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PRINCIPAL GROUPOIDS WITH CONTINUOUS WEAK MULTIPLICATION

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In the present paper by extending the original multiplication of a principal groupoid to a weak multiplication and introducing a notion of semitopological semigroupoid compactification we have proved that every principal groupoid with a continuous weak multiplication has a semitopological semigroupoid compactification.

INTRODUCTION

In an arbitary groupoid G, the pair (x, y) is composable if and only if the range of y is the domain of x. The set of all composable pairs is denoted by G^2 and is a subset of $G \times G$ which is not necessary equal to $G \times G$. In the present paper we introduce a set G^2_w of weakly composable pairs which satisfies the inclusions $G^2 \subseteq G^2_w \subseteq G \times G$. We then extend the original multiplication of G^2 to a weak multiplication "$\circ_w$" on G^2_w, where in the case G is a principal groupoid, $(G, \circ_w)$ becomes a semigroupoid. In order to justify our work we have presented examples of well-known principal groupoids for which the weak multiplication is continuous (see, Proposition 3.5, Proposition 3.7, and Remark 3.8). In the case where G is a principal groupoid with a continuous multiplication, we have defined a semigroup S_G and introduced the space of G-weakly almost periodic functions $W_G(S_G)$ on S_G and have shown that a semitopological subsemigroupoid G^w of $B(W_G(S_G))$ (the space of all bounded linear operators on $W_G(S_G)$ with the composition multiplication) defines a semigroupoid compactification for G (see, Definition 4.16).

The organisation of this paper is as follows. Section 2 is devoted to introducing a set G^2_w of weakly composable pairs of a groupoid G. In section 3, for a principal groupoid G, we extend the original multiplication of G^2 to a weak multiplication $\circ_w$ of G^2_w which makes $(G, \circ_w)$ into a semigroupoid. Finally in section 4 to a principal groupoid G we associate a semigroup S_G and in the case of the continuity of the weak multiplication we prove that $W_G(S_G)$, the space of G-weakly almost periodic function on S_G and $A_G(S_G)$, the space of G-almost periodic function on S_G are translation G-invariant norm closed subspaces

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of $C(S_G)$. In the main result of the paper (Theorem 4.17) we prove that for every principal groupoid G with a continuous weak multiplication, G^w defines a semigroupoid compactification for G.

1. Definition and Notation

Definition 1.1: A groupoid is a set G endowed with a product map $(x,y) \mapsto xy$ [: $G^2 \to G$] and inverse map $x \mapsto x^{-1}$ [: $G \to G$], where G^2 is a subset of $G \times G$ called the set of composable pairs, such that the following relation are satisfied:

1. $(x^{-1})^{-1} = x$
2. if $(x,y),(y,z) \in G^2$, then $(xy,z),(x,yz) \in G^2$ and $(xy)z = x(yz)$
3. for all $x \in G$, $(x^{-1},x) \in G^2$ and if $(x,y) \in G^2$, then $x^{-1}(xy) = y$
4. for all $x \in G$, $(x,x^{-1}) \in G^2$ and if $(z,x) \in G^2$, then $(zx)x^{-1} = z$.

If $x \in G$, $d(x) = x^{-1}x$ is the domain of x and $r(x) = xx^{-1}$ is its range. It is clear that $d(x) = r(x^{-1})$ for each $x \in G$, and $r(xy) = r(x)$, $d(xy) = d(y)$ for $(x,y) \in G^2$. The reader is invited to prove, using only the axioms, basic groupoid facts such as that $(x,y) \in G^2$ if and only if $r(y) = d(x)$ and that if $(x,y) \in G^2$, then $(y^{-1},x^{-1}) \in G^2$ and $(xy)^{-1} = y^{-1}x^{-1}$. The set $G^0 = d(G) = r(G)$ is the unit space of G, its elements are units in the sense that $xd(x) = x = r(x)x$. For $u,v \in G^0$, $G^u = r^{-1}(\{u\})$, $G_v = d^{-1}(\{v\})$ and $G^u_v = G^u \cap G_v$, also $r(u) = d(u) = u \quad (u \in G^0)$ and $x^{-1} = x$ whenever x is a unit. A groupoid G is called a *group bundle* if $r(x) = d(x)$ for each $x \in G$. The relation $u \sim v$ if and only if $G^u_v \neq \emptyset$ is an equivalence relation on G^0. In fact $u \sim v$ if and only if there exists one $x \in G$ with $r(x) = u, d(x) = v$. Its equivalence classes are called orbits, and the orbit of u is denoted by $[u]$. The orbit space is denoted by G^0/G, and the graph of the equivalence relation $\sim$ on G^0 is denoted by R_G. For $x \in G$ it is clear that $r(x) \sim d(x)$ and for $x,y \in G$ if $r(y) \sim d(x)$, then $r(x) \sim d(y)$.

Definition 1.2: Let G be a groupoid, we write θ for the mapping (r,d) from G into $G^0 \times G^0$ with $(r,d)(x) = (r(x),d(x))$. A groupoid G is called *principal* provided that θ is one to one, so that G is isomorphic to R_G. Also G is called *transitive* if θ is onto. A groupoid is transitive if and only if it has a single orbit (see [**4**, p. 6]).

Definition 1.3: A topological groupoid consists of a groupoid G and a topology compatible with the groupoid structure. This means that:

1. $x \mapsto x^{-1}$ [: $G \to G$] is continuous
2. $(x,y) \mapsto xy$ [: $G^2 \to G$] is continuous where G^2 has the induced topology from $G \times G$.

Consequences: $x \mapsto x^{-1}$ is a homeomorphism; r and d are continuous; If G is Hausdorff, G^0 is closed; if G^0 is Hausdorff, G^2 is closed in $G \times G$ (see [**4**, p. 16]).

We shall only consider topological groupoids with locally compact and Hausdorff topology in the sense of [**3**, Definition 2.2.1].

DEFINITION 1.4: A *semigroup* is a pair $(S,.)$, where S is a nonempty set and $(.)$ is an associative (binary) operation $(s,t) \mapsto s.t$ [: $S \times S \to S$]. Associativity means that $r.(s.t) = (r.s).t$ for each $r, s, t \in S$.

A semigroup S is called *semitopological* (respectively *topological*) if its binary operation is separately (respectively, jointly) continuous. For a semitopological semigroup S we denote respectively Bounded, Continuous bounded, Left uniformly continuous, Right uniformly continuous, Uniformly continuous, Almost periodic and weakly almost periodic functions on S by $B(S)$, $C(S)$, $LUC(S)$, $RUC(S)$, $UC(S)$, $AP(S)$ and $WAP(S)$. For more details on these spaces see [**1**]

2. Weakly composable elements

Recall that in a groupoid G, $(x,y) \in G^2$(the set of composable pairs) if and only if $r(y) = d(x)$. Also $\sim$ is an equivalence relation on the unit space G^0.

DEFINITION 2.1: In a groupoid G the pair (x,y) is called *weakly composable* if and only if $r(y) \sim d(x)$. We put

$$G_w^2 = \{(x,y) \in G \times G : \ r(y) \sim d(x)\}.$$

REMARK 2.2. If $(x,y) \in G^2$, then $r(y) = d(x)$ and therefore $r(y) \sim d(x)$, hence $(x,y) \in G_w^2$. That is $G^2 \subseteq G_w^2 \subseteq G \times G$. Also in a groupoid G it is not necessary that $(x,x) \in G^2$ $(x \in G)$, but since $r(x) \sim d(x)$ $(x \in G)$, therefore $(x,x) \in G_w^2$ $(x \in G)$. On the other hand if $(x,y) \in G^2$, then it is not necessary that $(y,x) \in G^2$, but $(x,y) \in G_w^2$ implies that $(y,x) \in G_w^2$.

REMARK 2.3. If G is a groupoid, then it is easy to check that G is a group bundle if and only if $G^2 = G_w^2$. Also $G_w^2 = G \times G$ if and only if G is transitive.

PROPOSITION 2.4. *For a locally compact groupoid G, the set G_w^2 with the induced topology from $G \times G$ is closed in $G \times G$ if and only if R_G (the graph of the equivalence relation $\sim$ on the unit space G^0) with the product topology induced from $G^0 \times G^0$ is closed in $G^0 \times G^0$.*

PROOF: Suppose that R_G is closed and $\big((x_\alpha, y_\alpha)\big)_{\alpha\in\Lambda}$ is a net in G_w^2 which converges to (x,y). Since r and d are continuous and $r(y_\alpha) \sim d(x_\alpha)$, hence $\Big(\big(r(y_\alpha), d(x_\alpha)\big)\Big)_{\alpha\in\Lambda}$ is a net in R_G which converges to $\big(r(y), d(x)\big)$. Since R_G is closed, therefore $r(y) \sim d(x)$ and this means that $(x,y) \in G_w^2$. For the converse, let $\big((u_\alpha, v_\alpha)\big)_{\alpha\in\Lambda}$ be a net in R_G which converges to (u,v). Since G^0 is closed and the topology of R_G is the product topology induced from $G^0 \times G^0$, hence $u, v \in G^0$. Note that $r(z) = z = d(z)$ for $z \in G^0$, therefore $(u_\alpha, v_\alpha) \in G_w^2$. Since G_w^2 is closed, therefore $(u,v) \in G_w^2$. That is $v = r(v) \sim d(u) = u$. ∎

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3. Extension of the multiplication from G^2 to G_w^2 for principal groupoids G

Definition 3.1: A *semigroupoid* S is a set endowed with a product map $(x, y) \mapsto xy$ [: $S * S \to S$], where $S * S$ is a subset of $S \times S$ called the set of composable pairs such that the following relations are satisfied:

$$\text{if } (x,y),\ (y,z) \in S * S, \text{ then } (xy, z),\ (x, yz) \in S * S \text{ and } (xy)z = x(yz).$$

A semigroupid S is called semitopological if its product is separately continuous whenever is defined.

Proposition 3.2. *Let G be a principal groupoid, then we can extend the original multiplication on G^2 to a weak multiplication "$\circ_w$" on G_w^2 such that $(G, \circ_w)$ is semigroupoid.*

Proof: Suppose that $(x, y) \in G_w^2$, therefore there exists $u \in G$ such that $r(u) = r(y)$ and $d(u) = d(x)$ or equivalently

$$(u^{-1}, y) \in G^2 \text{ and } (x, u^{-1}) \in G^2. \tag{1}$$

Since G is principal, u is unique. We define a weak multiplication "$\circ_w$" on G_w^2 by $x \circ_w y = xu^{-1}y$, where u satisfies (1). By (1) and the uniqueness of u we conclude that $x \circ_w y$ is well defined. If $(x, y) \in G^2$, then $r(d(x)) = d(x) = r(y)$ and $d(d(x)) = d(x)$. Since G is principal, then $u = d(x)$ is the unique element of G which satisfies (1) for $(x, y) \in G^2$. Therefore $x \circ_w y = x(d(x))^{-1}y = xd(x)y = xy$. Hence the weak multiplication is an extension of the original multiplication. Finally, we prove that if $(x, y),\ (y, z) \in G_w^2$, then $(x \circ_w y, z),\ (x, y \circ_w z) \in G_w^2$ and $x \circ_w (y \circ_w z) = (x \circ_w y) \circ_w z$. But $(x, y) \in G_w^2$ implies that there exists a unique $v \in G$ such that $r(v) = r(y)$ and $d(v) = d(x)$. Similarly $(y, z) \in G_w^2$ implies that there exists a unique $u \in G$ such that $r(u) = r(z)$ and $d(u) = d(y)$. Therefore, $x \circ_w y = xv^{-1}y$ and $y \circ_w z = yu^{-1}z$. Thus

$$d(v) = d(x) \text{ and } r(v) = r(y) = r(yu^{-1}z) = r(y \circ_w z). \tag{2}$$

Hence $(x, y \circ_w z) \in G_w^2$. Since G is principal, v is a unique element which satisfies the equality (2). Therefore

$$x \circ_w (y \circ_w z) = xv^{-1}(y \circ_w z) = xv^{-1}(yu^{-1}z). \tag{3}$$

Also we have

$$d(u) = d(y) = d(xv^{-1}y) = d(x \circ_w y) \text{ and } r(u) = r(z). \tag{4}$$

Hence $(x \circ_w y, z) \in G_w^2$ and u is the unique element of G which satisfies the equality (4), hence

$$(x \circ_w y) \circ_w z = (x \circ_w y)u^{-1}z = (xv^{-1}y)u^{-1}z. \tag{5}$$

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A combination of (3) and (5) yields $x \circ_w (y \circ_w z) = (x \circ_w y) \circ_w z$. ∎

We now discuss two important class of groupoids with their weakly composable pairs.

EXAMPLE 3.3. (Transformation groups [4, p. 6]). Suppose that the group S acts freely on the space U on the right. The image of the point $u \in U$ by the transformation $s \in S$ is denoted by $u.s$. We put $G = U \times S$ and define the following groupoid structure: $\big((u,s),(v,t)\big)$ is composable if and only if $v = u.s$, $(u,s)(u.s,t) = (u,st)$ and $(u,s)^{-1} = (u.s, s^{-1})$. Then $r(u,s) = (u,e)$ and $d(u,s) = (u.s,e)$. In this case G defines principal groupoid which is not a group bundle. It is easy to check that $\big((u,s),(v,t)\big) \in G^2_w$ if and only if $v = u.t'$ for some $t' \in S$ and $(u,s) \circ_w (u.t',t) = (u,t't)$.

EXAMPLE 3.4. (Graph of an equivalence relation [4, p. 7]). If U is a locally compact space and R is the graph of an equivalence relation $\sim$ on U, then R with the induced topology from $U \times U$ and the following groupoid structure is a topological groupoid. $\big((u,v),(v',w)\big) \in R^2$ if and only if $v = v'$ and in this case $(u,v)(v,w) = (u,w)$. Also $(u,v)^{-1} = (v,u)$. It is obvious that R defines a principal groupoid which is not a group bundle and it is easy to check that $\big((u,v),(v',w)\big) \in R^2_w$ if and only if $(v,v') \in R$. Also $(u,v) \circ_w (v',w) = (u,w)$.

Note that every principal topological groupoid G is isomorphic as a groupoid (not topological) to R_G, the graph of the equivalence relation on G^0 with product topology induced from $G^0 \times G^0$.

PROPOSITION 3.5. *If G is a principal compact Hausdorff groupoid, then G^2_w is closed with jointly continuous weak multiplication.*

PROOF: Since G is compact and $(r,d) : G \to G^0 \times G^0$ is continuous, therefore $(r,d)(G) = R_G$ is compact, hence is a closed subset of $G^0 \times G^0$, and by Proposition 2.4, G^2_w is closed. For the second part of the proof take $(x,y) \in G^2_w$ and we suppose that $\big((x_\alpha, y_\alpha)\big)_{\alpha\in\Lambda}$ is a net in G^2_w which converges to (x,y). So for each $\alpha \in \Lambda$ there exists a unique $u_\alpha \in G$ such that $r(u_\alpha) = r(y_\alpha), d(u_\alpha) = d(x_\alpha)$ and there exists a unique $u \in G$ with $r(u) = r(y)$, $d(u) = d(x)$. Thus

$$\begin{aligned}(r,d)(u_\alpha) &= (r(u_\alpha), d(u_\alpha)) \\ &= (r(y_\alpha), d(x_\alpha)) \\ &\longrightarrow (r(y), d(x)) \\ &= (r(u), d(u)) \\ &= (r,d)(u).\end{aligned}$$

Since G is principal, it follows that (r,d) is one-to-one and continuous from the compact Hausdorff groupoid G onto Hausdorff groupoid R_G, therefore it is a homeomorphism. Since $(r,d)(u_\alpha) \longrightarrow (r,d)(u)$, hence $u_\alpha \longrightarrow u$ and by the continuity of the original multiplication

$$x_\alpha \circ_w y_\alpha = x_\alpha u_\alpha^{-1} y_\alpha \longrightarrow x u^{-1} y = x \circ_w y.$$

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This completes the proof. ∎

REMARK 3.6. In a transformation group $G = U \times S$ (Example 3.3), if S is a locally compact group and U is a locally compact space, then G with the product topology is a locally compact topological groupoid (see [5, Example 2.25]). Since the inversion mapping $(u, s) \mapsto (u, s)^{-1} = (u.s, s^{-1})$ is continuous, we infer that the mapping: $(u, s) \mapsto u.s$ [: $U \times S \longrightarrow U$] is jointly continuous.

PROPOSITION 3.7. *Suppose S is a compact Hausdorff topological group and acts freely on a noncompact locally compact Hausdorff space U, then $G = U \times S$ with the product topology and the groupoid structure defined in Example 3.3 is a principal noncompact locally compact Hausdorff groupoid for which G^2_w is a closed subset of $G \times G$ and the weak multiplication "$\circ_w$" is jointly continuous.*

PROOF: Let $\Big(\big((u_\alpha, s_\alpha), (v_\alpha, t_\alpha)\big)\Big)_{\alpha\in\Lambda}$ be a net in G^2_w which converges to $\big((u, s), (v, t)\big)$. By the Example 3.3, $v_\alpha = u_\alpha.t'_\alpha$ for some $t'_\alpha \in S$ $(\alpha \in \Lambda)$. Since S is compact, $\{t'_\alpha\}$ has a limit point t'. So $v = u.t'$ and therefore $\big((u, s), (v, t)\big) \in G^2_w$. Since S acts freely on U, it is easy to check that $\{t_\alpha\}$ has exactly one limit point, hence $t_\alpha \longrightarrow t'$. Therefore

$$\begin{aligned}(u_\alpha, s_\alpha) \circ_w (v_\alpha, t_\alpha) &= (u_\alpha, s_\alpha) \circ_w (u_\alpha.t'_\alpha, t_\alpha)\\ &= (u_\alpha, t'_\alpha t_\alpha)\\ &\longrightarrow (u, t't)\\ &= (u, s) \circ_w (u.t', t)\\ &= (u, s) \circ_w (v, t).\end{aligned}$$

∎

REMARK 3.8. It is easy to check that in the principal topological groupoid R defined in Example 3.4, the weak multiplication is jointly continuous. Also if R is a closed subset of $U \times U$, then R^2_w is closed in $R \times R$.

In [5, Proposition 2.9] it is shown that each groupoid G may be written as the disjoint union $G = \bigcup_{[u]\in(G^0/G)} G^{[u]}$, where $G^{[u]} = \bigcup_{u\sim v} G^v$ is a transitive groupoid. It is easy to check that for a locally compact topological groupoid G, if R_G is a closed subset of $G \times G$, then $G^{[u]}$ is a closed subset of G for each $[u] \in G^0/G$.

PROPOSITION 3.9. *Let G be a principal groupoid, then for each $[u] \in G^0/G$, $G^{[u]} \times G^{[u]} \subseteq G^2_w$ and $(G^{[u]} \times G^{[v]}) \cap G^2_w = \emptyset$, whenever $[u] \neq [v]$. In addition $(G^{[u]}, \circ_w)$ $([u] \in G^0/G)$ is a semigroup and $(G, \circ_w)$ is a semigroupoid.*

PROOF: If $(x, y) \in G^{[u]} \times G^{[u]}$, then $u \sim r(x)$, $u \sim r(y)$. From these relations and that $r(x) \sim d(x)$ we conclude that $r(y) \sim d(x)$, hence $(x, y) \in G^2_w$. Also $r(x \circ_w y) = r(x)$ implies that $x \circ_w y \in G^{[u]}$. In Proposition 3.2 we proved that the weak multiplication is an associative operation, that is $(G^{[u]}, \circ_w)$ is a semigroup for each $[u] \in G^0/G$. Next suppose that $x \in G^{[u]}$, $y \in G^{[v]}$ with $[u] \neq [v]$ and $(x, y) \in G^2_w$, therefore $u \sim r(x)$, $v \sim r(y)$

and $r(y) \sim d(x)$. These relations together with $r(x) \sim d(x)$ imply that $u \sim v$, which contradicts the fact that $[u] \neq [v]$. Therefore (x, y) is weakly composable if and only if there exists a $[u] \in G^0/G$ with $x, y \in G^{[u]}$ and in this case $x \circ_w y \in G^{[u]}$. Now it is easy to check that $(G, \circ_w)$ with

$$G * G = \bigcup_{[u] \neq [v]} G^{[u]} \times G^{[v]}$$

defines a semigroupoid (see Definition 3.1). $\square$

REMARK 3.10. It is easy to check that when R_G is a closed subset of $G^0 \times G^0$, then $G^{[u]}$ ($[u] \in G^0/G$) is closed and therefore is a locally compact subset of G.

REMARK 3.11. Proposition 3.7 and Remark 3.8 show that there exist principal non-compact locally compact topological Hausdorff groupoids G for which G^2_w is closed and the weak multiplication is jointly continuous.

4. A SEMIGROUP ASSOCIATED TO A PRINCIPAL GROUPOID G

For a principal groupoid G, put $S_G = \prod_{[u] \in G^0/G} G^{[u]}$. For $X = (x_{[u]})$ and $Y = (y_{[u]})$ in S_G, where $x_{[u]}, y_{[u]} \in G^{[u]}$, define $X.Y = (x_{[u]} \circ_w y_{[u]})$. Since $x_{[u]}, y_{[u]} \in G^{[u]}$, hence by Proposition 3.9, the pair $(x_{[u]}, y_{[u]})$ is weakly composable, so this multiplication is well defined and $(S_G, .)$ is a semigroup. When G is a principal locally compact topological groupoid such that the weak multiplication is jointly continuous, then S_G with the product topology defines a topological semigroup. Also if G is principal locally compact groupoid with a jointly continuous weak multiplication and closed graph R_G, then by Remark 3.10, S_G with product topology defines a locally compact topological semigroup. If G is compact, then by Proposition 3.5, G^2_w is closed and weak multiplication is jointly continuous, hence $G^{[u]}$ is closed and so is compact. Therefore S_G is compact topological semigroup.

In the following we define an action of G on S_G which is important for our purpose.

DEFINITION 4.1: For $x \in G$ and $Y = (y_{[u]}) \in S$ we define a right action of G on S_G by

$$Y \circ x = Z = (z_{[u]}) \in S : \begin{cases} z_{[w]} = y_{[w]} \circ_w x & \text{if } x \in G^{[w]} \\ z_{[u]} = y_{[u]} & \text{for other components.} \end{cases}$$

Similarly we can define a left action of G on S_G.

LEMMA 4.2. *For a principal topological groupoid G the following relations are valid:*

1. *If $(x, y) \in G^2_w$ and $Z \in S$ then $Z \circ (x \circ_w y) = (Z \circ x) \circ y$. Also if $(x, y) \notin {G^2}_w$ and $Z \in S$ then $(Z \circ x) \circ y = (Z \circ y) \circ x$. Similar results are also valid for the left action.*

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2. *For $x, y \in G$ and $Z \in S$ we have $y \circ (Z \circ x) = (y \circ Z) \circ x$.*
3. *For $x \in G$ and $Z, W \in S$ we have $(Z.W) \circ x = Z.(W \circ x)$. Similarly $x \circ (W.Z) = (x \circ W).Z$.*
4. *If $(x, y) \in G^2$ and $(y, z) \in G^2$ then $(W \circ xy) \circ z = (W \circ x) \circ yz$ for each $W \in S$. Similar equalities are valid for the left action.*

PROOF: Straightforward. ∎

DEFINITION 4.3: Each $x \in G$ defines norm bounded linear operators L_x, R_x on $B(S_G)$ respectively by $f \mapsto L_x f$ and $f \to R_x f$, where

$$L_x f(Y) = f(x \circ Y) \text{ and } R_x f(Y) = f(Y \circ x) \quad (Y \in S).$$

Also each $X \in S$ defines the right (respectively, left) translation operator R_X (respectively, L_X) on $B(S_G)$ by $f \mapsto L_X f$ and $f \to R_X f$, where

$$L_X f(Y) = f(X.Y) \text{ and } R_X f(Y) = f(Y.X) \quad (Y \in S).$$

We consider the set of all norm bounded linear operators on $B(S_G)$ with multiplication "."(= composition) and this set with this binary operation is a semigroup.

LEMMA 4.4. *For $x, y \in G$ and $Z \in S$ we have*

1. $R_x.R_y = \begin{cases} R_{x \circ_w y} & \text{if } (x, y) \in G^2_w \\ R_y.R_x & \text{otherwise} \end{cases}.$
2. $R_x.L_y = L_y.R_x$ *for each $x, y \in G$.*
3. $L_Z.L_x = L_{x \circ Z}$; $R_Z.R_x = R_{Z \circ x}$; $R_Z.L_x = L_x.R_Z$ and $L_Z.R_x = R_x.L_Z$.

PROOF: (1) is proved by using part (1) of Lemma 4.2 and definitions of R_x, R_y. The part (2) is proved by (2) of Lemma 4.2. and similarly (3) is proved by (3) of Lemma 4.2. ∎

LEMMA 4.5. *Let G be a principal topological groupoid with continuous weak multiplication, then the following are valid:*

1. *For each $x \in G : Y \mapsto x \circ Y$ and $Y \mapsto Y \circ x$ [: $S \to S$] are continuous.*
2. *For $Y \in S$, the restriction of the mappings $x \mapsto Y \circ x$ and $x \mapsto x \circ Y$ to $G^{[u]}$ are continuous for each $[u] \in G^0/G$.*
3. *If G is a groupoid with finite orbit space G^0/G and closed graph R_G, then $(Y, x) \mapsto Y \circ x$ [: $S \times G \to S$] is jointly continuous.*

PROOF: (1) Let $Y_\alpha \longrightarrow Y$ in S_G, with $Y_\alpha = (y^\alpha_{[u]})$ and $Y = (y_{[u]})$ where $y^\alpha_{[u]}$, $y_{[u]} \in G^{[u]}$. Since the topology on S_G is the product topology, hence $y^\alpha_{[u]} \longrightarrow y_{[u]}$ $([u] \in G^0/G)$. Suppose $x \in G$, so $x \in G^{[w]}$ for exactly one $[w] \in G^0/G$. We have $(y^\alpha_{[w]}, x) \longrightarrow (y_{[w]}, x)$ in

G_w^2. Since the weak multiplication is continuous so $y_{[w]}^\alpha \circ_w x \longrightarrow y_{[w]} \circ_w x$ and therefore $Y_\alpha \circ x \longrightarrow Y \circ x$.

(2) The proof is straightforward.

(3) Let $Y_\alpha \longrightarrow Y$ in S_G and $x_\alpha \longrightarrow x$ in G, where $Y_\alpha = (y_{[u]}^\alpha)$ and $Y = (y_{[u]})$. Since R_G is closed, hence $G^{[u]}$ is a closed subset of G for each $[u] \in G^0/G$. From the fact that $G^{[u]} \cap G^{[v]} = \emptyset$ for $[u] \neq [v]$ and G^0/G is finite, it follows that if $x \in G^{[v]}$, then $G^{[v]}$ contains all x_α's except finitely many. We may assume that $G^{[v]}$ contains all of x_α. Since the topology in S_G is the product topology, therefore $y_{[v]}^\alpha \longrightarrow y_{[v]}$. Hence $y_{[v]}^\alpha \circ_w x_\alpha \longrightarrow y_{[v]} \circ_w x$ and therefore $Y_\alpha \circ x_\alpha \longrightarrow Y \circ x$. ∎

DEFINITION 4.6: A subset F of $B(S_G)$ is called *left* (respectively *right*) *translation G-invariant* if $L_x F$ (respectively $R_x F$) $\subseteq F$ for each $x \in G$. Also F is called *translation G- invariant* if it is both left and right translation G- invariant.

By Part (1) of Lemma 4.5 it is obvious if G is a principal locally compact groupoid with continuous weak multiplication, then $C(S_G)$ defines a translation G-invariant subspace of $B(S_G)$. If $F \subseteq B(S_G)$ is a left (respectively right) translation G-invariant, then it is clear that R_x and L_x ($x \in G$) are norm continuous linear operators on F. In the following we show that R_x and L_x are weak-weak continuous operators on $C(S_G)$.

LEMMA 4.7. *If G is a principal groupoid with associated semigroup S_G, then R_x and L_x are weak-weak continuous operators on $C(S_G)$.*

PROOF: We give the proof for R_x, the proof for L_x is similar. For $x \in G$ we define $\eta(x) : C(S_G)^* \to C(S_G)^*$ by $\eta(x)(\mu) = \eta(x) * \mu$ ($\mu \in C(S_G)^*$), where $(\eta(x) * \mu)(f) = \mu(R_x f)$ for each $f \in C(S_G)$. Now let $f_\alpha \longrightarrow f$ in $C(S_G)$ in the weak topology, then for each $\mu \in C(S_G)^*$

$$\begin{aligned} \mu(R_x f_\alpha) &= (\eta(x) * \mu)(f_\alpha) \\ &\longrightarrow (\eta(x) * \mu)(f) \\ &= \mu(R_x f). \end{aligned}$$

That is $R_x f_\alpha \longrightarrow R_x f$ in the weak topology. ∎

THEOREM 4.8. *Suppose that G is a principal topological groupoid with continuous weak multiplication, then $LUC(S_G), RUC(S_G), UC(S_G), AP(S_G)$ and $WAP(S_G)$ are translation G-invariant linear subspaces of $B(S_G)$.*

PROOF: By part (3) of Lemma 4.4 and part (1) of Lemma 4.5 the subspaces $LUC(S_G), RUC(S_G)$ and $UC(S_G)$ are translation G-invariant. We show that $AP(S_G)$ is also translation G-invariant, the proof for $WAP(S_G)$ is similar. Let $f \in AP(S_G)$ and $x \in G$, then $\{R_Z f : Z \in S\}$ is norm relatively compact. By part (3) of Lemma 4.4 and

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that L_x is norm continuous operator on $C(S_G)$, we conclude

$$\begin{aligned}\overline{\{R_Z(L_xf) : Z \in S\}}^{\|\cdot\|} &= \overline{\{(R_Z.L_x)(f) : Z \in S\}}^{\|\cdot\|} \\ &= \overline{\{(L_x.R_Z)(f) : Z \in S\}}^{\|\cdot\|} \\ &= \overline{\{L_x(R_Zf) : Z \in S\}}^{\|\cdot\|} \\ &= L_x(\overline{\{R_Zf : Z \in S\}}^{\|\cdot\|}),\end{aligned}$$

where $\overline{\{R_Z(L_xf) : Z \in S\}}^{\|\cdot\|}$ denotes the norm closure of $\{\{R_Z(L_xf) : Z \in S\}$.

Since $f \in AP(S_G)$ it follows that $\overline{\{R_Z(L_xf) : Z \in S\}}^{\|\cdot\|}$ is compact. That is $L_xf \in AP(S_G)$. Also by part (3) of Lemma 4.4 for every $f \in AP(S_G)$

$$\begin{aligned}\overline{\{R_Z(R_xf) : Z \in S\}}^{\|\cdot\|} &= \overline{\{(R_Z.R_x)f : Z \in S\}}^{\|\cdot\|} \\ &= \overline{\{R_{Zox}f : Z \in S\}}^{\|\cdot\|} \\ &\subseteq \overline{\{R_Xf : X \in S\}}^{\|\cdot\|}.\end{aligned}$$

That is $R_xf \in AP(S_G)$. ∎

DEFINITION 4.9: For $f \in C(S_G)$ we denote the set $\{R_xf \mid x \in G\}$ by $O_R^G(f)$. A function $f \in C(S_G)$ is called *G-weakly almost* (respectively, *G-almost*) *periodic* if $O_R^G(f)$ is relatively compact in the weak (respectively, norm) topology of $C(S_G)$. This means that $\overline{O_R^G(f)}^w$ (respectively, $\overline{O_R^G(f)}^{\|\cdot\|}$) is compact, where $\overline{O_R^G(f)}^w$ is closure of $O_R^G(f)$ in $C(S_G)$ in weak topology. We denote the class of all G-weakly almost (respectively G-almost) periodic functions on S_G by $W_G(S_G)$ (respectively, $A_G(S_G)$).

THEOREM 4.10. *For a principal topological groupoid G with a continuous weak multiplication, $W_G(S_G)$ and $A_G(S_G)$ are translation G-invariant norm closed linear subspaces of $C(S_G)$.*

PROOF: We give the proof only for $W_G(S_G)$, the proof for $A_G(S_G)$ is similar. Since $O_R^G(f+g) \subseteq O_R^G(f) + O_R^G(g)$ and the addition is obviously weak continuous from $C(S_G) \times C(S_G)$ into $C(S_G)$, the relative weak compactness of $O_R^G(f)$ and $O_R^G(g)$ imply that so is $O_R^G(f) + O_R^G(f)$. Hence if $f, g \in W_G(S_G)$, then $f + g \in W_G(S_G)$. Similarly from $O_R^G(\lambda f) = \lambda O_R^G(f)$ $(\lambda \in \mathbb{C},\ f \in W_G(S_G))$ it follows trivially that $\lambda f \in W_G(S_G)$. That is $W_G(S_G)$ is a linear subspace of $C(S_G)$. Since for $f \in W_G(S_G)$ and $x \in G$,

$$\begin{aligned}O_R^G(R_xf) &= \{R_y(R_xf) : y \in G\} \\ &= \{(R_y.R_x)(f) : y \in G\} \\ &= \{(R_y.R_x)(f) : (y,x) \in G_w^2\} \cup \{(R_y.R_x)(f) : (y,x) \notin G_w^2\}.\end{aligned}$$

Therefore by part (1) of Lemma 4.4

$$O_R^G(R_xf) = \{R_{y\circ_w x}f : y \in G^{[r(x)]}\} \cup \{(R_x.R_y)(f) : y \notin G^{[r(x)]}\}.$$

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Hence

$$\overline{O_R^G(R_x f)}^{\,w} = \overline{\{R_{y\circ_w x} f : y \in G^{[r(x)]}\}}^{\,w} \cup \overline{\{(R_x.R_y)(f) : y \notin G^{[r(x)]}\}.}^{\,w}$$

Since $f \in W_G(S_G)$, it follows that $\overline{\{R_{y\circ_w x} f : y \in G^{[r(x)]}\}}^{w}$ is compact. Also by Lemma 4.7, R_x is a weak continuous operator on $C(S_G)$, so the weak relatively compactness of $\{R_y f : y \notin G^{[r(x)]}\}$ implies that

$$\begin{aligned}\overline{\{(R_x.R_y)(f) : y \notin G^{[r(x)]}\}}^{\,w} &= \overline{\{R_x(R_y f) : y \notin G^{[r(x)]}\}}^{\,w} \\ &= R_x \overline{\{R_y f : \quad y \notin G^{[r(x)]}\}}^{\,w}.\end{aligned}$$

Hence $\overline{\{R_x R_y f : y \notin G^{[r(x)]}\}}^{w}$ is compact, and therefore $\overline{O_R^G(R_x f)}^{\,w}$ is compact, that is $R_x f \in W_G(S_G)$. Similarly by part (2) of Lemma 4.4 we have $O_R^G(L_x f) = L_x(O_R^G(f))$. Therefore

$$\overline{O_R^G(L_x f)}^{\,w} = \overline{L_x(O_R^G(f))}^{\,w} = L_x(\overline{O_R^G(f)}^{\,w}).$$

Since by Lemma 4.7, L_x is a weakly continuous operator on $C(S_G)$ and $f \in W_G(S_G)$, we infer that $L_x f \in W_G(S_G)$. That is $W_G(S_G)$ is translation G-invariant. Similar to [**1**, 2.5] by using Eberlein Smulian theorem we can show that $W_G(S_G)$ is norm closed. ∎

THEOREM 4.11. *If G is a principal topological groupoid with a continuous weak multiplication, then the following assertions are valid:*

1. *For each $f \in W_G(S_G)$ (respectively, $f \in A_G(S_G)$) the restriction of the mapping $x \mapsto R_x f$ to $G^{[u]}$ $[: G^{[u]} \to W_G(S_G)]$ (respectively, $[: G^{[u]} \to A_G(S_G)]$) is weak (respectively, norm) continuous for each $[u] \in G^0/G$.*
2. *If G is compact of finite orbit space G^0/G, then $O_R^G(f)$ is both weak and norm compact in $C(S_G)$ for each $f \in C(S_G)$.*
3. *If G is compact and the action defined in Definition 4.1 is jointly continuous, then $O_R^G(f)$ is both norm and weak relatively compact.*

PROOF: (1) Let $[u] \in G^0/G$ and $x_\alpha \longrightarrow x$ in $G^{[u]}$, then for $f \in W_G(S_G)$ (respectively, $f \in A_G(S_G)$) by part (2) of Lemma 4.5 for each $X \in S$

$$R_{x_\alpha} f(X) = f(X \circ x_\alpha) \longrightarrow f(X \circ x) = R_x f(X).$$

That is the mapping: $x \mapsto R_x f$ is pointwise continuous from $G^{[u]}$ into $W_G(S_G)$ (respectively, $A_G(S_G)$). Since $O_R^G(f)$ is weak (respectively, norm) relatively compact, therefore the mapping: $\{R_x f : x \in G^{[u]}\}$ is weak (respectively, norm) relatively compact. Hence the weak (respectively, norm) topology on $\{R_x f : x \in G^{[u]}\}$ coincides with the pointwise topology. Therefore $x \mapsto R_x f$ is weak (respectively, norm) continuous on $G^{[u]}$ for each $[u] \in G^0/G$.

(2) If G is compact of finite orbit space G^0/G, then we prove that for $f \in C(S_G)$ the restriction of the mapping: $x \mapsto R_x f$ to $G^{[u]}$ is weak and norm continuous for

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each $[u] \in G^0/G$. But this follows from [1, Lemma A.9] for weak continuous and from [1, Lemma B.3] for norm continuous applied to the jointly continuous mapping $(x, Y) \mapsto f(Y \circ x)$ [$: G^{[u]} \times S \to \mathbb{C}$]. Now since G is compact, by Proposition 3.5 G_w^2 is closed, hence $G^{[u]}$ is closed and so is compact for each $[u] \in G^0/G$. Therefore $\{R_x f : x \in G^{[u]}\}$ is both weak and norm compact for each $[u] \in G^0/G$. Now $O_R^G(f) = \bigcup_{[u]\in G^0/G} \{R_x f : x \in G^{[u]}\}$ is a finite union of weak and norm compact sets, so is both weak and norm compact.

(3) is similar to (2). ∎

As a result of the above theorem we obtain the following corollary.

Corollary 4.12. *Let G be a principal compact topological groupoid such that the orbit space G^0/G is finite or the action defined in Definition 4.1 is jointly continuous, then $W_G(S_G) = A_G(S_G) = C(S_G)$.*

Definition 4.13: We denote by $B(W_G(S_G))$ the linear space of all bounded linear operators of the Banach space $W_G(S_G)$. Its weak operator topology is, by definition, the (relative) product topology of $\prod_{f\in W_G(S_G)} X_f$, where each X_f is $W_G(S_G)$ in its weak topology. $B(W_G(S_G))$ is a closed subset of $\prod_{f\in W_G(S_G)} X_f$. It is easy to check that $B(W_G(S_G))$ with the weak operator topology and multiplication "." (=composition) is a semitopological semigroup.

Lemma 4.14. *If G is a principal topological groupoid, then $\{R_x : x \in G\} \subseteq B(W_G(S_G))$. Moreover $\{R_x : x \in G\}$ has a principal groupoid structure and (R_x, R_y) are weakly composable if and only if $(x, y) \in G_w^2$. In this case $R_x \circ_w R_y = R_{x\circ_w y} = R_x.R_y$.*

Proof: In Theorem 4.10 we proved that $W_G(S_G)$ is translation G-invariant, hence $\{R_x : x \in G\} \subseteq B(W_G(S_G))$. We define the following groupoid structure, (R_x, R_y) are composable if and only if $(x, y) \in G^2$ and define $R_x R_y = R_x.R_y$, where $R_x.R_y$ denotes the composition of the operators R_x and R_y in $B(W_G(S_G))$. In this case by part (1) of Lemma 4.4, $R_x.R_y = R_{xy}$, hence $\{R_x : x \in G\}$ is closed under this multiplication. Also define $(R_x)^{-1} = R_{x^{-1}}$, then $r(R_x) = R_x R_{x^{-1}} = R_{xx^{-1}} = R_{r(x)}$ and similarly $d(R_x) = R_{d(x)}$. It is easy to check that $\{R_x : x \in G\}$ with this structure defines a groupoid. If $(r, d)(R_x) = (r, d)(R_z)$, then $r(R_x) = r(R_z) = d(R_{z^{-1}})$, that is $(R_{z^{-1}}, R_x)$ is composable. Hence $(z^{-1}, x) \in G^2$, so $r(x) = d(z^{-1}) = r(z)$. Similarly $d(z) = d(x)$. Since G is principal, therefore $x = z$, and consequently $R_x = R_z$. That is, $\{R_x : x \in G\}$ is a principal groupoid. If $(x, y) \in G_w^2$, then there exists a $z \in G$ with $r(z) = r(y)$ and $d(z) = d(x)$. So

$$r(R_z) = R_{r(z)} = R_{r(y)} = r(R_y) \text{ and } d(R_z) = R_{d(z)} = R_{d(x)} = d(R_x).$$

That is, (R_x, R_y) is weakly composable. Conversely if (R_x, R_y) is weakly composable, then there exists a unique R_z with

$$r(R_z) = r(R_y) \text{ and } d(R_z) = d(R_x). \tag{6}$$

Equivalently $(R_{z^{-1}}, R_y)$ and $(R_x,\ R_{z^{-1}})$ are composable, hence $(z^{-1}, y), (x, z^{-1}) \in G^2$, this means that $(x, y) \in G^2_w$. Finally if (R_x, R_y) is weakly composable, then by Proposition 3.2 and part (1) of Lemma 4.4,

$$\begin{aligned} R_x \circ_w R_y &= R_x(R_z)^{-1}R_y \\ &= R_x R_{z^{-1}} R_y \\ &= R_x . R_{z^{-1}} . R_y \\ &= R_{xz^{-1}y} \\ &= R_{x \circ_w y} \\ &= R_x . R_y, \end{aligned}$$

where R_z satisfies the equality (6). ∎

REMARK 4.15. By Proposition 3.2 the set $\{R_x : x \in G\}$ with the weak multiplication is a semigroupoid. We denote by G^w the weak operator closure of $\{R_x : x \in G\}$ in $B(W_G(S_G))$.

DEFINITION 4.16: A semigroupoid compactification of a topological groupoid G is a pair (ψ, X), where X is a compact, Hausdorff, semitopological semigroupoid and $\psi : G \to X$ is such that $\psi(G)$ is a groupoid and ψ is a groupoid homomorphism with $\overline{\psi(G)} = X$ and the restriction of the mapping: $x \mapsto \psi(x)$ to $G^{[u]}$ is continuous for every $[u] \in G^0/\dot{G}$.

The following theorem whose proof which is adopted from [**2**] is indeed the main result of this paper.

THEOREM 4.17. *Let G be a principal topological groupoid with a continuous weak multiplication, then G^w defines a semitopological semigroupoid compactification for G.*

PROOF: For every $f \in W_G(S_G)$ let $X_f = W_G(S_G)$ and Y_f denotes the weak closure of $O^G_R(f)$ in $W_G(S_G)$. Then from the fact that $W_G(S_G)$ is norm closed (Theorem 4.10) (hence is weak closed in $C(S_G)$ [**2**, Corollary A.7]) it follows that Y_f is the weak closure of $O^G_R(f)$ in $C(S_G)$ and so is weak compact by the hypothesis on f. By Tychonoff Theorem $\prod_{f \in W_G(S_G)} Y_f$ is weak compact. From

$$\{R_x : x \in G\} \subseteq \prod_{f \in W_G(S_G)} Y_f \subseteq \prod_{f \in W_G(S_G)} X_f,$$

it follows that G^w is compact. Now we define the following semigroupoid structure on G^w. $(T, T') \in G^w * G^w$ if there are two nets $(R_{x_\alpha}), (R_{y_\beta})$ with $(x_\alpha, y_\beta) \in G^2_w$ for every α and β, $R_{x_\alpha} \longrightarrow T$ and $R_{y_\beta} \longrightarrow T'$ in weak operator topology. Since $B(W_G(S_G))$ with the composition operation is a semitopological semigroup and $R_{x_\alpha} \circ_w R_{y_\beta} = R_{x_\alpha} . R_{y_\beta}$, it follows that

$$T.T' = \lim_\alpha \lim_\beta R_{x_\alpha} \circ_w R_{y_\beta} = \lim_\beta \lim_\alpha R_{x_\alpha} \circ_w R_{y_\beta}.$$

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If we put $T \circ_w T' = T.T'$ for $(T, T') \in G^w * G^w$, then it is easy to check that $(G^w, \circ_w)$ is a semitopological semigroupoid. In Lemma 4.14 we showed that $R(G)$ is a principal groupoid. Now let $[u] \in G^0/G$ and $(x_\alpha)_{\alpha\in\Lambda}$ be a net with $x_\alpha \longrightarrow x$ in $G^{[u]}$, then by part (2) of Lemma 4.5, for $f \in W_G(S_G)$ and $X \in S_G$

$$\begin{aligned} \lim(R_{x_\alpha} f)(X) &= \lim f(X \circ x_\alpha) \\ &= f(X \circ x) \\ &= R_x f(X). \end{aligned}$$

That is the mapping: $x \mapsto R_x f$ $[: G^{[u]} \longrightarrow W_G(S_G)]$ is continuous for pointwise topology. Since $\overline{O_R^G(f)}^w$ is compact and weak topology of $\overline{O_R^G(f)}^w$ is stronger than the pointwise topology and the pointwise topology is Hausdorff, hence the two topologies coincide. Therefore for $f \in W_G(S_G)$ the mapping: $x \mapsto R_x f$ $\left[: G^{[u]} \longrightarrow W_G(S_G)\right]$ is weak continuous. Thus the mapping: $x \mapsto R_x\left[: G^{[u]} \longrightarrow B\big(W_G(S_G)\big)\right]$ is continuous. Since the topology in G^w is the weak operator topology. Finally it is easy to check that $x \mapsto R_x$ is both groupoid and semigroupoid homomorphism. ∎

REMARK 4.18. Similar to the proof of Theorem 4.17, when G is a groupoid of finite orbit space G^0/G and closed graph R_G, by using part (3) of Lemma 4.5 we can show that the mapping: $x \mapsto R_x$ in Theorem 4.17 is continuous on G.

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