm HG}(z)$. The ray Γ_k is oriented from 0 to ∞, and forms an angle with $\mathbb{R}^+$ which is a multiple of $\frac{\pi}{4}$.

In (B.2), $z^{-\beta}$ has a cut along $i\mathbb{R}^-$ so that $z^{-\beta} = |z|^{-\beta}e^{-\beta i \arg(z)}$ with $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$. As $z \to 0$, we have

$$\Phi_{\rm HG}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in II \cup III \cup VI \cup VII, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in I \cup IV \cup V \cup VIII, \end{cases}, \qquad \text{if } \operatorname{Re}\alpha = 0,$$

$$\Phi_{\rm HG}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & \text{if } z \in II \cup III \cup VI \cup VII, \\ \begin{pmatrix} \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \\ \mathcal{O}(z^{-\frac{\alpha}{2}}) & \mathcal{O}(z^{-\frac{\alpha}{2}}) \end{pmatrix}, & \text{if } z \in I \cup IV \cup V \cup VIII, \end{cases}, \qquad \text{if } \operatorname{Re}\alpha > 0, \tag{B.5}$$

$$\Phi_{\rm HG}(z) = \begin{pmatrix} \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \\ \mathcal{O}(z^{\frac{\alpha}{2}}) & \mathcal{O}(z^{\frac{\alpha}{2}}) \end{pmatrix}, \qquad \text{if } \operatorname{Re}\alpha < 0.$$

This model RH problem was first introduced and solved explicitly in [**37**] for the case $\alpha = 0$, and then in [**24**, **35**] for the general case. The constant matrices $\Phi_{{\rm HG},k}$ depend analytically on α and β (they can be found explicitly, see e.g. [**35**, eq. (56)]). Consider the matrix

$$\widehat{\Phi}_{\rm HG}(z) = \begin{pmatrix} \frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(1+\alpha)} G(\frac{\alpha}{2}+\beta, \alpha; z) e^{-\frac{i\pi\alpha}{2}} & -\frac{\Gamma(1+\frac{\alpha}{2}-\beta)}{\Gamma(\frac{\alpha}{2}+\beta)} H(1+\frac{\alpha}{2}-\beta, \alpha; ze^{-i\pi}) \\ \frac{\Gamma(1+\frac{\alpha}{2}+\beta)}{\Gamma(1+\alpha)} G(1+\frac{\alpha}{2}+\beta, \alpha; z) e^{-\frac{i\pi\alpha}{2}} & H(\frac{\alpha}{2}-\beta, \alpha; ze^{-i\pi}) \end{pmatrix} e^{-\frac{i\pi\alpha}{4}\sigma_3}, \tag{B.6}$$

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where G and H are related to the Whittaker functions:

$$G(a,\alpha;z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \quad H(a,\alpha;z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}}, \quad \mu = \frac{\alpha}{2}, \quad \kappa = \frac{1}{2} + \frac{\alpha}{2} - a. \tag{B.7}$$

The solution Φ_{HG} is given by

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1}, & \text{for } z \in I, \\ \widehat{\Phi}_{\mathrm{HG}}(z), & \text{for } z \in II, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_3, & \text{for } z \in III, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_3 J_4^{-1}, & \text{for } z \in IV, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1} J_6, & \text{for } z \in V, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1} J_7^{-1}, & \text{for } z \in VI, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_8^{-1}, & \text{for } z \in VII, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1}, & \text{for } z \in VIII. \end{cases} \tag{B.8}$$

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