BULL. AUSTRAL. MATH. SOC. 34K15, 34C10
VOL. 43 (1991) [147–152]

OSCILLATIONS FOR FIRST ORDER NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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In this paper, sufficient conditions for oscillations of the first order neutral differential equation with variable coefficients

$$(x(t) - cx(t-\tau))' + p(t)x(t-\sigma) - q(t)x(t-\mu) = 0$$

are obtained, where c, τ, σ and μ are positive constants, $p, q \in C\big([t_0, \infty), R^+\big)$.

1. INTRODUCTION

During the past decade the oscillation theory of first order neutral delay differential equations has been extensively developed by many authors. Particularly, we mention the papers by Grove, Kulennovic and Ladas [2], and Ladas and Sficas [4, 5], who investigated neutral delay differential equations with variable coefficients.

In this paper we consider the first order neutral delay differential equations with positive and negative variable coefficients

$$(x(t) - cx(t-\tau))' + p(t)x(t-\sigma) - q(t)x(t-\mu) = 0 \tag{1}$$

where $t \geqslant t_0$, c, τ, σ and μ are positive constants, $p, q \in C([t_0, \infty), R^+)$. Let $\phi \in C(t_0 - T, t_0], R)$, where $T = \max\{\tau, \sigma, \mu\}$. By a solution of equation (1) with initial function ϕ at t_0 we mean a function $x \in C([t_0 - T, \infty), R)$ such that $x(t) = \phi(t)$ for $t_0 - T \leqslant t \leqslant t_0$. Now $x(t) - cx(t-\tau)$ is continuously differentiable for $t \geqslant t_0$, and x satisfies equation (1) for all $t \geqslant t_0$. Using the method of steps, it follows that for any continuous initial function ϕ, there exists a unique solution of equation (1) valid for $t \geqslant t_0$.

It is customary to define a real valued function $x(t)$ on an interval of the form $[t_0, \infty)$ to be oscillatory if there exists a sequence of real numbers $\{t_m\} \to \infty$ as $m \to \infty$ such that $t_m \in [t_0, \infty)$ and $x(t_m) = 0$, $m = 1, 2, \ldots$. For our purpose, it is convenient to use the following definition established by Gopalsamy [1].

Received 19 March 1990

DEFINITION: A real valued differentiable function $x(t)$ defined on $[t_0, \infty)$ is said to be oscillatory on $[t_0, \infty)$, if there exists a sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that

$$t_m \in [t_0, \infty) \text{ and } x(t_m)x'(t_m) = 0, m = 1, 2, \ldots. \tag{2}$$

A real valued differentiable function $x(t)$ defined on $[t_0, \infty)$ is said to be nonoscillatory on $[t_0, \infty)$ if there exists a number $t_1 \geqslant t_0$, such that

$$x(t)x'(t) \neq 0, \qquad \forall\, t \geqslant t_1. \tag{3}$$

The following lemma will be used in the proof of our results, it can be found in [**3**].

LEMMA. *Let ρ be a positive constant, $h \in C([t_0, \infty), R^+)$, and assume that*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} h(s)ds > \frac{1}{e}.$$

Then (a) the differential difference inequaltiy

$$z'(t) + h(t)z(t-\rho) \leqslant 0, \quad t \geqslant t_0$$

has no eventually positive solutions;

(b) the differential difference inequality

$$z'(t) + h(t)z(t-\rho) \geqslant 0, \quad t \geqslant t_0$$

has no eventually negative solutions.

2. MAIN RESULTS

THEOREM 1. *Assume that*

(a) $c < 1$, $\sigma > \mu$;

(b) *there exists a $t_1 \geqslant t_0$, such that*

$$p(t) \geqslant q(t-(\sigma-\mu)), \quad \forall t \geqslant t_1 \geqslant t_0;$$

(c) $\liminf_{t\to\infty} \int_{t-\sigma}^{t} p(s)ds > 1/e$;

(d) $\int_{t_0}^{\infty} q(s)ds < \infty$.

Then every solution of equation (1) is oscillatory.

PROOF: Suppose equation (1) has a nonoscillatory solution $x(t)$, that is, there exists a number $t_1 \geqslant t_0$ such that (3) holds. Without loss of generality, suppose that $x(t) > 0$ for $t \geqslant t_2 \geqslant t_1$. We consider the following cases:

I. $x(t)$ is bounded on $[t_2, \infty)$. From (1) we have

$$(4)\quad [x(t) - cx(t-\tau) - \int_{t-(\sigma-\mu)}^{t} q(s)x(s-\mu)ds]' + [p(t) - q(t-(\sigma-\mu))]x(t-\sigma) = 0$$

for all $t \geqslant t_2 + \sigma$. Let

$$(5)\qquad z(t) = x(t) - cx(t-\tau) - \int_{t-(\sigma-\mu)}^{t} q(s)x(s-\mu)ds.$$

From condition (d), $\int_{t_0}^{\infty} q(s)ds$ converges, and $x(t)$ is bounded, so we have

$$\int_{t_2+\sigma}^{\infty} q(s)x(s-\mu)ds < \infty,$$

that is

$$\lim_{t\to\infty} \int_{t-(\sigma-\mu)}^{t} q(s)x(s-\mu)ds = 0.$$

Then, by (4), we get

$$(6)\qquad z'(t) + [p(t) - q(t-(\sigma-\mu))]x(t-\sigma) = 0, \quad t \geqslant t_2 + \sigma.$$

From condition (b) we have $z'(t) \leqslant 0$ for all $t \geqslant t_2 + \sigma$. Hence $z(t)$ is nonincreasing and bounded on $[t_2 + \sigma, \infty)$, so $\lim_{t\to\infty} z(t)$ exists. Denote

$$l = \lim_{t\to\infty} z(t).$$

By (5), we have

$$(7)\qquad \lim_{t\to\infty} x(t) = \frac{l}{1-c},$$

where $l > 0$ or $l = 0$.

If $l > 0$, integrating (1) from $t - \sigma$ to t, we get

(8)

$$x(t) - cx(t-\tau) - x(t-\sigma) + cx(t-\sigma-\tau) + \int_{t-\sigma}^{t} p(s)x(s-\tau)ds - \int_{t-\sigma}^{t} q(s)x(s-\mu)ds = 0.$$

From (7), there is a $t_3 \geqslant t_2 + \sigma$ such that

$$\frac{l}{2(1-c)} < x(t) < \frac{2l}{1-c}, \quad t \geqslant t_3.$$

https://doi.org/10.1017/S0004972700028872 Published online by Cambridge University Press

By (8), we have

(9)

$$x(t)-cx(t-\tau)-x(t-\sigma)+cx(t-\sigma-\mu)+\frac{l}{2(1-c)}\int_{t-\sigma}^{t}p(s)ds-\frac{2l}{1-c}\int_{t-\sigma}^{t}q(s)ds\leqslant 0$$

for all $t\geqslant t_3+2\sigma$. Taking the lower limit in (9) as $t\to\infty$, we get

$$\frac{l}{2(1-c)}\int_{t-\sigma}^{t}p(s)ds\leqslant 0,$$

which contradicts condition (c).

If $l=0$, we claim that

$$z(t)>0,\ \forall t\geqslant t_2+\sigma.$$

Indeed, since $z(t)$ is nonincreasing and bounded on $[t_2+\sigma,\infty)$, from conditions (c) and (d), we have

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t}[p(s)-q(t-(\sigma-\mu))]ds=\liminf_{t\to\infty}\int_{t-\sigma}^{t}p(s)ds\geqslant\frac{1}{e},$$

so $p(t)-q(t-(\sigma-\mu))\geqslant 0$ and is not identically and eventually zero, which means $z'(t)\leqslant 0$ for $t\geqslant t_2+\sigma$ and is not identically and eventually zero. Therefore, if there is a $t_4\geqslant t_2+\sigma$ such that $z(t_4)\leqslant 0$, then $\lim_{t\to\infty}z(t)<0$, which contradicts $\lim_{t\to\infty}z(t)=0$. So we must have

$$z(t)>0,\ t\geqslant t_2+\sigma. \tag{10}$$

On the other hand, by (6), condition (b) and the fact that $z(t)\leqslant x(t)$ for $t\geqslant t_2+2\sigma$, we have

$$z'(t)+[p(t)-q(t-(\sigma-\mu))]z(t-\sigma)\leqslant 0,\ t\geqslant t_2+2\sigma. \tag{11}$$

By the Lemma and conditions (c) and (d), (11) has no eventually positive solutions, which contradicts (10).

II. $x(t)$ is unbounded on $[t_2,\infty)$. From (1) we have

$$(x(t)-cx(t-\tau))'\leqslant q(t)x(t-\sigma),\ t\geqslant t_2+\sigma. \tag{12}$$

Integrating (12) from $t_2+\sigma$ to t, we get

$$x(t)-cx(t-\tau)\leqslant x(t_2+\sigma)-cx(t_2+\sigma-\tau)+\int_{t_2-\sigma}^{t}q(s)x(s-\mu)ds. \tag{13}$$

Since $x(t)$ is positive and unbounded, in view of (3), we must have $x'(t) > 0$ for $t \geqslant t_2 + \sigma$. So $x(t)$ is nondecreasing; then there exists a $t_5 \geqslant t_2 + \sigma$ such that

$$x(t) \geqslant x(t-\tau), \ \forall t \geqslant t_5.$$

Hence (13) implies that

$$(1-c)x(t) \leqslant x(t_2+\sigma) - cx(t_2+\sigma-\tau) + \int_{t_2+\sigma-\mu}^{t-\mu} q(s+\mu)x(s)ds$$
$$\leqslant x(t_2+\sigma) + \int_{t_2+\sigma-\mu}^{t} q(s+\mu)x(s)ds.$$

that is

$$x(t) \leqslant \frac{x(t_2+\sigma)}{1-c} + \int_{t_2+\sigma-\mu}^{t} \frac{1}{1-c} q(s+\mu)x(s)ds.$$

By the Gronwall-Bellman inequality, we have

$$x(t) \leqslant \frac{x(t_2+\sigma)}{1-c} \exp \int_{t_2+\sigma-\mu}^{t} \frac{1}{1-c} q(s+\mu)ds,$$

which contradicts condition (d). The proof of Theorem 1 is now complete. ∎

REMARK. Let $q(t) = 0$ in Theorem 1; then we get Theorem 7 of Ladas and Sficas [4].

Next we consider the following equation

$$(14) \qquad (x(t) - cx(t-\tau))' + \sum_{i=1}^{n} p_i(t)x(t-\sigma_i) - q(t)x(t-\mu) = 0,$$

where $c < 1$, τ, μ, σ_i $(i = 1, 2, \ldots, n)$ are positive constants, q, $p_i \in C([t_0, \infty), R^+)$, $i = 1, 2, \ldots, n$.

THEOREM 2. *Assume that*

(a) *there is a* $t_1 \geqslant t_0$, $i_0 \in \{1, 2, \ldots, n\}$, *such that* $\sigma_{i_0} > \mu$, *and*

$$p_{i_0}(t) \geqslant q(t - (\sigma_{i_0} - \mu)), \ \forall t \geqslant t_1;$$

(b) $\liminf_{t\to\infty} \int_{t-\sigma_{i_0}}^{t} p_{i_0}(s)ds > 1/e$;

(c) $\int_{t_0}^{\infty} q(s)ds < \infty$.

Then every solution of equation (14) is oscillatory.

PROOF: Suppose that equation (14) has a nonoscillatory solution $x(t)$. Without loss of generality, assume that $x(t) > 0$ for $t \geqslant t_2 \geqslant t_1 \geqslant t_0$; then from (14) we have

$$(x(t) - cx(t-\tau))' + p_{i_0}(t)x(t-\sigma_{i_0}) - q(t)x(t-\mu) \leqslant 0.$$

The rest of the proof is similar to that of Theorem 1, so we omit it here. ∎

https://doi.org/10.1017/S0004972700028872 Published online by Cambridge University Press

References

[1] K. Gopalsamy, 'Oscillations in neutral delay-differential equations', *J. Math. Phys. Sci.* **21** (1987), 23–34.

[2] E.A. Grove, M.R.S. Kulennovic and G. Ladas, 'Sufficient conditions for oscillation and nonoscillation of neutral equations', *J. Differential Equations* **68** (1987), 373–382.

[3] R.G. Koplatadze and T.A. Canturia, 'On oscillatory and monotonic solutions of first-order differential equations with retarded arguments', *Differensial'nye Uravneniya* **8** (1982), 1463–1465.

[4] G. Ladas and Y.G. Sficas, 'Oscillation of neutral delay differential equations', *Canad. Math. Bull.* **29** (1986), 438–445.

[5] G. Ladas and Y.G. Sficas, 'Asymptotic behaviour of oscillatory solutions', *Hiroshima Math. J.* **18** (1988), 351–359.

[6] L. Liao, 'Oscillatory properties of a class of first order retarded differential equations', in *Ordinary differential equations and control theory proceeding*, Editor Z. Deng et al., pp. 237–244, 1987.

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