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On invariant measures of 'satellite' infinitely renormalizable quadratic polynomials

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Abstract. Let $f(z) = z^2 + c$ be an infinitely renormalizable quadratic polynomial and J_∞ be the intersection of forward orbits of 'small' Julia sets of its simple renormalizations. We prove that if f admits an infinite sequence of satellite renormalizations, then every invariant measure of $f : J_\infty \to J_\infty$ is supported on the postcritical set and has zero Lyapunov exponent. Coupled with [13], this implies that the Lyapunov exponent of such f at c is equal to zero, which partly answers a question posed by Weixiao Shen.

Key words: Julia sets, infinitely renormalizable, iteration of complex polynomials, Lyapunov exponent, invariant measures
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1. *Introduction*

We consider the dynamics $f : \mathbb{C} \to \mathbb{C}$ of a quadratic polynomial. Up to a linear change of coordinates, f has the form $f_c(z) = z^2 + c$ for some $c \in \mathbb{C}$. In this paper, which is the sequel to [12], we assume that f is infinitely renormalizable. Moreover, in the main results we assume that f has infinitely many 'satellite renormalizations'; see, for example, [16] or below for definitions. The dynamics, geometry and topology of such a system can be very non-trivial, in particular, due to the fact that different renormalization levels are largely independent.

Historically, the first example of an infinitely renormalizable one-dimensional map was probably the Feigenbaum period-doubling quadratic polynomial f_{c_F}, where $c_F = -1.4\ldots$ [6]. The Julia set of f_{c_F} is locally connected [7], which follows from so-called 'complex bounds', a compactness property of renormalizations. This is a key tool since [28], in particular, in proving the Feigenbaum–Coullet–Tresser universality conjecture [15, 17, 28]. Perhaps more striking for us are Douady and Hubbard's examples of infinitely renormalizable quadratic polynomials with non-locally connected Julia sets [3, 4, 9–11, 18, 27]. As for the Feigenbaum polynomial f_{c_F}, all the renormalizations of such maps are satellite, although, contrary to f_{c_F}, the combinatorics is unbounded (which, in turn, implies that those maps cannot have complex bounds [1]).

The dynamics of every holomorphic endomorphism of the Riemann sphere $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ classically splits $\hat{\mathbb{C}}$ into two subsets: the Fatou set $F(g)$ and its complement, the Julia set $J(g)$, where $F(g)$ is the maximal (possibly empty) open set where the sequence of iterates g^n, $n = 0, 1, \ldots$ forms a normal (that is, a precompact) family. See, for example, [2, 20] for the Fatou–Julia theory and [26] for a recent survey.

If g is a polynomial, then the Julia set $J(g)$ coincides with the boundary of the basin of infinity $A(\infty) = \{z \in \mathbb{C} | \lim_{n\to\infty} g^n(z) = \infty\}$ of g. The complement $\mathbb{C} \setminus A(g)$ is called the filled Julia set $K(g)$ of the polynomial g. The compact $K(g) \subset \mathbb{C}$ is connected if and only if it contains all critical points of g in the complex plane.

A quadratic polynomial f_c with connected filled Julia set $K(f)$ is *renormalizable* if, for some topological disks $U \Subset V$ around the critical point 0 of f_c and some $p \geq 2$ (period of renormalization), the restriction $F := f_c^p : U \to V$ is a proper branched covering map (called a polynomial-like map) of degree 2 and the non-escaping set $K(F) = \{z \in U : F^n(z) \in U \text{ for all } n \geq 1\}$ (called the filled Julia set of the polynomial-like map F) is connected. The map $F : U \to V$ is then a *renormalization* of f_c and the set $K(F)$ is a *'small' (filled) Julia set* of f_c. By the theory of polynomial-like mappings [5], there is a quasiconformal homeomorphism of $\mathbb{C}$, which is conformal on $K(F)$, that conjugates F on a neighborhood of $K(F)$ to another uniquely defined quadratic polynomial $f_{c'}$ with connected filled Julia set. If $f_{c'}$ is renormalizable by itself, then f_c is called twice renormalizable, etc. If f_c admits infinitely many renormalizations, it is called *infinitely renormalizable*. The renormalization $F = f_c^p$ is *simple* if any two sets $f^i(K(f))$, $f^j(K(F))$, $0 \leq i < j \leq p-1$, are either disjoint or intersect each other at a unique point which does not separate either of them. A simple renormalization f^{p_n} is called *primitive* if all sets $f^i(K_n)$, $i = 0, \ldots, p_n - 1$, are disjoints and *satellite* otherwise.

To state our main result, Theorem 1.1, let $f(z) = z^2 + c$ be infinitely renormalizable. Then its Julia set $J = J(f)$ coincides with the filled Julia set $K(f)$ and is a nowhere dense compact full connected subset of $\mathbb{C}$. Let $1 = p_0 < p_1 < \cdots < p_n < \cdots$ be the sequence of consecutive periods of simple renormalizations of f and $J_n \ni 0$ denote the 'small' Julia set of the n-renormalization (where $J_0 = J$). Then p_{n+1}/p_n is an integer, $f^{p_n}(J_n) = J_n$, for any n, and f-orbits of J_n,

$$\mathrm{orb}(J_n) = \bigcup_{j\geq 0} f^j(J_n) = \bigcup_{j=0}^{p_n-1} f^j(J_n),$$

$n = 0, 1, \ldots$, form a strictly decreasing sequence of compact subsets of $\mathbb{C}$. Let

$$J_\infty = \bigcap_{n\geq 0} \mathrm{orb}(J_n)$$

be the intersection of the orbits of the 'small' Julia sets J_n. For every n, repelling periodic orbits of f are dense in $\mathrm{orb}(J_n)$, while each component of J_∞ is wandering. In particular, J_∞ contains no periodic points of f.

Let

$$P = \overline{\{f^n(0) | n = 1, 2, \dots\}}$$

be the postcritical set of f. Clearly,

$$P \subset J_\infty.$$

Moreover, the critical point 0 is recurrent, hence,

$$P = \omega(0),$$

where $\omega(z)$ is the omega-limit set of a point $z \in J$.

We prove in [12] that J_∞ cannot contain any hyperbolic set. On the other hand, a hyperbolic set of a rational map always carries an invariant measure with a positive Lyapunov exponent. So a generalization of [12] would be that J_∞ never carries such a measure. Here we prove this generalization for a class of 'satellite' infinitely renormalizable quadratic polynomials.

THEOREM 1.1. *Suppose that $f(z) = z^2 + c$ admits infinitely many satellite renormalizations. Then $f : J_\infty \to J_\infty$ has no invariant probability measure with positive Lyapunov exponent.*

Remark 1.1. Conjecturally, the same conclusion should hold for any infinitely renormalizable $f(z) = z^2 + c$. One can show this assuming that the Julia set of f is locally connected (for example, this is the case if f admits complex bounds). Indeed, if $f : J_\infty \to J_\infty$ had an invariant probability measure with positive Lyapunov exponent, then, taking a typical point of this measure and repeating the proof of [13, Corollary 5.5], we would conclude that the Julia set of f is not locally connected (in fact, J_∞ contains a non-trivial continuum). Thus the only open case remains when f has only finitely many satellite renormalizations and J_∞ contains a non-trivial continuum.

Remark 1.2. For every rational map $f : \mathbb{C} \to \mathbb{C}$ (in particular, quadratic polynomial) and every invariant probability measure supported on the Julia set of f, the Lyapunov exponent is non-negative: see [22] (compare a remark preceding Corollary 1.3). On the other hand, if f is hyperbolic or non-uniformly hyperbolic (topologically Collet–Eckmann), Lyapunov exponents for all invariant probability measures supported on the Julia set are positive and bounded away from 0; see [24].

Let us comment on the behavior of the restriction map $f : J_\infty \to J_\infty$ where f as in Theorem 1.1. First, by [12], the postcritical set P must intersect the omega-limit set $\omega(x)$ of each $x \in J_\infty$. At the same time, the dynamics and topology of the further restriction $f : P \to P$ can vary. Indeed, there are infinitely renormalizable quadratic polynomials f with all renormalizations being of satellite type such that at least one of the following statements holds. (A more complete description of $f : P \to P$ should follow from the methods developed in [3].)

(1) $f : P \to P$ is not minimal. This case occurs in Douady–Hubbard type examples. Indeed, by the basic construction [18], J_∞ then contains a closed invariant set X (which is the limit set for the collection of α-fixed points of renormalizations) such that $0 \notin X$. By [12], $X \cap P$ is non-empty. Thus $X \cap P$ is an invariant non-empty proper compact subset of P.

(2) P is a so-called 'hairy' Cantor set; in particular, P contains uncountably many non-trivial continua. This case occurs following [3].

(3) P is a Cantor set and $f : P \to P$ is minimal; this happens whenever f either admits complex bounds (which then imply $J_\infty = P$) or is robust [16]. (The 'robustness' can happen without 'complex bounds', which follows from [3] combined with [1].) Under either of the two conditions, $f : P \to P$ is a minimal homeomorphism, which is topologically conjugate to $x \mapsto x + 1$ acting on the projective limit of the sequence of groups $\{\mathbb{Z}/p_n\mathbb{Z}\}_{n=1}^\infty$; in particular, $f : P \to P$ (hence, also $f : J_\infty \to J_\infty$, which follows from Corollary 1.3) is uniquely ergodic in this case.

Theorem 1.1 yields the following dichotomy about the measurable dynamics of $f : J \to J$ on the Julia set J of f. Recall that, by [22], any invariant probability measure on the Julia set of a rational function has non-negative exponents.

COROLLARY 1.3. *Let μ be an invariant probability ergodic measure of $f : J \to J$. Then either*

(i) $\operatorname{supp}(\mu) \cap J_\infty = \emptyset$ *and its Lyapunov exponent* $\chi(\mu) > 0$, *or*
(ii) $\operatorname{supp}(\mu) \subset P$ *and* $\chi(\mu) = 0$.

In particular, the set $J_\infty \setminus P$ is 'measure invisible' (see also Proposition 6.1 which is a somewhat stronger version of Corollary 1.3).

COROLLARY 1.4. *If f admits infinitely many satellite renormalizations, then*

$$\limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \le 0 \quad \textit{for any } x \in J_\infty, \tag{1.1}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(c)| = 0. \tag{1.2}$$

For the proof of Corollaries 1.3–1.4, see §6. The proof of Theorem 1.1 occupies §§2–5.

As in [12], we make heavy use of a general result of [23] on the accessibility, although the main idea of the proof is different. Indeed, in [12] we utilize the fact that the map cannot be one-to-one on an infinite hyperbolic set. In the present paper, to prove Theorem 1.1 we assign, loosely speaking, an external ray to a typical point of a hypothetical measure with positive exponent such that the family of such rays is invariant and has a controlled geometry. Given a satellite renormalization f^{p_n}, we use the measure and the above family of rays to choose a point x and build a special domain that covers a 'small' Julia set $J_{n,x} \ni x$ such that there is a univalent pullback of the domain by f^{p_n} along the renormalization that enters into itself, leading to a contradiction. The choice of x is 'probabilistic', that is, made from sets of positive measure, and the construction of the domain differs substantially depending on whether all satellite renormalizations of f are doubling or not.

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2. *Preliminaries*

Here we collect, for further reference and use throughout the paper, necessary notation and general facts. Statements (A)–(D) below are slightly adapted versions of (A)–(D) in [12] which are either well known [16, 19] or proved here.

Let $f(z) = z^2 + c$ be infinitely renormalizable. We retain the notation of the Introduction.

(A) Let G be the Green function of the basin of infinity $A(\infty) = \{z | f^n(z) \to \infty, n \to \infty\}$ of f with the standard normalization at infinity $G(z) = \ln |z| + O(1/|z|)$. The external ray R_t of the argument $t \in \mathbf{S}^1 = \mathbb{R}/\mathbb{Z}$ is a gradient line to the level sets of G that has the (asymptotic) argument t at ∞. $G(z)$ is called the (Green) level of $z \in A(\infty)$ and the unique t such that $z \in R_t$ is called the (external) argument (or angle) of z. A point $z \in J(f)$ is accessible if there is an external ray R_t which lands at (that is, converges to) z. Then t is called an (external) argument (angle) of z.

Let $\sigma : \mathbf{S}^1 \to \mathbf{S}^1$ be the doubling map $\sigma(t) = 2t (\mathrm{mod}\ 1)$. Then $f(R_t) = R_{\sigma(t)}$.

Every point a of a repelling cycle O_a of period p is the landing point of an equal number v, $1 \le v < \infty$, of external rays where v coincides with the number of connected components of $J(f) \setminus \{a\}$. Their arguments are permuted by σ^p according to a rational rotation number r/q (written in lowest terms); v/q is the number of cycles of rays landing at a. If $v/q \ge 2$, there is an alternative [19]: if $r/q = 0/1$, then $v = 2$ so that each of two external rays landing at a is fixed by f^p; if $r/q \ne 0/1$, that is, $q \ge 2$, then $v = q$, that is, the arguments of q rays landing at a form a single cycle of σ^p.

(B) All periodic points of f are repelling. Given a small Julia set J_n containing 0, the sets $f^j(J_n)$, $0 \le j < p_n$, are called small Julia sets of level n. Each $f^j(J_n)$ contains $p_{n+1}/p_n \ge 2$ small Julia sets of level $n+1$. We have $J_n = -J_n$. Since all renormalizations are simple, for $j \ne 0$, the symmetric companion $-f^j(J_n)$ of $f^j(J_n)$ can intersect the orbit $\mathrm{orb}(J_n) = \bigcup_{j=0}^{p_n-1} f^j(J_n)$ of J_n only at a single point which is periodic. On the other hand, since only finitely many external rays converge to each periodic point of f, the set J_∞ contains no periodic points. In particular, each component K of J_∞ is wandering, that is, $f^i(K) \cap f^j(K) = \emptyset$ for all $0 \le i < j < \infty$. All this implies that $\{x, -x\} \subset J_\infty$ if and only if $x \in K_0 := \bigcap_{n=1}^\infty J_n$.

Given $x \in J_\infty$, for every n, let $j_n(x)$ be the unique $j \in \{0, 1, \ldots, p_n - 1\}$ such that $x \in f^{j_n(x)}(J_n)$. Let $J_{n,x} = f^{j_n(x)}(J_n)$ be a small Julia set of level n containing x and $K_x = \bigcap_{n \ge 0} J_{n,x}$, a component of J_∞ containing x.

In particular, $K_0 = \bigcap_{n\ge 0} J_n$ is the component of J_∞ containing 0 and $K_c = \bigcap_{n=1}^\infty f(J_n)$, the component containing c.

Note that either $p_n - j_n(x) \to \infty$ as $n \to \infty$ or $p_n - j_n(x) = N$ for some $N \ge 0$ and all n, that is, $f^N(x) \in K_0$. This is so since the sequence of the sets J_n is non-increasing, hence $J_{n,x}$ is non-increasing, hence $p_n - j_n(x)$ (the time to reach J_n) is non-decreasing.

The map $f : K_x \to K_{f(x)}$ is two-to-one if $x = 0$ and one-to-one otherwise. Moreover, for every $y \in J_\infty$, $f^{-1}(y) \cap J_\infty$ consists of two points if $y \in K_c$ and consists of a single point otherwise. Denote

$$J'_\infty = J_\infty \setminus \bigcup_{j=-\infty}^{\infty} f^j(K_0).$$

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We conclude that $f: J'_\infty \to J'_\infty$ is a homeomorphism. Given $x \in J'_\infty$ and $m > 0$, denote $x_m = f^m(x)$ and

$$x_{-m} = f|_{J'_\infty}^{-m}(x),$$

that is, the only point $f^{-m}(x) \cap J_\infty$.

(C) Given $n \geq 0$, the map $f^{p_n}: f(J_n) \to f(J_n)$ has two fixed points: the separating fixed point α_n (that is, $f(J_n) \setminus \{\alpha_n\}$ has at least two components) and the non-separating β_n (so that $f(J_n) \setminus \beta_n$ has a single component).

For every $n > 0$, there are $0 < t_n < \tilde{t}_n < 1$ such that two rays R_{t_n} and $R_{\tilde{t}_n}$ land at the non-separating fixed point $\beta_n \in f(J_n)$ of f^{p_n} and the component Ω_n of $\mathbb{C} \setminus (R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n)$ which does not contain 0 has two characteristic properties [19]:

(i) Ω_n contains c and is disjoint with the forward orbit of β_n.

(ii) For every $1 \leq j < p_n$, consider arguments (angles) of external rays which land at $f^{j-1}(\beta_n)$. The angles split $\mathbf{S}^1$ into finitely many arcs. Then the length of any such arc is bigger than the length of the arc

$$S_{n,1} = [t_n, \tilde{t}_n] = \{t : R_t \subset \Omega_n\}.$$

Denote

$$t'_n = t_n + \frac{\tilde{t}_n - t_n}{2^{p_n}}, \quad \tilde{t}'_n = \tilde{t}_n - \frac{\tilde{t}_n - t_n}{2^{p_n}}.$$

The rays $R_{t'_n}, R_{\tilde{t}'_n}$ land at a common point $\beta'_n \in f^{-p_n}(\beta_n) \cap \Omega_n$. Introduce an (unbounded) domain U_n with boundary consisting of two curves $R_{t_n} \cup R_{\tilde{t}_n} \cup \beta_n$ and $R_{t'_n} \cup R_{\tilde{t}'_n} \cup \beta'_n$. Then $c \in U_n$ and $f^{p_n}: U_n \to \Omega_n$ is a two-to-one branched covering. Also,

$$f(J_n) = \{z : f^{kp_n}(z) \in \overline{U}_n, G(f^{kp_n}(z)) < 10, k = 0, 1, \ldots\}.$$

Let

$$s_{n,1} = [t_n, t'_n] \cup [\tilde{t}'_n, \tilde{t}_n]$$

so that $s_{n,1} \subset S_{n,1}$ and the argument of any ray to $f(J_n)$ lies in $s_{n,1}$.

Let us iterate this construction. Given $1 \leq j \leq p_n$, let $S_{n,j}$ be one of the two arcs of $\mathbf{S}^1$ with end points

$$t_{n,j} = \sigma^{j-1}(t_n), \tilde{t}_{n,j} = \sigma^{j-1}(\tilde{t}_n)$$

such that arguments of any ray to $f^j(J_n)$ lies in $S_{n,j}$. Let

$$s_{n,j} = \sigma^{j-1}(s_{n,1}) = [t_{n,j}, t'_{n,j}] \cup [\tilde{t}'_{n,j}, \tilde{t}_{n,j}].$$

where $t'_{n,j} = \sigma^{j-1}(t'_n)$, $\tilde{t}'_{n,j} = \sigma^{j-1}(\tilde{t}'_n)$. Then

$$s_{n,j} \subset S_{n,j}$$

and the argument of any ray to $f^j(J_n)$ lies in fact in $s_{n,j}$. Note that

$$t'_{n,j} - t_{n,j} = \tilde{t}_{n,j} - \tilde{t}'_{n,j} = \frac{\tilde{t}_n - t_n}{2^{p_n - j + 1}} < \tilde{t}_n - t_n < 1/2. \tag{2.1}$$

So $\sigma^{j-1}: s_{n,1} \to s_{n,j}$ is a homeomorphism and $s_{n,j}$ has two components ('windows') $[t_{n,j}, t'_{n,j}]$ and $[\tilde{t}'_{n,j}, \tilde{t}_{n,j}]$ of equal length.

Let $U_{n,j} = f^{j-1}(U_n)$ and $\beta_{n,j} = f^{j-1}(\beta_n)$. The domain $U_{n,j}$ is bounded by two rays $R_{t_{n,j}} \cup R_{\tilde{t}_{n,j}}$ converging to $\beta_{n,j}$ and completed by $\beta_{n,j}$ along with two rays $R_{t'_{n,j}} \cup R_{\tilde{t}'_{n,j}}$ completed by their common limit point $f^{j-1}(\beta'_n)$ where $t'_{n,j} = \sigma^{j-1}(t'_n)$, $\tilde{t}'_{n,j} = \sigma^{j-1}(\tilde{t}'_n)$.

By (i) and (ii), for a fixed n, the domains $U_{n,j}$, $1 \le j \le p_n$, are pairwise disjoint.

Let $U_{n,j-p_n}$ be a component of $f^{-(p_n-j)}(U_n)$ which is contained in $U_{n,j}$. Then

$$f^{p_n}: U_{n,j-p_n} \to U_{n,j} \tag{2.2}$$

is a two-to-one branched covering and

$$f^{j-1}(J_n) = \{z : f^{kp_n}(z) \in \overline{U}_{n,j-p_n}, G(f^{kp_n}(z)) < 10, k = 0, 1, \ldots\}.$$

Let $s^1_{n,j}$ be the set of arguments of rays entering $U_{n,j-p_n}$. Then $s^1_{n,j}$ consists of four components so that the σ^{p_n} map each of these components homeomorphically onto one of the 'windows' of $s_{n,j}$.

Furthermore, let

$$\Omega_{n,j} = f^{j-1}(\Omega_n).$$

Unlike the map (2.2), the map

$$f^{p_n}: U_{n,j} \to \Omega_{n,j} \tag{2.3}$$

is a two-to-one branched covering only assuming $f^{j-1}: \Omega_n \to \Omega_{n,j}$ is a homeomorphism, which holds if and only if $\sigma^{j-1}: S_{n,1} \to \sigma^{j-1}(S_{n,1})$ is a homeomorphism. In the latter case,

$$\sigma^{j-1}(S_{n,1}) = S_{n,j}.$$

Primitive versus satellite renormalizations. Let $n \ge 2$ and r_n/q_n be the rotation number of β_n. The next claim is well known; we include the proof for the reader's convenience.

LEMMA 2.1.

(1) *The renormalization f^{p_n} is primitive if and only if $r_n/q_n = 0/1$, the period of β_n is p_n and β_n is the landing point of exactly two rays and they are fixed by f^{p_n}.*

(2) *The points β_n, $n = 1, 2, \ldots$, are all different.*

(3) *f^{p_n} is satellite if and only if the α-fixed point α_{n-1} of $f^{p_{n-1}}: f(J_{n-1}) \to f(J_{n-1})$ coincides with the β-fixed point β_n of $f^{p_n}: f(J_n) \to f(J_n)$. In particular, $\bigcup_{j=0}^{q_n-1} f^{jp_{n-1}}(f(J_n)) \subset f(J_{n-1})$ and $p_n = q_n p_{n-1}$. Moreover, each of the p_{n-1} points of the orbit of β_n is the landing point of precisely q_n rays which are permuted by $f^{p_{n-1}}$ according to the rotation number r_n/q_n. Completed by the landing point, they split $\mathbb{C}$ into q_n 'sectors' such that the closure of each of them contains a unique 'small' Julia set of level n sharing a common point with the boundary of the 'sector'.*

Proof. (1) f^{p_n} is satellite if and only if $f(J_n)$ meets at β_n some other iterate of J_n, hence, $r_n/q_n \neq 0$, and vice versa.

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(2) Assume $\beta := \beta_n = \beta_m$ for some $0 \leq n < m$. As $p_n < p_m$, the period of β_m is smaller than p_n. It follows that $f(J_n)$ contains two small Julia sets of level m that meet at β, hence, β separates $f(J_n)$, a contradiction as β_n does not.

(3) By (1), f^{p_n} is satellite if and only if $r_n/q_n \neq 0$. Let $\tilde{p}_{n-1} = p_n/q_n$. Then $\tilde{p}_{n-1}$ is an integer and is equal to the period of β_n. It follows that the p_n sets $f(J_n)$, $f^2(J_n), \ldots, f^{p_n}(J_n)$ are split into $\tilde{p}_{n-1}$ connected closed subsets E_i, $i = 1, \ldots, \tilde{\tilde{p}}_{n-1}$ where $E_1 = \bigcup_{j=0}^{q_n-1} f^{j\tilde{p}_{n-1}}(f(J_n))$ and $E_i = f^{i-1}(E_1)$, $i = 1, 2, \ldots, \tilde{p}_{n-1}$. Moreover, $0 \in E_{p_{n-1}}$ and $f(E_i) = E_{i+1}, i = 1, \ldots, \tilde{p}_{n-1} - 1$, $f(E_{\tilde{p}_{n-1}}) = E_1$. By [16, Theorem 8.5], $f^{\tilde{p}_{n-1}}$ is a simple renormalization and the E_i, $i = 1, \ldots, \tilde{p}_{n-1}$, are subsets of its $\tilde{p}_{n-1}$ small Julia sets. Since $1 = p_0 < p_1 < \cdots$ are all consecutive periods of simple renormalizations, then $\tilde{p}_{n-1} = p_k$ for some $k < n$. Therefore, the β_n-fixed point of $f^{p_n} : f(J_n) \to f(J_n)$ is α_k-fixed point of $f^{p_k} : f(J_{p_k}) \to f(J_{p_k})$. As all renormalizations are simple, if $k < n - 1$ then that would imply that $\beta_n = \beta_{n-1} = \cdots = \beta_{k+1}$, a contradiction with (2). The claim about 'sectors' follows since each map f^j is one-to-one in a neighborhood of β_n and the closure of Ω_n contains a single 'small' Julia set $f(J_n)$ of level n sharing a common point with $\partial\Omega_n$. □

We need a more refined estimate provided the renormalization is not doubling. Assume f^{p_n} is satellite so that $p_{n-1} = p_n/q_n$, with $q_n \geq 2$, and the rotation number of β_n is $r_n/q_n \neq 0/1$.

LEMMA 2.2. *Assume f^{p_n} is satellite and $q_n = p_n/p_{n-1} \geq 3$, that is, f^{p_n} is not doubling. Then*

$$\sigma^{j-1} : S_{n,1} \to \sigma^{j-1} S_{n,1} \quad \textit{is a homeomorphism for } j = 1, \ldots, p_{n-1}(q_n - 2). \tag{2.4}$$

In particular, given $\zeta \in (0, 1/3)$, the length of $\sigma^{j-1} S_{n,1}$ tends to zero as $n \to \infty$ uniformly in $j = 1, \ldots, [\zeta p_n]$ (where $[x]$ is the integer part of $x \in \mathbb{R}$).

Moreover, for every $1 \leq j \leq p_{n-1}(q_n - 2)$, $S_{n,j} = \sigma^{j-1}(S_{n,1})$ and the map $f^{p_n} : U_{n,j} \to \Omega_{n,j}$ is a two-to-one branched covering such that

$$f^j(J_n) = \{z : f^{kp_n}(z) \in \overline{U}_{n,j}, G(f^{kp_n}(z)) < 10, k = 0, 1, \ldots\}.$$

Proof. Let $g = f^{p_{n-1}} : U_{n-1} \to \Omega_{n-1}$. Then g is a two-to-one covering of degree 2 and the critical value c.

(1) Recall that $s_{n-1,1} = [t_{n-1}, t'_{n-1}] \cup [\tilde{t}'_{n-1}, \tilde{t}_{n-1}]$ consists of two 'windows' so that $\sigma^{p_{n-1}}$ is an orientation-preserving homeomorphism of either 'window' onto $S_{n-1,1} = [t_{n-1}, \tilde{t}_{n-1}]$.

(2) Consider the q_n rays $L_1, \ldots, L_{q_n}$ to α_{n-1}. The map g is a local homeomorphism near α_{n-1} which permutes the rays to α_{n-1} according to the rotation number $\nu := r_n/q_n \neq 0, 1/2$. In particular, g maps any pair of adjacent rays to α_{n-1} onto another pair of adjacent rays to α_{n-1} (see Figure 1).

(3) Not all arguments of these rays lie in a single 'window' I of $s_{n-1,1}$ because otherwise, by (1), the set of those arguments would lie in the non-escaping set of an orientation-preserving homeomorphism $\sigma^{p_{n-1}} : I \to S_{n,1}$, which consists of a fixed point of this map, a contradiction with the fact that $q_n > 1$.

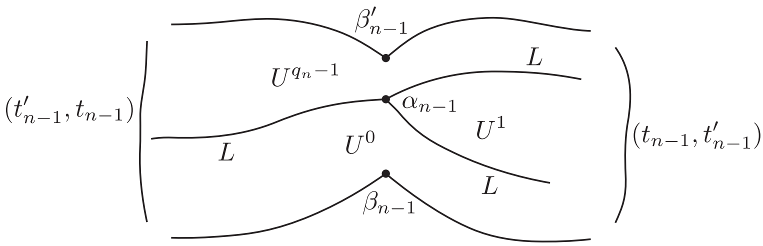

FIGURE 1. $q_n = 3$.

(4) The rays L_j split U_{n-1} into q_n disjoint domains U^j, $j = 0, 1, \ldots, q_n - 1$. By the 'ideal boundary' $\hat{\partial} U^j$ of U^j we will mean the usual (topological) boundary ∂U^j (in our case, the set of boundary rays completed by their landing points) along with the 'boundary at infinity' which is the set of arguments of rays entering U^j. Then define $\hat{g}$ on $\hat{\partial} U^j$ to be g on ∂U^j and $\sigma^{p_{n-1}}$ on the 'boundary at infinity' of U^j.

(5) By (3), one of the U^j, called U^0, has β_{n-1} in its boundary, and another one, called U^{q_n-1}, has β'_{n-1} in its boundary. In particular, the boundary of any other U^j, $j \neq 0, q_n - 1$, consists of a pair of adjacent rays to α_{n-1} whose arguments belong to a single 'window' of $s_{n-1,1}$. Therefore, by (1), the rest of the indices $j = 1, \ldots, q_n - 2$ can be ordered in such a way that $\hat{g} : \hat{\partial} U^j \to \hat{\partial} U^{j+1}$ is a one-to-one map for $j = 1, \ldots, q_n - 3$ (note that the 'boundary at infinity' of each U^j, $1 \leq j \leq q_n - 2$, consists of a single 'arc at infinity'). Therefore, $g : U^j \to U^{j+1}$ is a homeomorphism for $j = 1, \ldots, q_n - 3$. The map $\hat{g}$ on $\hat{\partial} U^{q_n-2}$ is also a one-to-one map on its image $W = g(U^{q_n-2})$ where W is bounded by two adjacent rays to α_{n-1}. W cannot contain U^0 because otherwise W would contain β'_{n-1}, a contradiction. Thus W must contain β'_{n-1}. That is, $g(U^{q_n-2})$ covers U^{q_n-1}.

Thus, for $j = 1, \ldots, q_n - 3$, $g : U^j \to U^{j+1}$ is a homeomorphism, and $g : U^{q_n-2} \to W$ is also a homeomorphism where the image $W = g(U^{q_n-2})$ covers U^{q_n-1} and has two common rays with the boundary of U^{q_n-1}.

(6) The critical value c of g has a unique preimage by g (the critical point of g). As $c \in \Omega_n \subset \Omega_{n-1}$ and Ω_n is bounded by two adjacent rays to α_{n-1}, $c \in U^i$ for some $i \in \{1, \ldots, q_n - 1\}$. If $i > 1$, then $i - 1 \geq 1$ while g would not be a homeomorphism of U^{i-1} on its image. This shows that $c \in U^1 = \Omega_n$.

Concluding, $U^j = g^{j-1}(\Omega_n)$, $j = 1, \ldots, q_n - 2$, in particular,

$$\Omega_n, g(\Omega_n), \ldots, g^{q_n-3}(\Omega_n) \subset U_{n-1}$$

and $g^{q_n-2} : \Omega_n \to g^{q_n-2}(\Omega_n)$ is a homeomorphism, that is, (2.4) holds. It implies the rest. □

(D) Given a compact set $Y \subset J(f)$ denote by $(\tilde{Y})_f$ (or simply $\tilde{Y}$, if the map is fixed) the set of arguments of the external rays which have their limit sets contained in Y. It follows from (C) that $\tilde{K}_c = \bigcap_{n=1}^{\infty} \{[t_n, t'_n] \cup [\tilde{t}'_n, \tilde{t}_n]\}$, that is, it is either a single-point set or a two-point set.

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Since $\tilde{K}_c$ contains at most two angles, K_c contains at most two different accessible points. More generally, given $x \in J'_\infty$, let

$$s_{n,j_n(x)} = [t_{n,j_n(x)}, t'_{n,j_n(x)}] \cup [\tilde{t}'_{n,j_n(x)}, \tilde{t}_{n,j_n(x)}].$$

Then $s_{n+1,j_{n+1}(x)} \subset s_{n,j_n(x)}$ so that

$$s_{\infty,x} := \bigcap_{n>0} s_{n,j_n(x)}$$

is not empty and consists of either one or two components. Since $p_n - j_n(x) \to \infty$ for $x \in J'_\infty$ we conclude using (2.1), that $s_{\infty,x}$ consists of either a single point or two different points. In particular, for any component K of J_∞ which is not one of $f^{-j}(K_0)$, $j \geq 0$, there are either one or two rays tending to K.

From now on, μ is an f-invariant probability ergodic measure supported in J_∞: supp $\mu \subset J_\infty$, and having a positive Lyapunov exponent

$$\chi(\mu) := \int \log |f'| \, d\mu > 0.$$

(E). We start with the following basic statement. Parts (i)–(ii) are easy consequences of the invariance of μ and (B), while (iii) is a part of Pesin's theory as in [**25**, Theorem 11.2.3] coupled with the structure of $f : J_\infty \to J_\infty$ (see (B)). Recall that $J'_\infty = J_\infty \setminus \bigcup_{j=-\infty}^{\infty} f^j(K_0)$.

PROPOSITION 2.3.
(i) *For every n and $0 \leq j < p_n$, $\mu(f^j(J_n)) = 1/p_n$.*
(ii) *μ has no atoms and $\mu(K) = 0$ for every component K of J_∞.*
(iii) *$\mu(J'_\infty) = 1$ and $f : J'_\infty \to J'_\infty$ is a μ-measure preserving homeomorphism.*

There exists a measurable positive function $\tilde{r}(x) > 0$ on J'_∞ such that for μ-almost every $x \in J'_\infty$, and all $n \in \mathbb{N}$, if x_{-n} is the unique point of J'_∞ with $f^n(x_{-n}) = x$, then a (univalent) branch $g_n : B(x, \tilde{r}(x)) \to \mathbb{C}$ of f^{-n} is well defined such that $g_n(x) = x_{-n}$,

Remark 2.4. The branch g_n of f^{-n} depends on n and x_{-n} but it should be clear from the context which points x and x_{-n} are meant.

Using Birkhoff's ergodic theorem and Egorov's theorem, Proposition 2.3 immediately implies (e_1)–(e_3) of the next corollary. The proof of (e_4)–(e_5) is given immediately after the result.

COROLLARY 2.5. *For every $\epsilon > 0$, there exist a closed set $E'_{\epsilon/2} \subset J'_\infty$ and constants $\rho = \rho(\epsilon) > 0$, $\kappa = \kappa(\epsilon) \in (0, 1)$ such that the following statements hold.*

(e_1) $\mu(E'_{\epsilon/2}) > 1 - \epsilon/2$.

(e_2) *There exists another closed set $\hat{E}_{\epsilon/2}$ such that $E'_{\epsilon/2} \subset \hat{E}_{\epsilon/2} \subset J'_\infty$ as follows. For every $x \in \hat{E}_{\epsilon/2}$ and every $m > 0$ there exists a (univalent) branch $g_m : B(x, 3\rho) \to \mathbb{C}$ of f^{-m} such that $g_m(x) = x_{-m}$ and $|g'_m(x_1)/g'_m(x_2)| < 2$, for every $x_1, x_2 \in B(x, 2\rho)$. Moreover, $m^{-1} \ln |g'_m(x)| \to -\chi(\mu)$ as $m \to \infty$ uniformly in $x \in E'_{\epsilon/2}$.*

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(e$_3$) *For every $x \in E'_{\epsilon/2}$ there exists a sequence of positive integers $n_j = n_j(x)$, $j = 1, 2, \ldots$, such that $j/n_j \geq \kappa$ and $f^{n_j}(x) \in \hat{E}_{\epsilon/2}$ for all j.*

(e$_4$) *Given $x \in J_\infty$ and $n \geq 0$, let $j_n(x)$ be the unique $1 \leq j < p_n$ such that $x \in f^j(J_n)$. Then $p_n - j_n(x) \to \infty$ as $n \to \infty$ uniformly in $x \in E'_{\epsilon/2}$.*

(e$_5$) *For $s_{n,j_n(x)} = [t_{n,j_n(x)}, t'_{n,j_n(x)}] \cup [\tilde{t}'_{n,j_n(x)}, \tilde{t}_{n,j_n(x)}]$, we have $s_{n+1,j_{n+1}(x)} \subset s_{n,j_n(x)}$ and*

$$|t_{n,j_n(x)} - t'_{n,j_n(x)}| = |\tilde{t}'_{n,j_n(x)} - \tilde{t}_{n,j_n(x)}| \to 0$$

as $n \to \infty$ uniformly in $x \in E'_{\epsilon/2}$.

Proof of (e$_4$)–(e$_5$). Assuming the contrary in (e$_4$), we find some N and sequences $(n_k) \subset \mathbb{N}$ and (x_k), $x_k \in E'_{\epsilon/2}$, such that $p_{n_k} - j_{n_k}(x_k) = N$ (see (B)), hence $x_k \in f^{-N}(J_{n_k})$, for all k. Since $E_{\epsilon/2}$ is closed, one can assume $x_k \to x \in E'_{\epsilon/2} \subset J'_\infty$. Hence, $x \in f^{-N}(K_0)$, a contradiction. Now for (e$_5$), using (e$_4$), $t'_{n,j_n(x)} - t_{n,j_n(x)} = \tilde{t}_{n,j_n(x)} - \tilde{t}'_{n,j_n(x)} < 1/2^{p_n - j_n(x)} \to 0$ uniformly in x.

3. *External rays to typical points*

We define a *telescope* essentially following [23]. Given $x \in J(f)$, $r > 0$, $\delta > 0$, $k \in \mathbb{N}$ and $\kappa \in (0, 1)$, an (r, κ, δ, k)-telescope at $x \in J$ is a collection of times $0 = n_0 < n_1 < \cdots < n_k$ and disks $B_l = B(f^{n_l}(x), r)$, $l = 0, 1, \ldots, k$, such that, for every $l > 0$: (i) $l/n_l > \kappa$; (ii) there is a univalent branch $g_{n_l} : B(f^{n_l}(x), 2r) \to \mathbb{C}$ of f^{-n_l} so that $g_{n_l}(f^{n_l}(x)) = x$ and, for $l = 1, \ldots, k$, $d(f^{n_{l-1}} \circ g_{n_l}(B_l), \partial B_{l-1}) > \delta$ (clearly, here $f^{n_{l-1}} \circ g_{n_l}$ is a branch of $f^{-(n_l - n_{l-1})}$ that maps $f^{n_l}(x)$ to $f^{n_{l-1}}(x)$). The trace of the telescope is a collection of sets $B_{l,0} = g_{n_l}(B_l)$, $l = 0, 1, \ldots, k$. We have $B_{k,0} \subset B_{k-1,0} \subset \cdots \subset B_{1,0} \subset B_{0,0} = B_0 = B(x, r)$.

By the first point of intersection of a ray R_t, or an arc of R_t, with a set E we mean a point of $R_t \cap E$ with the minimal level (if it exists).

THEOREM 3.1. [23] *Given $r > 0$, $\kappa \in (0, 1)$, $\delta > 0$ and $C > 0$, there exist $M > 0$, $\tilde{l}, \tilde{k} \in \mathbb{N}$ and $K > 1$ such that for every (r, κ, δ, k)-telescope the following statements hold. Let $k > \tilde{k}$. Let $u_0 = u$ be any point at the boundary of B_k such that $G(u) \geq C$. Then there are indexes $1 \leq l_1 < l_2 < \cdots < l_j = k$ such that $l_1 < \tilde{l}$, $l_{i+1} < K l_i$, $i = 1, \ldots, j - 1$, as follows. Let $u_k = g_{n_k}(u) \in \partial B_{k,0}$ and let γ_k be an infinite arc of an external ray through u_k between the point u_k and ∞. Let $u_{k,k} = u_k$ and, for $l = 1, \ldots, k - 1$, let $u_{k,l}$ be the first point of intersection of γ_k with $\partial B_{l,0}$. Then, for $i = 1, \ldots, j$,*

$$G(u_{k,l_i}) > M 2^{-n_{l_i}}.$$

The next proposition, a corollary of Theorem 3.1, is a key one.

PROPOSITION 3.1. *Given $\epsilon > 0$, there exists a closed set E_ϵ as follows. First, $\mu(E_\epsilon) > 1 - \epsilon$ and $E_\epsilon \subset E'_{\epsilon/2}$, where $E'_{\epsilon/2}$ is the set defined in (E) and satisfies* (e$_1$)–(e$_5$)*. There exists $r = r(\epsilon) > 0$ and, for each $\nu > 0$, there is $C(\nu) > 0$ as follows.*

(1) *Let $x \in E_\epsilon$. Then x is the landing point of an external ray $R_{t(x)}$ of the argument $t(x)$. Moreover, the first intersection of $R_{t(x)}$ with $\partial B(x, \nu)$ has level at least $C(\nu)$.*

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(2) *For each n, a branch* $g_n : B(x, 2r) \to \mathbb{C}$ *of* f^{-n} *is well defined such that* $g_n(x) = x_{-n}$, $|g_n'(x_1)/g_n'(x_2)| < 2$, *for every* $x_1, x_2 \in B(x, r)$ *and* $n^{-1} \ln |g_n'(x)| \to -\chi(\mu)$ *as* $m \to \infty$ *uniformly in* $x \in E_\epsilon$.

(3) *If* $x' = g_n(x) \in E_\epsilon$, *then* $f^n(R_{t(x')}) = R_{t(x)}$.

Proof. Statements (1)–(2) will hold already for the set $E'_{\epsilon/2}$ which follows from Theorem 3.1 as in [23] and uses only the fact that μ has a positive exponent; (3) will follow in our case as we shrink the set $E'_{\epsilon/2}$ a bit since each point $x \in J'_\infty$ admits at most two external arguments.

Here are details. Let $r = \rho(\epsilon)$ and $\kappa = \kappa(\epsilon)$ as in the properties (e_2)–(e_3) of the set $E'_{\epsilon/2}$. Then, by (e_2)–(e_3), there is $\delta > 0$ such that, for each k, every $x \in E'_{\epsilon/2}$ admits an (r, κ, δ, k)-telescope with the times $0 = n_0 < n_1 < n_2 < \cdots < n_k$ that appear in property (e_3) of $E'_{\epsilon/2}$. On the other hand, there exists $L_r > 0$ such that for every $z \in J(f)$ there is a point $u(z) \in \partial B(z, r)$ with the level $G(u(z)) > L_r$. This is so due to $\bigcup_{L>0}\{G(z) \geq L\} = A_\infty$. Given this $C = L_r$, let M, $\tilde{l}$ and $\tilde{k}$ be as in Theorem 3.1.

Let $x \in E'_{\epsilon/2}$ and $n_1 < n_2 < \cdots < n_k < \cdots$ as in (e_3). Fix $k > \tilde{k}$. Let $B_{k,0}(x) \subset B_{k-1,0}(x) \subset \cdots \subset B_{1,0}(x) \subset B_{0,0}(x)$ be the corresponding trace. By Theorem 3.1, there are $1 \leq l_{1,k}(x) < l_{2,k}(x) < \cdots < l_{j_k^x,k}(x) = k$ such that $l_{1,k}(x) < \tilde{l}$, $l_{i+1,k}(x) < K l_{i,k}(x)$, $i = 1, \ldots, j_k^x - 1$. Let $\gamma_k(x)$ be an arc of an external ray between the point $u_k(x) := g_{n_k}(u(f^{n_k}(x))$ and ∞. Let $u_{k,l}(x)$ be the first intersection of $\gamma_k(x)$ with $\partial B_{l,0}(x)$. Then, for $i = 1, \ldots, j_k^x - 1$,

$$G(u_{k,l_{i,k}}(x)) > M 2^{-n_{l_{i,k}(x)}} > M 2^{-l_{i,k}(x)/\kappa}. \tag{3.1}$$

For all $i = 1, \ldots, j_k^x - 1$,

$$i \leq l_{i,k}(x) < K^i \tilde{l}. \tag{3.2}$$

Denote by $t_k(x)$ the argument of an external ray that contains the arc $\gamma_k(x)$.

Now, given a sequence

$$k_1 < k_2 < \cdots < k_m < \cdots \tag{3.3}$$

such that $k_1 > \tilde{k}$, we get a sequence of arguments $t_{k_m}(x)$ and a sequence of arcs $\gamma_{k_m}(x)$ of external rays of the corresponding arguments $t_{k_m}(x)$. Passing to a subsequence in the sequence (k_m), if necessary, one can assume that $t_{k_m}(x) \to \tilde{t}(x)$, for some argument $\tilde{t}(x)$.

Fix any $\nu \in (0, r)$ and choose $\tilde{k}_0 > \tilde{k}$ such that

$$2 \exp(-K^{\tilde{k}_0 - 2} \tilde{l} \chi(\mu)) < \nu,$$

and let

$$C(\nu) = M (2^{-1/\kappa})^{\tilde{l} K^{\tilde{k}_0}}.$$

Then, by Theorem 3.1, for each $k_m > k_0$, the first intersection of the ray $R_{t_{k_m}}(x)$ with the boundary of $B(x, \nu)$ has level at least $C(\nu)$. It follows, for any $0 < C < C(\nu)$, that the sequence of arcs of the rays $R_{t_{k_m}(x)}$ between the levels C and $C(\nu)$ does not exit $B(x, \nu)$

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for all $k_m > k_0$. As $t_{k_m}(x) \to \tilde{t}(x)$, it follows that the arc of the ray $R_{\tilde{t}(x)}$ between levels C and $C(\nu)$ stays in $B(x, \nu)$ too. As $\nu > 0$ and $C \in (0, C(\nu))$ can be chosen arbitrary small, $R_{\tilde{t}(x)}$ must land at x and satisfy (1) with $t(x)$ replaced by $\tilde{t}(x)$.

Let us call the above procedure of getting $\tilde{t}(x)$ from the constants r, L_r, the point $x \in E'_{\epsilon/2}$, and the sequence (3.3) the $(r, L_r, x, (k_m))$-*procedure*.

Note that (2) is property (e_2) of the set $E'_{\epsilon/2}$.

In order to satisfy property (3), we shrink the set $E'_{\epsilon/2}$ and correct $\tilde{t}(x)$, changing it to some $t(x)$ (if necessary) as follows. Using Birkhoff's ergodic theorem and Egorov's theorem, choose a closed subset E_ϵ of $E'_{\epsilon/2}$ such that $\mu(E_\epsilon) > 1 - \epsilon$ and, for each $x \in E_\epsilon$, the set $\mathcal{N}(x) := \{N \in \mathbb{N} : f^N(x) \in E'_{\epsilon/2}\}$ is infinite. Note that $\mathcal{N}(x) \subset \{n_k\}_{k=1}^\infty$. We have proved that, for each $N \in \mathcal{N}(x)$, (1) holds for the point $f^N(x)$ instead of x; in particular, $\tilde{t}(f^N(x))$ is an argument of $f^N(x)$. On the other hand, by (D), each $y \in E_\epsilon$ admits at most two external arguments, hence, all possible external arguments of the forward orbit $f^n(x)$, $n \geq 0$, belong to at most two different orbits of $\sigma : S^1 \to S^1$. Hence, there is one of those orbits, $O = \{\sigma^n(t(x))\}_{n\geq 0}$ for some $t(x)$, such that the intersection $O \cap \{\tilde{t}(f^N(x)) : N \in \mathcal{N}(x)\}$ is an infinite set, so that $\tilde{t}(f^{n_{k_m(x)}}(x)) = \sigma^{n_{k_m(x)}}(t(x))$ for an infinite sequence $(k_m(x))_{m\geq 1}$.

Let us start over with the $(r/2, C(r/2), x, (k_m(x)))$-procedure for the point x and the sequence $\{k_j(x)\}$. Then, by construction, $t_{k_m(x)} = t(x)$ for all m, hence (1) holds with $t(x)$ instead of the previous $\tilde{t}(x)$. If $y \in E_\epsilon$ is any other point of the grand orbit $\{f^n(x) : n \in \mathbb{Z}\}$ (remember that $f : J'_\infty \to J'_\infty$ is invertible), the $(r/2, C(r/2), y, (k_m))$-procedure works for y with the same (perhaps, truncated) sequence $k_1(x) < k_2(x) < \cdots$, which ensures that (3) holds (for the corrected arguments) too. □

Remark 3.2. Given $t(x)$, we cannot just set $t(f^n(x)) = \sigma^n(t(x))$ to satisfy property (3) because this would change κ in the definition of the telescope, so we might lose property (1). Notice that correcting (flipping) $\tilde{t}(x)$ to $t(x)$ does not change $C(\nu)$ The same goes for flipping any $t(y)$ in the grand orbit of x. But the flipping can make $f^\ell(R_{t(y)}) = R_{t(f^N(x)}$ for $f^\ell(y) = f^N(x)$ where $N = n_{k_m}$ with $G(R_{t(f^\ell(y))}) \cap \partial B(f^\ell(y), r/2) > L_{r/2}$, thus yielding (3).

4. *Lemmas*

Recall that for any $z \in J'_\infty$ we define $z_m = f^m(z)$ for any $m \in \mathbb{Z}$. This makes sense since f is invertible on J'_∞; see (E).

LEMMA 4.1. *Let $z(k) \in \bigcup_{j=0}^{p_{n_k}-1} f^j(J_{n_k})$, where $n_k \nearrow \infty$.*

(a) *If $z(k) \to z$ then $z \in J_\infty$.*

(b) *$z \in J_{n,x} \cap J'_\infty$ yields $z_{\pm p_n} \in J_{n,x}$. If, in addition to (a), $z(k) \in J'_\infty$ for all k and $w(k) \to w$ where $w(k) = z(k)_{ep_{n_k}}$, where e is always either 1 or -1, then z and w are in the same component of J_∞.*

(c) *If $z(k) \in E_\epsilon$ for all k and $t(z(k)) \to t$ (where E_ϵ, $t(z(k))$ are defined in Proposition 3.1), then the ray R_t lands at the limit point z. In particular, given $\sigma > 0$, there is $\Delta(\sigma) > 0$ such that $|x_1 - x_2| < \sigma$ for some $x_1, x_2 \in E_\epsilon$ whenever $|t(x_1) - t(x_2)| < \Delta(\sigma)$.*

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Proof. (a) Assume the contrary. Then there is n such that $d := d(z, \bigcup_{j=0}^{p_n-1} f^j(J_n)) > 0$. As, for any $n_k \geq n$, $z(k) \in \bigcup_{j=0}^{p_{n_k}-1} f^j(J_{n_k})$, where the latter union is a subset of $\bigcup_{j=0}^{p_n-1} f^j(J_n)$, the distance between z and z_k is at least d, a contradiction.

(b) $z_{\pm p_n} \in J_{n,x}$ by combinatorics and definitions of points z_m. In particular, for every k, $z(k)$ and $w(k)$ are in the same component $f^{j_k}(J_{n_k})$ of $\bigcup_{j=0}^{p_{n_k}-1} f^j(J_{n_k})$. By (a), any limit set A of the sequence of compacts $(f^{j_k}(J_{n_k}))$ in the Hausdorff metric is a subset of J_∞. On the other hand, A is connected as each set $f^{j_k}(J_{n_k})$ is connected. This proves (b).

(c) We prove only the first claim as the second one directly follows from it. Fix any $\nu \in (0, r)$ and choose k_0 such that for any $k > k_0$, $B(z(k), \nu) \subset B(z, 11/10\nu)$. Then, by Proposition 3.1, part (1), for each $k > k_0$, the first intersection of the ray $R_{t(z_k)}$ with the boundary of $B(z, (11/10)\nu)$ has level at least $C(\nu)$. It follows, for any $0 < C < C(\nu)$, that the sequence of arcs of the rays $R_{t_{z_k}}$ between the levels C and $C(\nu)$ does not exit $B(z, (11/10)\nu)$ for all $k > k_0$. As $\nu > 0$ and $C \in (0, \tilde{C}(\nu))$ can be chosen arbitrary small, R_t must land at z. □

By Lemma 4.1(c), if the arguments $t(x), t(x')$ of $x, x' \in E_\epsilon$ are close then x, x' are close as well.

Definition 4.2. Given ϵ and ρ, we define δ as follows. First, for $\hat{r} \in (0, 1)$ and $\hat{C} > 0$, we define $\hat{\delta} = \hat{\delta}(\hat{r}, \hat{C}) > 0$. Namely, let $C_0 > 0$ be so that the distance between the equipotential of level C_0 and $J(f)$ is bigger than 1. Then $\hat{\delta} = \hat{\delta}(\hat{r}/2, \hat{C}) > 0$ is such that for any $C \in [\hat{C}, C_0]$, if w_1, w_2 lie on the same equipotential Γ of level C and the difference between external arguments of w_1, w_2 is less than $\hat{\delta}$, then the length of the shortest arc of the equipotential Γ between w_1 and w_2 is less than $\hat{r}/2$. Apply Lemma 4.1(c) with $\sigma = \rho/4$ and find the corresponding $\Delta(\rho/4)$. Let

$$\delta = \delta(\epsilon, \rho) := \min\left\{\hat{\delta}(\rho, C(\rho/2)), \Delta\left(\frac{\rho}{4}\right)\right\},$$

where $C(\nu)$ is defined in Proposition 3.1.

In the next two lemmas we construct curves with special properties. The idea is as follows. Let $x \in E_\epsilon \cap J_{n,x}$. Then $x_{-p_n} \in J_{n,x}$. It is easy to get a curve γ in $A(\infty)$. Begin with an arc from a point $b \in R_{t(x)}$ to $g_{p_n}(b)$ and then iterate this arc by g_{p_n}. In this way we get a curve γ such that $g^{p_n}(\gamma) \subset \gamma$, hence γ lands at a fixed point a of f^{p_n}. We show in the next lemma (in a more general setting) that if both points x, x_{-p_n} are either in the range of the covering (2.2) (condition (I)) or in the range of the covering (2.3) (condition (II)) then $a \in J_{n,x}$. This implies that a has to be the β-fixed point of $f^{p_n} : J_{n,x} \to J_{n,x}$. In Lemma 4.5, assuming additionally that f^{p_n} is satellite, we 'rotate' the curve γ by $g_{p_{n-1}}$ to put the set $J_{n,x}$ in a 'sector' bounded by γ and by its 'rotation'. In Lemmas 4.7–4.8 we consider the case of doubling for which condition (II) usually does not hold.

In what follows, we use the following notation: given $p, q \in \mathbb{N}$, let

$$E_{\epsilon,p,q} = \bigcap_{j=0}^{q-1} f^{jp}(E_\epsilon).$$

This is a closed subset of E_ϵ of points x such that $x_{-jp} \in E_\epsilon$ for $j = 0, 1, \ldots, q-1$. As $f : J'_\infty \to J'_\infty$ is a μ-automorphism, $\mu(E_{\epsilon,p,q}) > 1 - q\epsilon$. Notice that this bound is independent of p.

For every $n > 0$ consider the closed set $E_{\epsilon,p_n,q}$. Let $x \in E_{\epsilon,p_n,q}$. Denote for brevity

$$x^k := x_{-kp_n} \quad \text{and} \quad R^k := R_{t(x^k)}, \quad k = 0, 1, \ldots, q-1.$$

By Lemma 4.1(b), $x^k \in J_{n,x}$. Hence, $t(x^k) \in s_{n,j_n(x)} \subset S_{n,j_n(x)}$, $0 \le k \le q-1$.

Recall that for a semi-open curve $l : [0, 1) \to \mathbb{C}$, we say that l lands at, or tends to, or converges to a point $z \in \mathbb{C}$ if there exists $\lim_{t\to 1} l(t) = z$. Then $l(0)$, z are endpoints of the curve and $l(0)$ is called also the starting point of l.

LEMMA 4.3. *Fix $\epsilon > 0$ and consider the set E_ϵ with the corresponding constant $r(\epsilon) > 0$. Fix $\rho \in (0, r(\epsilon)/3)$. Let $\delta := \delta(\epsilon, \rho)$ from Definition 4.2. For every $q \ge 2$ there exist $\tilde{n}$, $\tilde{C}$ as follows.*

For every $n > \tilde{n}$ consider the closed set $E_{\epsilon,p_n,q}$. Fix $0 \le i < j \le q-1$. Assume, for an arbitrary n as above, that either (I) $t(x^j)$ and $t(x^i)$ belong to a single component of $s_{n,j_n(x)}$, or (II) the map $\sigma^{j_n(x)-1} : S_{n,1} \to S_{n,j_n(x)}$ is a homeomorphism and the length of the arc $S_{n,j_n(x)}$ is less than δ.

Then the following statements hold.

(a) *The map $f^{(j-i)p_n} : g_{(j-i)p_n}(B(x^i, \rho)) \to B(x^i, \rho)$ has a unique fixed point $a = a_n$ and $a \in J_{n,x}$.*

(b) *There is a semi-open simple curve*

$$\gamma_{p_n,q,i,j}(x) \subset B(x^i, \rho) \cap A(\infty)$$

such that:

(1) *it lands at a and $g_{(j-i)p_n}(\gamma_{p_n,q,i,j}(x)) \subset \gamma_{p_n,q,i,j}(x)$. Another end point b of $\gamma_{p_n,q,i,j}(x)$ lies in R^i and $G(b) > \tilde{C}/2$,*

(2) *$\gamma_{p_n,q,i,j}(x) = \bigcup_{l\ge 0} g^l_{(j-i)p_n}(L_0 \cup L_1)$ where the 'fundamental arc' $L_0 \cup L_1$ consists of an arc L_0 of an equipotential of the level at least $\tilde{C}/2$ that joins a point $b \in R^i$ with a point $b_1 \in R^j$, being extended by an arc L_1 of the ray R^j between points b_1 and $g_{(j-i)p_n}(b) \in R^j$; in particular, the Green function $G(y)$ at a point y is not increasing as y moves from b to a along $\gamma_{p_n,q,i,j}(x)$,*

(3) *the point a is the landing point of a ray $R(a)$ which is fixed by $f^{(j-i)p_n}$ and which is homotopic to $\gamma_{p_n,q,i,j}(x)$ through a family of curves in $A(\infty)$ with the fixed end point a.*

(4) *the arguments of all points of the curve $g_{(j-i)p_n}(\gamma_{p_n,q,i,j}(x))$ lie in a single component of $s^1_{n,j_n(x)}$ in case (I) and in a single component of $s_{n,j_n(x)}$ in case (II) (recall that $s^1_{n,j_n(x)}$ has four components and $s_{n,j_n(x)}$ has two components, see §2, (C)).*

Besides,

$$|a - x^j| \to 0 \quad \text{and} \quad \log \frac{|(g_{(j-i)p_n})'(x^j)|}{|(g_{(j-i)p_n})'(a)|} \to 0 \tag{4.1}$$

as $n \to \infty$, uniformly in x^j and q.

(c) *If* $j - i = 1$ *then* $a = \beta_{n,j_n(x)}$ *where* $\beta_{n,j_n(x)} = f^{j_n(x)-1}(\beta_n)$*, the non-separating fixed point of* $f^{p_n} : J_{n,x} \to J_{n,x}$*. Moreover,*

$$\chi(\beta_{n,j_n(x)}) := \frac{1}{p_n} \log |(f^{p_n})'(\beta_{n,j_n(x)})| = \frac{1}{p_n} \log |(f^{p_n})'(\beta_n)| \to \chi(\mu)$$

as $n \to \infty$.

Remark 4.4. Note that $a \notin J_\infty$ while $x, x^1, \ldots, x^{q-1} \in J_\infty$.

Proof. Fix n_0 such that, for every $n > n_0$ and $x \in E_\epsilon$, the length of each 'window' of $s_{n,j_n(x)}$ is less than δ. Therefore, for $n > n_0$, in either case (I), (II),

$$|t(x^i) - t(x^j)| < \delta, \tag{4.2}$$

which implies, in particular, that $|x^i - x^j| < \rho/4$.

Denote $G_n := g_{(j-i)p_n}$, which is a holomorphic univalent function in $B(x^i, \rho)$. Since g_m are uniform contractions there is n_1 such that $G_n(\overline{B(x^i, \rho)}) \subset B(x^i, \rho/2)$ whenever $n > n_1$. Let $\tilde{n} = \max\{n_0, n_1\}$.

Let also $\tilde{C} = C(\rho/2)$, where $C(\nu)$ is defined in Proposition 3.1.

Let $a = a_n$ be the unique fixed point of the latter map G_n. We construct the curve $\gamma_{p_n,q,i,j}(x)$ to the point a as follows. Firstly, joint a point $b \in R^i$, $G(b) = (3/4)\tilde{C}$, to a point $b_1 \in R^j$ by an arc L_0 of the equipotential $\{G(z) = (3/4)\tilde{C}\}$. By the choice of $\delta > 0$, $L_0 \subset B(x^i, \rho)$. Secondly, connect b_1 to the point $g_{(j-i)p_n}(b) \in R^j$ by an arc $L_1 \subset R^j$. Now let $\gamma_{p_n,q,i,j}(x) = \bigcup_{l \geq 0} g^l_{(j-i)p_n}(L_0 \cup L_1)$. Then properties (1), (2) in (b) are immediate and (3) follows from general properties of conformal maps. Now, by Proposition 3.1(2) and (4.2), for all n big enough, $x^j = g_{(j-i)p_n}(x^i) \in g_{(j-i)p_n}(B(x^i, \rho)) \subset B(x^i, \rho)$; moreover, the modulus of the annulus $B(x^i, \rho) \setminus g_{(j-i)p_n}(B(x^i, \rho))$ tends to ∞ as $n \to \infty$. Therefore, (4.1) follows from Koebe, see e.g. [**2**, Section 1.1], and Proposition 3.1(2).

It remains to show property (3) and that $a \in J_{n,x}$. Consider case (II), which is equivalent to saying that the map $\sigma^{p_n} : s \to S_{n,j_n(x)}$ is a homeomorphism on each of two components s of $s_{n,j_n(x)}$. Let Λ be the set of arguments of points of the curve $\Gamma := g_{(j-i)p_n}(\gamma_{p_n,q,i,j}(x))$. Let s be a component that contains $t(x^j)$. Assume, by contradiction, that Λ contains t which is in the boundary of s. Then t is the argument of a point of $G_n^l(L_0)$, for some $l \geq 1$, and hence $\sigma^{l(j-i)p_n}(t)$ is simultaneously the argument of a point of L_0 and in the boundary of $S_{n,j_n(x)}$, a contradiction. Case (I) is similar. Property (3) is verified. In fact, we have proved more: for $k = 0, 1, \ldots, j - i - 1$, the set $\sigma^{kp_n}(\Lambda)$ is a subset of a single (depending on k) component of $s_{n,j_n(x)}$ in case (II) and a single component of $s^1_{n,j_n(x)}$ in case (I). This implies that all points $f^{kp_n}(a)$, $0 \leq k \leq j - i - 1$, of the cycle of f^{p_n} containing a belong to the closure of $U_{n,j_n(x)}$ in the case (II) and to the closure of $U_{n,j_n(x)-p_n}$ in the case (I). Therefore, this cycle lies in $J_{n,x}$, in particular, $a \in J_{n,x}$.

Proof of (c). If $j - i = 1$ then a is a fixed point of $f^{p_n} : J_{n,x} \to J_{n,x}$ and, moreover, the ray $R(a)$ lands at a and is fixed by f^{p_n}. Hence, the rotation number of a with respect to the map $f^{p_n} : J_{n,x} \to J_{n,x}$ is zero. On the other hand, $\beta_{n,j_n(x)}$ is the only such a fixed point, that is, $a = \beta_{n,j_n(x)}$ as claimed. Then (4.1) implies that $\chi(\beta_{n,j_n(x)}) \to \chi(\mu)$. □

For the rest of the paper, let us fix Q, ϵ, r, ρ, $\tilde{n}$, $\tilde{C}$ and δ as follows: $Q \in \mathbb{N}$, $Q > 3$, *is such that*

$$Q > 4 \log 2/\chi(\mu).$$

This choice is motivated by the following fact [8, 14, 21]: if a repelling fixed point z of f^n is the landing point of q rays, then $\chi(z) := (1/n) \log |(f^n)'(z)| \le (2/q) \log 2$. Hence, if $\chi(z) > \chi(\mu)/2$, then $q < Q$.

Furthermore, fix $\epsilon > 0$ such that $2^{100} Q\epsilon < 1$, apply Proposition 3.1 and Lemma 4.3 and find, first, $r = r(\epsilon)$, then fix $\rho \in (0, r/32)$ and find the corresponding $\tilde{n}$, $\tilde{C}$ and δ.

Let

$$X_n = E_{\epsilon,p_n,4} \cap E_{\epsilon,p_{n-1},Q} = \bigcap_{i=0}^{3} f^{ip_n}(E_\epsilon) \bigcap_{k=0}^{Q-1} f^{kp_{n-1}}(E_\epsilon).$$

Let us analyze several possibilities.

LEMMA 4.5. *There is $n_* > \tilde{n}$ as follows. Let $n > n_*$ and $x \in X_n$. Consider $J_{n,x} = f^{j_n(x)}(J_n) \subset f^{j_{n-1}(x)}(J_{n-1})$ so that $x \in J_{n,x}$.*

Let $x^0 = x$ and $x^1 = x_{-p_n}$. Assume that either (I) $t(x^0)$, $t(x^1)$ belong to a single component of $s_{n,j_n(x)}$, or (II) the map $\sigma^{j_n(x)-1} : S_{n,1} \to S_{n,j_n(x)}$ is a homeomorphism and the length of the arc $S_{n,j_n(x)}$ is less than δ.

Then the following statements hold.

(i) $\chi(\beta_{n,j_n(x)}) = \chi(\beta_n) \to \chi(\mu)$ *as* $n \to \infty$ *and* $\chi(\beta_n) > \chi(\mu)/2$ *for* $n > n_*$.

(ii) *Assume that f^{p_n} is satellite, that is (by Lemma 2.1). β_n has period p_{n-1}, $q_n \ge 2$ with rotation number r_n/q_n of β_n, and $\beta_{n,j_n(x)}$ is the α (that is, separating) fixed point of $f^{p_{n-1}} : J_{n-1,x} \to J_{n-1,x}$. Then $q_n < Q$ and*

$$|\beta_{n,j_n(x)} - x_{-kp_{n-1}}| \to 0,\ n \to \infty,\ \text{uniformly in } x \in X_n,\ 1 \le k \le q_n. \tag{4.3}$$

There exist two simple semi-open curves $\gamma(x)$ and $\tilde{\gamma}(x)$ that satisfy the following properties.

(1) *$\gamma(x)$ and $\tilde{\gamma}(x)$ tend to $\beta_{n,j_n(x)}$ and $\gamma(x), \tilde{\gamma}(x) \subset B(x^0, \rho) \cap A(\infty)$.*

(2) *$\gamma(x), \tilde{\gamma}(x)$ consist of arcs of equipotentials and external rays. The starting point $b_1 = b_1(x)$ of $\gamma(x)$ lies in an arc of $R_{t(x^1)}$ and the starting point $\tilde{b}_1 = \tilde{b}_1(x)$ of $\tilde{\gamma}(x)$ lies in an arc of $R_{t(\tilde{x})}$ where $\tilde{x} = x_{-ip_{n-1}}$ for some $i = i(x) \in \{1, \dots, q_n - 1\}$, such that levels of b_1 and $\tilde{b}_1$ are equal and at least $\tilde{C}/4$.*

(3) *One of the two curves (say, $\gamma(x)$) is homotopic, through curves in $A(\infty)$ tending to $\beta_{n,j_n(x)}$, to the ray $R_{t_{n,j_n(x)}} = f^{j_n(x)-1}(R_{t_n})$, and the other one to the ray $R_{\tilde{t}_{n,j_n(x)}} = f^{j_n(x)-1}(R_{\tilde{t}_n})$.*

(4) *$\gamma(x), \tilde{\gamma}(x) \subset U_{n-1,j_{n-1}(x)}$.*

(5) *$\gamma(x) \subset U_{n,j_n(x)}$, $\tilde{\gamma}(x) \subset U_{n,j_n(\tilde{x})}$, in particular, $\gamma(x), \tilde{\gamma}(x)$ are disjoint, being completed by their common limit point $\beta_{n,j_n(x)}$ and two other arcs: an arc of the ray $R_{t(x^1)}$ from $b_1 \in \gamma(x)$ to ∞ and an arc of the ray $R_{t(\tilde{x})}$ from $\tilde{b}_1 \in \tilde{\gamma}(x)$ to ∞, they split the plane into two domains such that one of them contains $I := J_{n,x} \setminus \beta_{n,j_n(x)}$ and the other one contains all $q_n - 1$ other different iterates $f^{kp_{n-1}}(I)$, $1 \le k \le q_n - 1$. The intersection of closures of all those q_n sets consists of the fixed point $\beta_{n,j_n(x)}$ of $f^{p_{n-1}}$.*

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Remark 4.6. Beware that the point x that determines both curves $\gamma(x)$, $\tilde{\gamma}(x)$ does not belong to either of these curves.

Proof. Statement (i) follows from Lemma 4.3 where we take $i = 0$, $j = 1$. Fix $n_* > \tilde{n}$ such that $\chi(\beta_n) > \chi(\mu)/2$ for all $n > n_*$.

Let us prove (ii). Here we build a 'flower' of arcs at the fixed β of the satellite f^{p_n}, starting with an arc which is fixed by f^{p_n}, and then 'rotate' this arc by a branch of $f^{-p_{n-1}}$ (for which the same β point is also a fixed point; see (C)). Let $\gamma'(x) := \gamma_{p_n,1,0,1}(x)$ where the latter curve is defined in Lemma 4.3. Then properties (1)–(3) of the curve $\gamma(x)$ are also satisfied for $\gamma'(x)$. In particular, $\gamma'(x)$ is homotopic to $R_{t_{n,j_n(x)}}$.

As both $\tilde{t}_{n,j_n(x)}, t_{n,j_n(x)}$ are external arguments of $\beta_{n,j_n(x)}$ which is a p_{n-1}-periodic point of f, there is $i \in \{1, \ldots, q_n - 1\}$ such that $\sigma^{ip_{n-1}}(\tilde{t}_{n,j_n(x)}) = t_{n,j_n(x)}$. Now we use that $x \in E_{\epsilon,p_{n-1},Q}$ and that $q_n < Q$ to prove (4.3). Indeed, for each $k = \{1, \ldots, q_n\}$, since $f : J'_\infty \to J'_\infty$ is a homeomorphism and $x_{-kp_{n-1}} \in E_\epsilon$, we have $g_{p_n} = g_{(q_n-k)p_{n-1}} \circ g_{kp_{n-1}}$. Hence, if $\beta' = g_{kp_{n-1}}(\beta_{n,j_n(x)})$, then $\beta_{n,j_n(x)} = g_{(q_{n-1}-k)p_{n-1}}(\beta')$, implying that $\beta' = f^{(q_n-k)p_{n-1}}(\beta_{n,j_n(x)}) = \beta_{n,j_n(x)}$. Then $\beta_{n,j_n(x)}, x_{-kp_{n-1}} \in g_{kp_{n-1}}(B(x, \rho))$ which, along with Proposition 3.1, part (2), implies (4.3).

In turn, (4.3) implies that, provided n is big, $g_{kp_{n-1}} : B(y, \rho/2) \to B(y, \rho/2)$ uniformly in $k = 0, 1, \ldots, q_n$ where y is either $\beta_{n,j_n(x)}$ or $x_{-kp_{n-1}}$.

Now we consider a curve $g_{i\tilde{p}_n}(\gamma'(x))$ that starts at $x_{-i\tilde{p}_n}$ and tends to $\beta_{n,j_n(x)}$. By Proposition 3.1 coupled with (4.3), one can join $x_{-ip_{n-1}}$ by an arc of the ray $R_{t(x_{-ip_{n-1}})}$ inside of $B(x, \rho/2)$ up to a point of level $\tilde{C}/4$. This will be the required curve $\tilde{\gamma}(x)$. To get the curve $\gamma(x)$ we modify $\gamma'(x) = \gamma_{p_n,1,0,1}(x) = \bigcup_{l\geq 0} g^l_{p_n}(L_0 \cup L_1)$ by cutting off the arc L_0 of an equipotential: $\gamma(x) = \gamma'(x) \setminus L_0$ (see Lemma 4.3 for details about L_0). Properties (1)–(5) follow. □

Given a point $x = x^0$ and n such that $x \in f^j(J_n) \cap E_{\epsilon,p_n,1}$, where $j = j_n(x)$, let $x^1 = x_{-p_n}$ and $t(x^0)$, $t(x^1)$ be the arguments of x^0, x^1 as in Proposition 3.1. We call x *n-friendly* if $t(x^0)$ and $t(x^1)$ lie in the same component of $s_{n,j}$ and *n-unfriendly* otherwise (or simply *friendly* and *unfriendly* if n is clear from the context). The name reflects the fact that for an n-friendly point x condition (I) of Lemma 4.5 always holds for $x^1 = x$ and $x^2 = x_{-p_n}$, so Lemma 4.5 always applies.

When the rotation number of α_n is equal to $1/2$ we have the following lemma.

LEMMA 4.7. *There is $\tilde{C}_3 > 0$ (depending only on fixed ϵ and ρ) as follows. Suppose that, for some $n > \tilde{n}$, the rotation number of the separating fixed point α_n is equal to $1/2$. Let $z = z^0 \in f^j(J_n) \cap E_{\epsilon,p_n,3}$ and $z^i = z_{-ip_n}$, $i = 1, 2, 3$. Assume that all three points z^0, z^1, z^2 are n-unfriendly.*

Then there exist two (semi-open) curves $\gamma_n^{1/2}(z)$ and $\tilde{\gamma}_n^{1/2}(z)$ consisting of arcs of rays and equipotentials with the following properties.

(i) $\gamma_n^{1/2}(z) \subset B(z, \rho)$, $\tilde{\gamma}_n^{1/2}(z) \subset B(z^1, \rho)$. *Moreover, the arguments of points of $\gamma_n^{1/2}(z)$ lie in one 'window' of $s_{n,j}$ while the arguments of points of $\tilde{\gamma}_n^{1/2}(x)$ lie in the other 'window' of $s_{n,j}$.*

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(ii) $\gamma_n^{1/2}(z)$ *and* $\tilde{\gamma}_n^{1/2}(z)$ *converge to a common point* $\alpha^*_{n,j}$ *which is a fixed point of* $f^{p_n}: f^j(J_n) \to f^j(J_n)$ *(that is,* $\alpha^*_{n,j}$ *is either the non-separating fixed point* $\beta_{n,j}$ *or the separating fixed point* $\alpha_{n,j}$.

(iii) *The starting points of* $\gamma_n^{1/2}(z)$, $\tilde{\gamma}_n^{1/2}(z)$ *have equal Green level. which is bigger than* $\tilde{C}_3$.

(iv) $z^k - \alpha^*_{n,j} \to 0$, $0 \le k \le 3$, *as* $n \to \infty$.

Proof. As $z \in E_\epsilon$, the lengths of the 'windows' of $s_{n,j_n(z)}$ tend uniformly to zero as $n \to \infty$. It follows from the definition of friendly and unfriendly points that $t(z^0)$, $t(z^2)$ are in one 'window' of $s_{n,j}$ and $t(z^1)$, $t(z^3)$ are in the other 'window' of $s_{n,j}$. Therefore, condition (I) of Lemma 4.3 holds for each pair z^0, z^2 and z^1, z^3. Now, apply Lemma 4.3 to $z \in E_{\epsilon,p_n,3}$, first with $i=0$, $j=2$, and then with $i=1$, $j=3$. Let $\gamma_n^{1/2}(z) = \gamma_{p_n,3,0,2}(z)$ and $\tilde{\gamma}_n^{1/2}(z) = \gamma_{p_n,3,1,3}(z)$. Then (i) and (iii) hold. To check (ii), note that these curves converge to some points $\alpha, \tilde{\alpha} \in f^j(J_n)$ which are fixed by f^{2p_n} On the other hand, since the rotation number of α_n is $1/2$, $f^{p_n}: f^j(J_n) \to f^j(J_n)$ has no 2-cycle. Therefore, one must have either $\alpha = \tilde{\alpha} = \beta_{n,j}$ or $\alpha = \tilde{\alpha} = \alpha_{n,j}$, that is, (ii) holds too. As $t(z^0) - t(z^2) \to 0$ and $t(z^1) - t(z^3) \to 0$ as $n \to \infty$, $z^0 - z^2$, $z^1 - z^3 \to 0$ also, by Lemma 4.1. Besides, by (4.1), $z^2 - \alpha$, $z^3 - \tilde{\alpha} \to 0$ as $n \to \infty$. As $\alpha = \tilde{\alpha} = \alpha^*_{n,j}$, (iv) also follows. □

The following lemma is a consequence of Lemmas 4.3 and 4.7.

LEMMA 4.8. *Let* $n > \tilde{n}$. *Assume that* f^{p_n} *is satellite and doubling, that is,* $\beta_n = \alpha_{n-1}$ *and the rotation number of* α_{n-1} *is equal to* $1/2$ *(in particular,* $p_n = 2p_{n-1}$*). For some* $1 \le j \le p_{n-1}$, *denote* $J := f^j(J_{n-1})$. *Let* $J^1 := f^j(J_n)$, $J^0 := f^{j+p_{n-1}}(J_n)$ *be the two small Julia sets of the next level n which are contained in J (note that* J^0 *contains the critical point and* J^1 *contains the critical value of the map* $F := f^{p_{n-1}}: J \to J$*). Let* $x \in J^1 \cap E_\epsilon$ *be such that all five of its forward iterates* $x_{kp_{n-1}} = F^k(x) \in E_\epsilon$, $k = 1, 2, 3, 4, 5$. *Then there exist two simple semi-open curves* $\Gamma_n^{1/2}(x)$, $\Gamma_n^{1/2}(x)$ *consisting of arcs of rays and equipotentials that satisfy essentially conclusions of the previous lemma where n is replaced by* $n-1$, *that is, the following statements hold.*

(i) $\Gamma_n^{1/2}(x), \tilde{\Gamma}_n^{1/2}(x) \subset B(x, 3/2\rho)$. *Moreover, the arguments of points of* $\Gamma_n^{1/2}(x)$ *lie in one 'window' of* $s_{n-1,j_{n-1}(x)}$ *while the arguments of points of* $\tilde{\Gamma}_n^{1/2}(x)$ *lie in the other 'window' of* $s_{n-1,j_{n-1}(x)}$.

(ii) $\Gamma_n^{1/2}(x)$ *and* $\tilde{\Gamma}_n^{1/2}(x)$ *converge to a common point* $\beta^*_{n-1,j_{n-1}(x)}$ *which is a fixed point of* $f^{p_{n-1}}: f^j(J_{n-1}) \to f^j(J_{n-1})$ *(that is,* $\beta^*_{n-1,j_{n-1}(x)}$ *is either the non-separating fixed point* $\beta_{n-1,j_{n-1}(x)}$ *or the separating fixed point* $\alpha_{n-1,j_{n-1}(x)}$.

(iii) *The starting points of* $\Gamma_n^{1/2}(x)$, $\tilde{\Gamma}_n^{1/2}(x)$ *have equal Green level which is bigger than* $\tilde{C}_3$.

(iv) $x_{kp_{n-1}} - \beta^*_{n-1,j_{n-1}(x)} \to 0$, $0 \le k \le 3$ *as* $n \to \infty$ *uniformly in x.*

Remark 4.9. The condition $F^k(x) \in E_\epsilon$, $0 \le k \le 5$, is equivalent to $x \in f^{-5p_{n-1}}(E_{\epsilon,p_{n-1},6})$.

Proof. To fix the idea, let us replace $f^{p_{n-1}}: f^j(J_{n-1}) \to f^j(J_{n-1})$, using a conjugacy to a quadratic polynomial, by a quadratic polynomial (also denoted by F) so that now

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$F: J \to J$ where $J = J(F)$ and F^2 is satellite with two small Julia sets J^0, J^1 that meet at the α-fixed point of F and rays of arguments $1/3, 2/3$ land at α. Here $0 \in J^0$, $F(0) \in J^1$, $F: J^1 \to J^0$ is a homeomorphism, while $F: J^0 \to J^1$ is a two-to-one map. If a ray R_t of F has its accumulation set in J^1 then $t \in [1/3, 5/12] \cup [7/12, 2/3]$ and if R_t accumulates in J^0 then $t \in [1/6, 1/3] \cup [2/3, 5/6]$. This implies that if R_t lands at $x \in J^1$ and t lies in one of the two 'windows' $[0, 1/2)$, $(1/2, 1]$ then $R_{\sigma(t)}$ lands at J^0 where $\sigma(t)$ must be in a different 'window' (in other words, the points of J^0 are 'unfriendly'). Returning to $f^{p_{n-1}}$, this means, for $x \in J^1$, that $t(x)$, $t(F(x))$ are always in different components (where by 'component' we mean a component of $s_{n-1,j}$). Besides, for $y \in J_\infty \cap J$, y and $F(y)$ are always in different J^i, $i = 0, 1$. This leaves us with just the following possibilities.

(i) $t(F(x))$, $t(F^2(x))$ are in different components. This implies that $t(x)$, $t(F(x))$ are in different components and $t(F(x))$, $t(F^2(x))$ are in different components, that is, points $F^3(x)$, $F^2(x)$, $F(x)$ are all unfriendly.

(ii) $t(F(x))$, $t(F^2(x))$ are in the same components. There are two subcases.

(ii$'$) $t(F^3(x))$, $t(F^4(x))$ are in different components, that is, (i) holds with x replaced by $F^2(x)$, which implies that $F^5(x)$, $F^4(x)$, $F^3(x)$ are all unfriendly.

(ii$''$) $t(F^3(x))$, $t(F^4(x))$ are in the same component which then means that $F^2(x)$ and $F^4(x)$ are both friendly.

In cases (i) and (ii$'$), apply Lemma 4.7 with $n-1$ instead of n to $z = F^3(x)$ and to $z = F^5(x)$, respectively, letting $\Gamma_n^{1/2}(x) = \gamma_{n-1}^{1/2}(F^3(x))$, $\tilde\Gamma_n^{1/2}(x) = \tilde\gamma_{n-1}^{1/2}(F^3(x))$ and $\Gamma_n^{1/2}(x) = \gamma_{n-1}^{1/2}(F^5(x))$, $\tilde\Gamma_n^{1/2}(x) = \tilde\gamma_{n-1}^{1/2}(F^5(x))$, respectively. In case (ii$''$), apply Lemma 4.3 with p_{n-1}, $q = 1$, $i = 0$, $j = 0$, first to the point $F^2(x)$ and then to the point $F^4(x)$, letting $\Gamma_n^{1/2}(x) = \gamma_{p_{n-1},1,0,1}(F^2(x))$, $\tilde\Gamma_n^{1/2}(x) = \gamma_{p_{n-1},1,0,1}(F^4(x))$. □

5. *Proof of Theorem 1.1*

Every invariant probability measure with positive Lyapunov exponent has an ergodic component with positive exponent. So let μ be such an ergodic f-invariant measure component supported in J_∞. First, we have the following general remark.

Remark 5.1. Given $x \in J'_\infty$ such that $\tilde r(x) > 0$ as in Proposition 2.3, and given n, the set $J_{n,x} = f^{j_n(x)}(J_n)$ cannot be covered by $B(x, \tilde r(x))$ because otherwise the branch $g_{p_n}: B(x, \tilde r(x)) \to \mathbb{C}$ of f^{-p_n}, which sends x to $x_{-p_n} \in J_{n,x}$, meets the critical value along the way so cannot be well defined. Thus $\operatorname{diam} J_{n,x} > \tilde r(x)$, for each n, and $\operatorname{diam} K_x = \lim \operatorname{diam} J_{n,x} \geq \tilde r(x)$. In particular, $\operatorname{diam} J_{n,x} \geq r(\epsilon)$ for all $x \in E_\epsilon$ and n.

We need to prove that f has finitely many satellite renormalizations. Assuming the contrary, let $\mathcal{S}$ be an infinite subsequence such that f^{p_n} is a satellite renormalization of f for each $n \in \mathcal{S}$.

We arrive at a contradiction by considering, roughly speaking, two alternative situations. In the first one, we find a point $x \in E_\epsilon$, n, and two curves in $B \cap A(\infty)$ where $B := B(x, \tilde r(x))$ that tend to the β-fixed points of $J_{n,x}$ such that the other ends of the curves can be joined by an arc of equipotential in B, thus 'surrounding' $J_{n,x}$ by a 'triangle' in B which would be a contradiction as in Remark 5.1. The second situation is when the first one does not occur. Then we use several curves to 'surround' $J_{n,x}$ by a 'quadrilateral' in B,

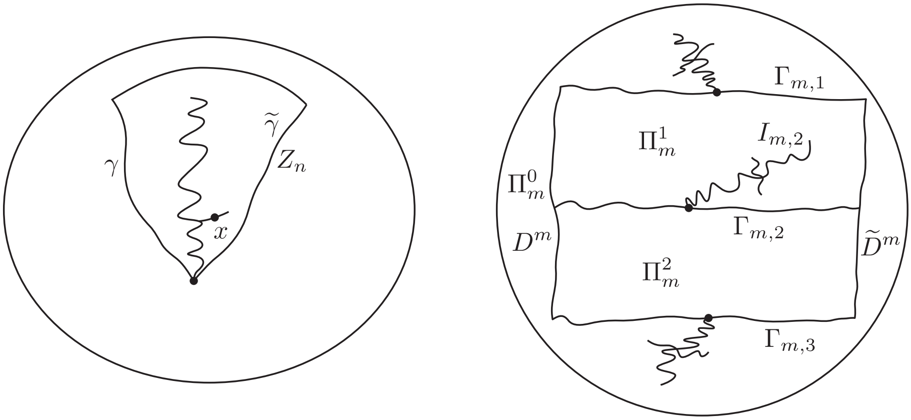

FIGURE 2. Left: cases A and B1. Right: case B2.

ending by the same conclusion. The curves we use have been constructed in Lemmas 4.5 and 4.8.

The first situation happens in cases A and B1 below, and the second one in B2, see Figure 2.

Case A: *$\mathcal{S}$ contains an infinite sequence of indices of non-doubling renormalizations.* Passing to a subsequence, one can assume that f^{p_n} is satellite and not doubling for every $n \in \mathcal{S}$.

Fix $\zeta = 1/4$. By Lemma 2.2, for each $n \in \mathcal{S}$ and each $j = 1, \ldots, [\zeta p_n]$, the map $\sigma^{j-1} : S_{n,1} \to S_{n,j}$ is a homeomorphism and the length $|S_{n,j}| \to 0$ as $n \to \infty$ uniformly in j. Fix N such that $|S_{n,j}| < \delta$ for each $n > N$, $n \in \mathcal{S}$. For $n \in \mathcal{S}$, let

$$\mathcal{C}_n = \{f^j(J_n) | 1 \le j \le [\zeta p_n]\}.$$

Let $n, m \in \mathcal{S}$, $m < n$. Denote $p = p_m$, $\tilde{p} = p_n$, $q = p_n/p_m$. The intersection $\mathcal{C}_n \cap \mathcal{C}_m$ contains all $f^{j+kp}(J_n)$ with $1 \le j \le [\zeta p]$, $j + kp \le [\zeta \tilde{p}]$. Hence,

$$\#(\mathcal{C}_n \cap \mathcal{C}_m) \ge \sum_{j=1}^{[\zeta p]} \left[\zeta q - \frac{j}{p} \right] \ge [\zeta q - 1][\zeta p]$$
$$\ge \tilde{p} \left(\frac{\zeta p - 1}{p} \frac{\zeta q - 1}{q} - \frac{\zeta}{q} \right) \sim \zeta^2 \tilde{p}$$

as $p, q \to \infty$. Therefore, fixing $\kappa = \zeta^2/2 = 1/8$, there are m_0, k_0 such that for each $n, m \in \mathcal{S}, m > m_0, n > m + k_0$,

$$\mu(\mathcal{C}_n \cap \mathcal{C}_m) > \kappa.$$

Fix such n, m. Assume also that $m > \max\{N, n_*\}$ where n_* is defined in Lemma 4.5 and recall the set

$$X_n = E_{\epsilon, p_n, 4} \cap E_{\epsilon, p_{n-1}, Q} = \bigcap_{i=0}^{3} f^{ip_n}(E_\epsilon) \bigcap_{k=0}^{Q-1} f^{kp_{n-1}}(E_\epsilon).$$

Since $\mu(X_n) > 1 - (Q+4)\epsilon > 1 - \kappa$, there is $x \in X_n \cap \mathcal{C}_n \cap \mathcal{C}_m$ and, by the choice of n, the assumption (II) of Lemma 4.5 holds for x. Therefore, there exist two simple semi-open curves $\gamma(x)$ and $\tilde{\gamma}(x)$ that satisfy the following properties: $\gamma(x)$ and $\tilde{\gamma}(x)$ tend to $\beta_{n,j_n(x)}$, $\gamma(x), \tilde{\gamma}(x) \subset B(x,\rho) \cap A(\infty)$, and $\gamma(x), \tilde{\gamma}(x)$ consist of arcs of equipotentials and external rays; the starting point b_1 of $\gamma(x)$ and the starting point $\tilde{b}_1$ of $\tilde{\gamma}(x)$ have equal levels which is at least $\tilde{C}/4$; $\gamma(x), \tilde{\gamma}(x) \subset U_{n-1,j_{n-1}(x)}$; finally, being completed by their common limit point $\beta_{n,j_n(x)}$ and arcs of rays from $b_1 \in \gamma(x)$ to ∞ and from $\tilde{b}_1 \in \tilde{\gamma}(x)$ to ∞, they split the plane into two domains such that one of them contains $I := J_{n,x} \setminus \beta_{n,j_n(x)}$ and the other one contains all other iterates $f^{kp_{n-1}}(I)$, $1 \le k \le q_n - 1$. Now, since $U_{n-1,j_{n-1}(x)} \subset U_{m,j_m(x)}$ and by the choice of m, the distance between the arguments of the points b_1 and $\tilde{b}_1$ inside of $S_{n-1,j_{n-1}(x)}$ is less than δ. By the definition of δ, b_1 and $\tilde{b}_1$ can be joined by an arc A_n of equipotential inside of $B(x,\rho) \cap U_{n-1,j_{n-1}(x)}$. Consider a Jordan domain Z_n with the boundary consisting of the arc A_n and semi-open curves $\gamma(x), \tilde{\gamma}(x)$ completed by their common limit point $\beta_{n,j_n(x)}$. Then $Z_n \subset B(x,\rho)$. By the properties of the curves, $Z_n \cup \beta_{n,j_n(x)}$ contains either $J_{n,x}$ or its iterate $f^{kp_{n-1}}(J_{n,x})$, for some $1 \le k \le q_n - 1$, in a contradiction with Remark 5.1.

We now turn to the complement to case A.

Case B: *for all big n, every satellite renormalization f^{p_n} is doubling*, that is, $\beta_n = \alpha_{n-1}$ and $p_n = 2p_{n-1}$ for every $n \in \mathcal{S}$. Let $Y_{n-1} = E_{\epsilon,p_{n-1},6}$ and $\tilde{Y}_{n-1} = f^{-5p_{n-1}}(Y_{n-1})$. Note that $\mu(Y_{n-1}) = \mu(\tilde{Y}_{n-1}) > 1 - 6\epsilon$.

For every $n \in \mathcal{S}$, let

$$L_n = \left\{0 < j < p_{n-1} | \mu(f^j(J_{n-1}) \cap \tilde{Y}_{n-1}) > \frac{1 - 2^{12}\epsilon}{p_{n-1}}\right\}.$$

As $\mu(\tilde{Y}_{n-1}) > 1 - 6\epsilon$, it follows that

$$\#L_n > (1 - 3/2^{11})p_{n-1}.$$

Since we are in case B, each $f^j(J_{n-1})$ contains precisely two small Julia sets $f^j(J_n)$, $f^{j+p_{n-1}}(J_n)$ of the next level n, each of them of measure $1/(2p_{n-1})$. Hence, the measure of intersection of each of these small Julia sets with $\tilde{Y}_{n-1}$ is bigger than $(1/2 - 2^{10}\epsilon)/p_{n-1} > 0$. By Lemma 4.8, choosing for every $j \in L_n$ a point $x_j \in f^j(J_{n-1}) \cap \tilde{Y}_{n-1}$, we get a pair of curves $\Gamma_n^{1/2}(x_j), \tilde{\Gamma}_n^{1/2}(x_j)$ consisting of arcs of rays and equipotentials as follows. (i) $\Gamma_n^{1/2}(x_j), \tilde{\Gamma}_n^{1/2}(x_j) \subset B(x_j, 3/2\rho)$. Moreover, arguments of points of $\Gamma_n^{1/2}(x_j)$ lie in one 'window' of $s_{n-1,j}$ while arguments of points of $\tilde{\Gamma}_n^{1/2}(x_j)$ lie in another 'window' of $s_{n-1,j}$. (ii) $\Gamma_n^{1/2}(x_j)$ and $\tilde{\Gamma}_n^{1/2}(x_j)$ converge to a common point $\beta^*_{n-1,j}$ which is a fixed point of $f^{p_{n-1}} : f^j(J_{n-1}) \to f^j(J_{n-1})$ (that is, $\beta^*_{n-1,j}$ is either the non-separating fixed point $\beta_{n-1,j}$ or the separating fixed point $\alpha_{n-1,j}$). (iii) The start points of $\Gamma_n^{1/2}(x_j), \tilde{\Gamma}_n^{1/2}(x_j)$ have equal Green level which is bigger than $\tilde{C}_3$. (iv) $x_j - \beta^*_{n-1,j} \to 0$ as $n \to \infty$ uniformly in j and x_j. We add one more property as follows. Let

$$\Gamma_{n,j} = \Gamma_n^{1/2}(x_j) \cup \beta^*_{n-1,j} \cup \tilde{\Gamma}_n^{1/2}(x_j).$$

Then (v) $\Gamma_{n,j}$ is a simple curve; the level of $z \in \Gamma_{n,j} \setminus \{\beta^*_{n-1,j}\}$ is positive and decreases (not strictly) from $\tilde{C}_3$ to zero along $\Gamma_n^{1/2}(x_j)$ and then increases from zero to $\tilde{C}_3$ along $\tilde{\Gamma}_n^{1/2}(x_j)$; moreover, if $j_1, j_2 \in L_n$, $j_1 \neq j_2$, then $\Gamma_{n,1}$, Γ_{n,j_2} are either disjoint or meet at the unique common point $\beta_{n-1,j_1} = \beta_{n-1,j_2}$ and then disjoint with all others $\gamma_{n-1,j}$, $j \neq j_1, j_2$. This is because, by property (i), $\Gamma_{n,j} \subset \overline{U_{n-1,j}}$ where (by (C), §2) any two $\overline{U_{n-1,j}}$, $\overline{U_{n-1,\tilde{j}}}$, $j \neq \tilde{j}$, are either disjoint or meet at $\beta := \beta_{n-1,j} = \beta_{n-1,\tilde{j}}$ in which case $f^{p_{n-1}}$ is satellite. In the case considered, any satellite is doubling so $\beta \neq \beta_{n-1,i}$ for all i different from $j, \tilde{j}$.

We assign, for use below, a 'small' Julia set $I_{n,j}$ to each $\Gamma_{n,j}$ as follows: by the construction, $\beta^*_{n-1,j}$ is either the β-fixed point of $f^j(J_{n-1})$ or the α-fixed point of $f^j(J_{n-1})$. In the former case let $I_{n,j} = f^j(J_{n-1})$, and in the latter case $I_{n,j} = f^j(J_n)$ (one of the two small Julia sets of the next level n that are contained in $f^j(J_{n-1})$. Observe that $I_{n,j} \cap \Gamma_{n-1,j} = \{\beta^*_{n-1,j}\}$ and is disjoint with any other $\Gamma_{n,j'}$ provided $\Gamma_{n,j}$, $\Gamma_{n,j'}$ are disjoint.

There are two subcases B1–B2 to distinguish depending on whether arguments of end points of $\Gamma_{m,j}$ become close or not. If they do, then one can join the end points of some $\Gamma_{n,j}$ by an arc of equipotential inside of $B(x_j, 2\rho) \supset \Gamma_{m,j}$ to surround a small Julia set as in case A, which would lead to a contradiction. If they do not, the construction is more subtle: we build a domain ('quadrilateral') in $B(x_j, 2\rho)$ bounded by two disjoint curves as above completed by two arcs of equipotential that join ends of different curve, so that the quadrilateral obtained again contains a small Julia set.

Case B1: $\liminf_{n\in\mathcal{S}, j\in L_n} |S_{n-1,j}| < \delta$. By property (i) listed above and the definition of δ, there are a sequence $(n_k) \subset \mathcal{S}$, $j_k \in L_{n_k}$ and x_{j_k} as above, such that two ends of each curve Γ_{n_k,j_k} can be joined inside of $B(x_{j_k}, \rho)$ by an arc A^k of equipotential of fixed level $\tilde{C}_3$ such that all arguments of points in A^k belong to S_{n_k-1,j_k}. Then we arrive at a contradiction as in case A.

Case B2: $|S_{n-1,j}| \geq \delta$ for all big $n \in \mathcal{S}$ and all $j \in L_n$. Fix $n, m \in \mathcal{S}$, $m - n \geq 3$. Define a subset of L_n as follows:

$$L_n^m = \left\{0 < j < p_{n-1} | \mu(f^j(J_{n-1}) \cap (\tilde{Y}_{n-1} \cap \tilde{Y}_{m-1})) > \frac{1 - 2^{12}\epsilon}{p_{n-1}}\right\}.$$

As $\mu(\tilde{Y}_{n-1} \cap \tilde{Y}_{m-1}) > 1 - 12\epsilon$,

$$\#L_n^m > (1 - 3/2^{10}) p_{n-1}.$$

For each $j \in L_n^m$ we define further

$$L_{n,j}^m = \left\{0 < k < p_{n-1} | f^k(J_{m-1}) \subset f^j(J_{n-1}), \mu(f^k(J_{m-1}) \cap (\tilde{Y}_{n-1} \cap \tilde{Y}_{m-1})) > \frac{1 - 2^{16}\epsilon}{p_{m-1}}\right\}.$$

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Then

$$\#L^m_{n,j} \geq 5,$$

as otherwise $\#L^m_{n,j} \leq 4$ and therefore $(1-2^{12}\epsilon)/p_{n-1} < 4/p_{m-1} + (p_{m-1}/p_{n-1} - 4)(1-2^{16}\epsilon)/p_{m-1} = 2^{18}\epsilon/p_{m-1} + (1-2^{16}\epsilon)/p_{n-1}$, that is, $p_{m-1}/p_{n-1} < 2^{18}\epsilon/(2^{16}\epsilon - 2^{12}\epsilon) = 4/(1-2^{-4}) < 8$, a contradiction because $p_{m-1}/p_{n-1} \geq 2^{m-n} \geq 2^3$.

Fix $j \in L^m_n$. Thus $L^m_{n,j}$ contains five pairwise different indices k_i, $1 \leq k \leq 5$. As $L^m_{n,j} \subset L_m$, we find five curves Γ_{m-1,k_i}. By property (v), if two of them meet, they are disjoint with all others. Therefore, there are at least three of them, denoted by Γ_{m-1,r_i}, $i = 1, 2, 3$, which are pairwise disjoint. Let w_i, $\tilde{w}_{m,i}$ be the two ends of Γ_{m-1,r_i}.

For each $i = 1, 2, 3$, the arguments of points of $w_{m,i}$, $\tilde{w}_{m,i}$ lie in different 'windows' of s_{m-1,r_i}. On the other hand, by the choice of j, $s_{m-1,r_i} \subset s_{n-1,j} \subset S_{n-1,j}$. As n is big enough, the lengths of the 'windows' of $s_{n-1,j}$ are less than δ. But since we are in case B2, the length of $S_{n-1,j}$ is bigger than δ. One can assume, therefore, that, for $i = 1, 2, 3$, the arguments of $w_{m,i}$ lie in one window of $s_{n-1,j}$ while the arguments of $\tilde{w}_{m,i}$ are in the other window. Therefore, differences of arguments of all $w_{m,i}$ tend to zero as $m \to \infty$, and the same for $\tilde{w}_{m,i}$. As all $w_{m,i}, \tilde{w}_{m,i} \in E_\epsilon$, this implies by Lemma 4.1 that $\max_{1\leq i,l\leq 3} |w_{m,i} - w_{m,l}| \to 0$. This, along with property (iv), implies that $\gamma_{m-1,r_i} \subset B(w_{m,1}, 2\rho)$, $i = 1, 2, 3$, for all big m. Since, for big m, differences of arguments of all $w_{m,i}$ are less than δ, and the same for $\tilde{w}_{m,i}$, one can join all $w_{m,i}$ by an arc D^m of equipotential of level $\tilde{C}_3$ and all $\tilde{w}_{m,i}$ by an arc $\tilde{D}^m$ of equipotential of the same level $\tilde{C}_3$ such that $D^m, \tilde{D}^m \subset B(w_1, 2\rho)$. Let the end points of D^m be, say, $w_{m,1}$ and $w_{m,3}$, so that $w_{m,2} \in D^m$ is in between. Since all three curves Γ_{m-1,r_i}, $i = 1, 2, 3$, are pairwise disjoint, the end points of $\tilde{D}^m$ then have to be $\tilde{w}_{m,1}$ and $\tilde{w}_{m,3}$, so that $\tilde{w}_{m,2} \in \tilde{D}^m$ is in between. Therefore, we get a 'big' quadrilateral $\Pi^0_m \subset B(w_{m,1}, 2\rho)$ bounded by $D^m, \tilde{D}^m, \Gamma_{m,1}, \Gamma_{m,3}$ where $\Gamma_{m,i} := \Gamma_{m-1,r_i}$, $i = 1, 2, 3$. The curve $\Gamma_{m,2}$ splits Π^0_m into two 'small' quadrilaterals Π^1_m, Π^2_m with a common curve $\Gamma_{m,2}$ in their boundaries. Recall now that the curve $\Gamma_{m,2}$ comes with a small Julia set $I_{m,2}$ of level either $m-1$ or m, such that $I_{m,2} \cap \Gamma_{m,2}$ is a single point while $I_{m,2}$ is disjoint with $\Gamma_{m,1}, \Gamma_{m,3}$. Therefore, $I_{m,2} \subset \Pi^0_m \subset B(w_{m,1}, 2\rho)$, a contradiction with Remark 5.1.

6. *Proof of Corollaries 1.3 and 1.4*

Corollary 1.3 follows directly from the following proposition

PROPOSITION 6.1. *Let f be an infinitely renormalizable quadratic polynomial. Then the following conditions are equivalent.*

(1) *$f : J_\infty \to J_\infty$ has no invariant probability measure with positive exponent.*

(2) *For every neighborhood W of P and every $\alpha \in (0, 1)$ there exist m_0 and n_0 such that, for each $m \geq m_0$ and $x \in \operatorname{orb}(J_n)$ with $n \geq n_0$,*

$$\frac{\#\{i \mid 0 \leq i < m,\ f^i(x) \in W\}}{m} > \alpha.$$

Additionally, $f : P \to P$ has no invariant probability measure with positive exponent.

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(3) *Every invariant probability measure of $f : J_\infty \to J_\infty$ is, in fact, supported on P and has zero exponent.*

(4) *For every invariant probability ergodic measure μ of f on the Julia set J of f, either* $\operatorname{supp}(\mu) \cap J_\infty = \emptyset$ *and its Lyapunov exponent* $\chi(\mu) > 0$*, or* $\operatorname{supp}(\mu) \subset P$ *and* $\chi(\mu) = 0$.

Proof. (1)⇒(2). Assume the contrary. Let $E = \mathbb{C} \setminus W$. Since W is a neighborhood of a compact set P, the Euclidean distance $d(E, P) > 0$. By a standard normality argument, as all periodic points of f are repelling, there are $\lambda > 1$ and $k_0 > 0$ such that $|(f^k)'(y)| > \lambda$ whenever $y, f^k(y) \in E$ and $k \geq k_0$. As (2) does not hold, find $\alpha \in (0, 1)$, a sequence $n_k \to \infty$, points $x_k \in \operatorname{orb}(J_{n_k})$ and a sequence $m_k \to \infty$ such that, for each k,

$$\frac{\#\{i : 0 \leq i < m_k, f^i(x_k) \in E\}}{m_k} \geq \beta := 1 - \alpha.$$

Fix a big k such that $\beta m_k > 3k_0$ and consider the times $0 \leq i_1^k < i_2^k < \cdots i_{l_k}^k < m_k$ where $l_k/m_k \geq \beta$ such that $f^i(x_k) \in E$. Let $z_k = f^{i_1^k}(x_k)$ so that $z_k \in E \cap \operatorname{orb}(J_n)$. Therefore, by the choice of λ and k_0, $|(f^{m_k - i_1^k})'(z_k)| \geq \tilde{\lambda}^{m_k} \geq \tilde{\lambda}^{m_k - i_1}$ where $\tilde{\lambda} = \lambda^{\beta/2k_0} > 1$. In this way we get a sequence of measures $\mu_k = (1/m_k - i_1^k) \sum_{i=0}^{m_k - i_1^k - 1} \delta_{f^i(z_k)}$ such that the Lyapunov exponent of μ_k is at least $\log \tilde{\lambda} > 0$. Passing to a subsequence, one can assume that $\{\mu_k\}$ converges weak-* to a measure μ. Then μ is an f-invariant probability measure on $J_\infty = \cap \operatorname{orb}(J_n)$ with the exponent at least $\log \tilde{\lambda} > 0$, a contradiction with (1).

(2)⇒(3). By Birkhoff's ergodic theorem along with [22].

(3)⇒(4). Let μ be as in (4) and $\overline{U} \cap P = \emptyset$ for some open set U with $\mu(U) > 0$. Let $F : U \to U$ be the first return map equipped with the induced invariant measure μ_U. By Birkhoff's ergodic theorem and by an argument as in (1)⇒(2), the exponent $\chi_F(\mu_U)$ of F with respect to μ_U is strictly positive. Hence, $\chi(\mu) = \mu(U)\chi_F(\mu_U)$ is positive too. This proves the implication.

And (4) obviously implies (1). □

Proof of Corollary 1.4. If $\overline{\chi}(x)$ were strictly positive, for some $x \in J_\infty$, that would imply, by a standard argument (see the proof of Corollary 1.3), the existence of an f-invariant measure with positive exponent supported in $\omega(x) \subset J_\infty$, with a contradiction to Theorem 1.1. This proves (1.1). By [13], $\liminf_{n\to\infty}(1/n) \log |(f^n)'(c)| \geq 0$. On the other hand, by (1.1), $\overline{\chi}(c) \leq 0$, which proves (1.2). □

Acknowledgements. The conclusion (1.2) of Corollary 1.4 that the Lyapunov exponent at the critical value equals zero partly answers a question by Weixiao Shen (Shen asked the following question, in relation to Corollary 5.5 of [13]: is it possible that the upper Lyapunov exponent at a critical value v of a polynomial g is positive, assuming that g is infinitely renormalizable around v? See also Remark 1.1). which inspired the present work as well as the previous paper [12]. The authors thank the referee for careful reading the paper and many helpful comments. The first author was partially supported by ISF grant 1226/17, Israel. The second author was partially supported by the National Science Centre, Poland, Grant OPUS 21 'Holomorphic dynamics, fractals, thermodynamic formalism', 2021/41/B/ST1/00461.

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