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MASS CONCENTRATION WITH MIXED NORM FOR A NONELLIPTIC SCHRÖDINGER EQUATION

SEHEON HAM

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Abstract

This paper is concerned with a mass concentration phenomenon for a two-dimensional nonelliptic Schrödinger equation. It is well known that this phenomenon occurs when the L^4-norm of the solution blows up in finite time. We extend this result to the case where a mixed norm of the solution blows up in finite time.

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1. Introduction

We begin with the two-dimensional initial value problem for a nonelliptic nonlinear Schrödinger equation defined by

$$\begin{cases} iu_t + \Box u + \gamma |u|^2 u = 0 \\ u(0, x) = u_0(x) \in L^2(\mathbb{R}^2) \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}\backslash\{0\}$ and $\Box = \partial_{x_1}\partial_{x_2}$. The solution of the linear version of (1.1) (that is, with $\gamma = 0$) can be written as

$$e^{it\Box}u_0(x) = \int_{\mathbb{R}^2} e^{2\pi i(x\cdot\xi - 2\pi t\xi_1\xi_2)}\widehat{u_0}(\xi)\,d\xi.$$

Note that (1.1) is invariant under the scaling

$$u(t, x_1, x_2) \mapsto (\lambda\mu)^{1/2} u(\lambda\mu t, \lambda x_1, \mu x_2)$$

for any $\lambda, \mu > 0$. So, we would have to consider rectangles instead of squares when we decompose $\mathbb{R}^2$.

It is well known that, in (1.1), there exist maximal existence times $T_{\min}, T_{\max} \in (0, \infty]$ and a unique solution

$$u \in C((-T_{\min}, T_{\max}), L^2(\mathbb{R}^2)) \cap L^q_{\mathrm{loc}}((-T_{\min}, T_{\max}), L^r(\mathbb{R}^2))$$

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for any admissible pair (q, r). Recall that (q, r) is called an admissible pair for (1.1) if $q, r \geq 2$, $1/q = 1/2 - 1/r$ and $(q, r) \neq (2, \infty)$. Also $\|u(t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}$ for all $t \in (-T_{\min}, T_{\max})$, regardless of γ. However, unlike the case of the Schrödinger equation, we do not know whether this nonelliptic equation has a blow-up solution related to a given initial datum.

In [12], Rogers and Vargas proved that if $\|u\|_{L^4_{t,x}([0,T_{\max})\times\mathbb{R}^2)} = \infty$ for some $T_{\max} < \infty$, then

$$\limsup_{t\nearrow T_{\max}} \sup_{\substack{\text{a rectangle } R\\ |R|\leq T_{\max}-t}} \int_R |u(t,x)|^2\,dx > \varepsilon \tag{1.2}$$

where ε is a positive constant depending only on γ and $\|u_0\|_{L^2(\mathbb{R}^2)}$. When $\|u\|_{L^4_{t,x}((-T_{\min},0]\times\mathbb{R}^2)}$ blows up, there is also a result similar to (1.2). In this note, we shall show that there is also a mass concentration phenomenon for (1.1) when the mixed norm $\|u\|_{L^q_t L^r_x}$ blows up in finite time.

In the elliptic case, Bourgain [2] proved the mass concentration phenomenon for an L^2-critical nonlinear Schrödinger equation with spatial dimension two. This result was extended to higher-dimensional cases by Bégout and Vargas [1]. They made use of bilinear extension (adjoint restriction) estimates for the paraboloid due to Tao [14] in order to get a refinement of the Strichartz estimate which is an essential ingredient in their argument. Moreover, the case where a mixed norm $L^q_t L^r_x$ of the solution blows up is considered in [4]. In this case, they utilize a mixed-norm generalization of the bilinear extension estimates for the paraboloid due to Lee and Vargas [10]. A similar result for the higher-order Schrödinger equation, $iu_t + (-\triangle)^{\alpha/2}u = \pm|u|^{2\alpha/d}u$, can be found in [5].

Our result may be stated as follows.

Theorem 1.1. *Let (q, r) be an admissible pair with $q \leq r \leq 6$. Also let u be the solution to (1.1). If $\|u\|_{L^q_t L^r_x([0,T_{\max})\times\mathbb{R}^2)} = \infty$ for some $0 < T_{\max} < \infty$ and $\|u\|_{L^q_t L^r_x([0,t]\times\mathbb{R}^2)} < \infty$ for all $t \in (0, T_{\max})$, then*

$$\limsup_{t\nearrow T_{\max}} \sup_{\substack{\text{a rectangle } R\\ |R|\leq T_{\max}-t}} \int_R |u(t,x)|^2\,dx > \varepsilon$$

where ε is a constant depending only on γ and $\|u_0\|_{L^2}$.

The proof of Theorem 1.1 basically follows the argument of Rogers and Vargas [12] which was partially based on a modification of the method of Bougain [2] and some new ideas essential for handling the hyperbolical situation. In the same manner, decomposing $\mathbb{R}^2$ into rectangles, we obtain a separation condition which satisfies the hypothesis of [10, Theorem 2.3], and then we define a more general function space $X^{q,r}_p$ than X_p in [12] (see Definition 3.1 below). Since [10, Theorem 2.3] is valid not only for paraboloid cases but also for some hyperbolic cases, a refinement of Strichartz estimates in [12] could be extended to our mixed-norm case. This refinement is especially meaningful in that it enables the decomposition of initial data $u_0(x)$ into

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a finite sequence of functions, which will be described precisely in Lemma 3.7. We will also make use of some mixed-norm estimates on the space $X_p^{q,r}$ which are adapted from the results in [4].

It is worthwhile to make the following remarks which allow us to restrict the range of an admissible pair (q, r) to $q \le r \le 6$.

REMARK 1.2. It suffices to consider only the case $q \le r$. To see this, observe that if $\|u\|_{L_t^q L_x^r([0,T_{\max})\times\mathbb{R}^2)} = \infty$ for $q \ge r$, then $\|u\|_{L^4_{t,x}([0,T_{\max})\times\mathbb{R}^2)} = \infty$ from interpolation with the mass conservation $\|u\|_{L_t^\infty L_x^2} = \|u_0\|_{L_x^2(\mathbb{R}^2)}$. Indeed, let (q_0, r_0) be an admissible pair with $q_0 \ge r_0$ such that

$$\frac{1}{q_0} = \frac{1-\theta}{\infty} + \frac{\theta}{4} \quad \text{and} \quad \frac{1}{r_0} = \frac{1-\theta}{2} + \frac{\theta}{4}$$

for some $\theta \in (0, 1)$. If $\|u\|_{L_t^{q_0} L_x^{r_0}} = \infty$, then $\|u\|_{L^4_{t,x}} = \infty$ follows from

$$\|u\|_{L_t^{q_0} L_x^{r_0}} \le \Big(\sup_t \|u\|_{L_x^2}\Big)^{1-\theta} \|u\|^{\theta}_{L^4_{t,x}} \quad \text{and} \quad \sup_t \|u\|_{L_x^2} = \|u_0\|_{L_x^2} \ne 0$$

by Hölder's inequality and the conservation of charge. Hence, there exists a mass concentration phenomenon by the result in [12].

REMARK 1.3. For the local well-posedness of (1.1) in the mixed-norm space $L_t^q L_x^r$, we would check if the inhomogeneous part of the solution is a contraction map. Actually, by Duhamel's principle, the solution to (1.1) is given by

$$u(t, x) = e^{it\Box} u_0(x) + i\gamma \int_0^t e^{i(t-s)\Box} |u(s)|^2 u(s)\, ds. \tag{1.3}$$

Using (1.3), the inhomogeneous Strichartz estimate in Lemma 2.1 below and Hölder's inequality, it follows that for any admissible pairs (q, r) and $(\tilde q, \tilde r)$,

$$\begin{aligned}
&\Big\| \int_0^T e^{i(t-s)\Box} [|u(s)|^2 u(s) - |v(s)|^2 v(s)]\, ds \Big\|_{L_t^q L_x^r} \\
&\quad \le \| |u|^2 u - |v|^2 v \|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\
&\quad = \| (|u|^2 - |v|^2) u + |v|^2 (u - v) \|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\
&\quad = \| (|u| - |v|)(|u| + |v|) u + |v|^2 (u - v) \|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\
&\quad \le \| |u - v| ((|u| + |v|)|u| + |v|^2) \|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\
&\quad \le C \| (|u|^2 + |v|^2) |u - v| \|_{L_t^{\tilde q'} L_x^{\tilde r'}} \\
&\quad \le C \| |u|^2 + |v|^2 \|_{L_t^{\frac{3}{2}\tilde q'} L_x^{\frac{3}{2}\tilde r'}} \|u - v\|_{L_t^{3\tilde q'} L_x^{3\tilde r'}} \\
&\quad \le C (\|u\|^2_{L_t^{3\tilde q'} L_x^{3\tilde r'}} + \|v\|^2_{L_t^{3\tilde q'} L_x^{3\tilde r'}}) \|u - v\|_{L_t^{3\tilde q'} L_x^{3\tilde r'}}.
\end{aligned}$$

The conditions $q = 3\tilde q'$ and $r = 3\tilde r'$ imply that $1/6 \le 1/r \le 1/3$. For this range of r, (1.1) is locally well-posed in the mixed norm space $C([0, T]; L^2(\mathbb{R}^2)) \cap L^q([0, T]; L^r(\mathbb{R}^2))$ for a small time $T < T_{\max}$.

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REMARK 1.4. For observing a mass concentration phenomenon, the inhomogeneous part of the solution does not play a primary role as long as the integral part in (1.3) can be controlled by the solution $u(t, x)$. For example, there may be a mass concentration phenomenon for the hyperbolic-elliptic type Davey–Stewartson system, with subsonic wave packet, which is defined by

$$iu_t - \partial_{x_1}^2 u + \partial_{x_2}^2 u = (\pm|u|^2 + \mathcal{B}(|u|^2))u$$

where

$$\widehat{\mathcal{B}(f)}(\xi_1, \xi_2) = \frac{-\gamma\xi_1^2}{\xi_1^2 + \xi_2^2}\hat{f}(\xi_1, \xi_2) \quad \text{and} \quad \gamma > 0.$$

A detailed discussion of the Davey–Stewartson system may be found in [13].

In practice, it suffices to show that

$$\| \pm |u|^2 u + \mathcal{B}(|u|^2)u\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}((T_0,T_1)\times\mathbb{R}^2)} \le C\|u\|_{L_t^q L_x^r}^3$$

for $q = 3\tilde{q}$ and $r = 3\tilde{r}$.

Note that $\|\mathcal{B}(f)\|_{L_x^p} \le C\|f\|_{L_x^p}$ for $1 < p < \infty$ by the Marcinkiewicz multiplier theorem. Thus

$$\begin{aligned}\|\mathcal{B}(|u|^2)u\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} &\le \|\mathcal{B}(|u|^2)\|_{L_t^{q/2} L_x^{r/2}} \|u\|_{L_t^q L_x^r} \\ &\le \Big\| C\| |u|^2 \|_{L_x^{r/2}} \Big\|_{L_t^{q/2}} \|u\|_{L_t^q L_x^r} \le C\|u\|_{L_t^q L_x^r}^3.\end{aligned}$$

Using the triangle inequality, we obtain the desired result.

This paper is organized as follows. In Section 2 we obtain some Strichartz estimates for the operator $e^{it\Box}$, which is proved in the same manner as in the case of the Schrödinger operator $e^{it\Delta}$. In Section 4 we give a proof of Theorem 1.1. In Section 3 we prove some useful and technical lemmas which are used in Section 4.

2. Strichartz estimates

In this section a brief review of Strichartz estimates will be given. The following argument may be found in [3, 7] or [15].

To get the dual operator of $e^{it\Box}$, we need the following calculation:

$$\begin{aligned}\langle e^{it\Box}u_0(x), v(t, x)\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{it\Box}u_0(x)\overline{v(t, x)}\, dx\, dt \\ &= \iiiint u_0(y)e^{-2\pi i y\cdot\xi} e^{2\pi i(x\cdot\xi - 2\pi t\xi_1\xi_2)}\overline{v(t, x)}\, d\xi\, dy\, dx\, dt \\ &= \int u_0(y)\overline{\iint \hat{v}(t, \xi)e^{2\pi i(y\cdot\xi + 2\pi t\xi_1\xi_2)}\, d\xi\, dt}\, dy.\end{aligned}$$

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Hence, the dual operator of $e^{it\Box}F(x)$ is $\int e^{-it\Box}F_t(x)\,dt$, where $F = F(t,x) = F_t(x)$.

Our claim is that

$$\Big\|\int_{\mathbb{R}} e^{-it\Box}F_t\,dt\Big\|_{L^2_x(\mathbb{R}^2)} \lesssim \|F\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^2)}$$

for every $F \in L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^2)$ and any admissible pair (q,r). Here and throughout this paper, q' denotes the conjugate exponent of q defined by $1/q + 1/q' = 1$.

Let (q,r) be an admissible pair with $2 \le q, r \le \infty$ and $(q,r) \ne (2,\infty)$. Then

$$\begin{aligned}
\Big\|\int_{\mathbb{R}} e^{-is\Box}F_s\,ds\Big\|^2_{L^2_x} &= \iint e^{-is\Box}F_s\,dt\,\overline{\int e^{-it\Box}F_t\,dt}\,dx \\
&= \iint \langle e^{-is\Box}F_s, e^{-it\Box}F_t\rangle\,ds\,dt \\
&= \iint \langle e^{i(t-s)\Box}F_s, F_t\rangle\,ds\,dt \\
&= \iiint e^{i(t-s)\Box}F_s\overline{F_t(x)}\,dx\,dt\,ds \\
&= \iiint e^{i(t-s)\Box}F_s\,ds\overline{F_t(x)}\,dt\,dx \\
&\le \int\Big(\int\Big|\int e^{i(t-s)\Box}F_s\,ds\Big|^q\,dt\Big)^{1/q}\Big(\int |F|^{q'}\,dt\Big)^{1/q'}\,dx \\
&\le \Big\|\int e^{i(t-s)\Box}F_s\,ds\Big\|_{L^q_tL^r_x}\|F\|_{L^{q'}_tL^{r'}_x}.
\end{aligned}$$

Now, by Minkowski's inequality,

$$\begin{aligned}
\Big\|\int e^{i(t-s)\Box}F_s\,ds\Big\|_{L^q_tL^r_x} &= \Big(\int\Big(\int\Big|\int e^{i(t-s)\Box}F_s\,ds\Big|^r\,dx\Big)^{q/r}\,dt\Big)^{1/q} \\
&\le \Big\|\int_{\mathbb{R}} \|e^{i(t-s)\Box}F_s\|_{L^r_x(\mathbb{R}^2)}\,ds\Big\|_{L^q_t(\mathbb{R})}.
\end{aligned}$$

Let us assume for the moment that $\|e^{it\Box}F_t\|_{L^r_x(\mathbb{R}^2)} \le C|t|^{-2(1/2-1/r)}\|F_t\|_{L^{r'}_x(\mathbb{R}^2)}$ for $2 \le r \le \infty$. Whenever (q,r) is an admissible pair with $2 < q < \infty$ and $2 < r < \infty$, it follows by the Hardy–Littlewood–Sobolev inequality that there is a positive constant C such that

$$\begin{aligned}
\Big\|\int_{\mathbb{R}} \|e^{i(t-s)\Box}F_s\|_{L^r_x(\mathbb{R}^2)}\,ds\Big\|_{L^q_t(\mathbb{R})} &\le C\Big\|\int_{\mathbb{R}} |t-s|^{-2(1/2-1/r)}\|F_s\|_{L^{r'}_x(\mathbb{R}^2)}\,ds\Big\|_{L^q_t(\mathbb{R})} \\
&\le C\|\|F_s\|_{L^{r'}_x(\mathbb{R}^2)}\|_{L^{q'}_t(\mathbb{R})} = C\|F\|_{L^{q'}_tL^{r'}_x}.
\end{aligned}$$

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We need to show that $\|e^{it\Box}F_t\|_{L^r_x} \le (2\pi|t|)^{-2(1/2-1/r)}\|F_t\|_{L^{r'}_x}$. We begin with

$$\begin{aligned} e^{it\Box}F_t(x) &= \int \hat{F}_t(\xi)e^{2\pi i(x\cdot\xi-2\pi t\xi_1\xi_2)}\,d\xi \\ &= \iint F(t,y)e^{-2\pi iy\cdot\xi}\,dy\,e^{2\pi i(x\cdot\xi-2\pi t\xi_1\xi_2)}\,d\xi \\ &= \iint e^{2\pi i(x-y)\cdot\xi}e^{-4\pi it\xi_1\xi_2}\,d\xi F(t,y)\,dt. \end{aligned}$$

If we simplify the phase by making a change of variables,

$$\begin{aligned} \int e^{2\pi ix\cdot\xi}e^{-4\pi it\xi_1\xi_2}\,d\xi &= 2\int e^{2\pi ix\cdot(\zeta_1+\zeta_2,\zeta_1-\zeta_2)}e^{-4\pi it(\zeta_1^2-\zeta_2^2)}\,d\zeta_1\,d\zeta_2 \\ &= 2\int e^{2\pi i(x_1+x_2)\zeta_1}e^{-4\pi it\zeta_1^2}\,d\zeta_1\int e^{2\pi i(x_1-x_2)\zeta_2}e^{-4\pi it\zeta_2^2}\,d\zeta_2 \\ &= 2\Big(\frac{1}{4\pi it}\Big)^{1/2}e^{i(x_1+x_2)^2/4t}\Big(\frac{1}{4\pi it}\Big)^{1/2}e^{i(x_1-x_2)^2/(-4t)} \\ &= \frac{1}{2\pi it}e^{(i/t)(x_1x_2)}. \end{aligned}$$

Therefore,

$$e^{it\Box}F_t(x) = \frac{1}{2\pi it}\int_{\mathbb{R}^2} e^{i(x_1-y_1)(x_2-y_2)/t}F(t,y)\,dt.$$

Hence, we get two estimates as follows:

$$\begin{aligned} \|e^{it\Box}F_t\|_{L^\infty_x} &\le (2\pi t)^{-1}\|F_t\|_{L^1_x} \\ \|e^{it\Box}F_t\|_{L^2_x} &\le \|F_t\|_{L^2_x}. \end{aligned}$$

By interpolating these two estimates, it follows that for $2 \le r \le \infty$,

$$\|e^{it\Box}F_t\|_{L^r(\mathbb{R}^2)} \le (2\pi|t|)^{-2(1/2-1/r)}\|F_t\|_{L^{r'}(\mathbb{R}^2)}.$$

Hence we have the following lemma.

Lemma 2.1 (An inhomogeneous Strichartz estimate). *Let (q,r) and $(\tilde{q},\tilde{r})$ be admissible pairs satisfying $2 \le q, r, \tilde{q}, \tilde{r} \le \infty$, $(q,r) \ne \infty$ and $(\tilde{q},\tilde{r}) \ne \infty$. Then for every $F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^2)$ and $u_0 \in L^2_x(\mathbb{R}^2)$,*

$$\Big\|\int_{\mathbb{R}} e^{-is\Box}F_s\,ds\Big\|_{L^2_x(\mathbb{R}^2)} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^2)} \quad \text{(dual homogeneous)} \tag{2.1}$$

and by duality,

$$\|e^{it\Box}u_0\|_{L^q_tL^r_x} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^2)} \quad \text{(homogeneous)}. \tag{2.2}$$

Moreover, for $t_0 < t$,

$$\Big\|\int_{t_0}^t e^{i(t-s)\Box}F_s\,ds\Big\|_{L^q_tL^r_x} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}} \quad \text{(inhomogeneous)}. \tag{2.3}$$

All omitted constants are positive and depend only on (q,r) or $(\tilde{q},\tilde{r})$.

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In fact,

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\Box} F_s \, ds \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

by (2.1) and (2.2). Then (2.3) follows from the Christ–Kiselev lemma in [6].

3. Proofs of lemmas

The purpose of this section is to prove Lemmas 3.7 and 3.9. We begin with a new function space whose definition is adapted from that of X_p in [11, 12].

Definition 3.1. For each $k, l \in \mathbb{Z}$, we break $\mathbb{R}^2$ up into rectangles $R^j_{k,l}$ such that

$$R^j_{k,l} = [j_1 2^{-k}, (j_1+1)2^{-k}] \times [j_2 2^{-l}, (j_2+1)2^{-l}]$$

where $j = (j_1, j_2) \in \mathbb{Z}^2$. We define a function space $X^{q,r}_p$ by

$$X^{q,r}_p = \left\{ f : \|f\|_{X^{q,r}_p} = \left[\sum_{k,l} \left(\sum_j \left(2^{(k+l)(1/p-1/2)} \left(\int_{R^j_{k,l}} |f|^p \right)^{1/p} \right)^r \right)^{q/r} \right]^{1/q} < \infty \right\}$$

for $1 \le p, q, r \le \infty$. When an index is ∞, we adopt the usual supremum norm interpretation for the corresponding norm.

Then we can observe the following properties of $X^{q,r}_p$.

Lemma 3.2. *If $p < 2 < \min\{q, r\}$, then for some $0 < \theta < 1$, there exists a constant C such that*

$$\|f\|_{X^{q,r}_p} \le C \sup_{j,k,l} \left(|R^j_{k,l}|^{(1/2-1/p)} \left(\int_{R^j_{k,l}} |f|^p \right)^{1/p} \right)^{\theta} \|f\|^{1-\theta}_{L^2}.$$

Proof. For $q \le r$, we have $\|f\|_{X^{q,r}_p} \le \|f\|_{X^{q,q}_p}$. Clearly,

$$\|f\|_{X^{\infty,\infty}_p} \le \sup_{j,k,l} 2^{(k+l)(1/p-1/2)} \left(\int_{R^j_{k,l}} |f|^p \right)^{1/p}. \tag{3.1}$$

If we show that

$$\|f\|_{X^{s,s}_p} \le C \|f\|_{L^2} \tag{3.2}$$

for $p < 2 < s$, then, by interpolation between (3.1) and (3.2),

$$\|f\|_{X^{q,r}_p} \le \|f\|_{X^{q,q}_p} \le \left(\sup_{j,k,l} 2^{(k+l)(1/p-1/2)} \left(\int_{R^j_{k,l}} |f|^p \right)^{1/p} \right)^{1-s/q} \|f\|^{s/q}_{L^2}$$

as long as we choose s smaller than q.

To prove (3.2), we may assume that $\|f\|_{L^2} = 1$. We decompose f into f^m and f_m where $f^m = f\chi_{\{|f| \ge 2^{(k+l)/2}\}}$ and $f_m = f\chi_{\{|f| < 2^{(k+l)/2}\}}$, respectively.

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First, for $p < 2 < s$, there is a constant $C_1 = C_1(p)$ such that

$$\begin{aligned}\sum_j \sum_{k,l}\Big(2^{(k+l)(1/p-1/2)}\Big(\int_{R^j_{k,l}} |f^m|^p\Big)^{1/p}\Big)^s &\le \Big(\sum_j \sum_{k,l} 2^{(k+l)(1/p-1/2)p} \int_{R^j_{k,l}} |f^m|^p\Big)^{s/p} \\ &= \Big(\int \sum_{k,l} 2^{(k+l)(1/p-1/2)p} |f^m|^p\Big)^{s/p} \\ &= \Big(\int \sum_{|f|\ge 2^{(k+l)/2}} 2^{(k+l)(1/p-1/2)p} |f|^p\Big)^{s/p} \\ &\le C_1\Big(\int |f|^{2p(1/p-1/2)+p}\Big)^{s/p} = C_1\|f\|_{L^2}^{2s/p} \le C_1.\end{aligned}$$

Using Hölder's inequality, we also know that there is a constant $C_2 = C_2(s)$ such that

$$\begin{aligned}\sum_j \sum_{k,l}\Big(2^{(k+l)(1/p-1/2)}\Big(\int_{R^j_{k,l}} |f_m|^p\Big)^{1/p}\Big)^s &\le \sum_j \sum_{k,l} 2^{(k+l)(1/s-1/2)s} \int_{R^j_{k,l}} |f^m|^s \\ &= \int_{\mathbb{R}^2} \sum_{k,l} 2^{(k+l)(1/s-1/2)s} |f_m|^s \\ &= \int_{\mathbb{R}^2} \sum_{|f|<2^{(k+l)/2}} 2^{(k+l)(1/s-1/2)s} |f|^s \\ &\le C_2 \int_{\mathbb{R}^2} |f|^{2s(1/s-1/2)+s} = C_2\|f\|_{L^2}^2 = C_2.\end{aligned}$$

As a result, we can choose a constant $C = C(p, s)$ satisfying $\|f\|_{X_p^{s,s}} \le C\|f\|_{L^2}$ when $p < 2 < s$.

On the other hand, for the case $r \le q$, we have $X_p^{r,r} \subset X_p^{q,r}$ and so we obtain again the estimate (3.2) for any $2 < s < r$. This completes the proof. □

To prove Lemma 3.5 stated below, we need some results about bilinear extension estimates on the saddle surface. Fortunately, [10, Theorem 2.3], which is a sort of mixed-norm generalization of the results in [8, 14, 17], is useful in our case. The following theorem is taken from [10].

THEOREM 3.3 [10]. *Assume that $n \ge 2$. Let ϕ_1 and ϕ_2 be smooth functions defined on $[-1, 1]^{n-1}$. Define an operator $E_i f(x, t)$, for $i = 1, 2$, by*

$$E_i f(x, t) = \int_{[-1,1]^{n-1}} e^{i(x\cdot\xi + t\phi_i(\xi))} f(\xi)\, d\xi.$$

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Also, denote the Hessian matrix of ϕ *by* $H\phi$. *If* $\det H\phi_i \neq 0$ *on* $[-1, 1]^{n-1}$ *and for all* $\xi, \zeta \in [-1, 1]^{n-1}$,

$$|\langle H\phi_i^{-1}(\nabla\phi_1(\xi) - \nabla\phi_2(\zeta)), \nabla\phi_1(\xi) - \nabla\phi_2(\zeta)\rangle| \geq c > 0, \tag{3.3}$$

then for $2 < q, r$ *satisfying* $2/q < \min(1, n/4)$ *and* $4/q < n(1 - 2/r)$, *there is a constant* C *such that*

$$\|E_1(f_1)E_2(f_2)\|_{L_t^{q/2}L_x^{r/2}} \leq C\|f_1\|_{L^2}\|f_2\|_{L^2}. \tag{3.4}$$

Remark 3.4. From [9, Theorem 5.1], the condition $4/q < n(1 - 2/r)$ could be extended to $2/q < 2 - 1/r$ when $n = 3$. Then the range of p in (3.5) or (3.6) is also extended. In particular, we may substitute $\frac{16}{13}$ for the infimum $\frac{12}{7}$ of p^* in Lemma 3.5. Nevertheless, the conditions in Theorem 3.3 are enough to prove Lemma 3.5.

Note that the line segment $1/q = 1/2 - 1/r$ with $3 < r \leq 6$ is contained in the area given by $1/q < 3/8$ and $1/q < \frac{3}{4}(1 - 2/r)$. By interpolation between (3.4) and a trivial $L^1 - L^\infty$ estimate, we can conclude that there is a constant C such that

$$\|E_1(f_1)E_2(f_2)\|_{L_t^{q_0/2}L_x^{r_0/2}} \leq C\|f_1\|_{L^p}\|f_2\|_{L^p} \tag{3.5}$$

for some $1 < p < 2$ determined by a given (q_0, r_0) pair satisfying $1/q_0 = 1/2 - 1/r_0$ and $3 < r_0 \leq 6$ (see Figure 1). More precisely, $1/p = 1 - \theta/2$ with $\theta = q/q_0 = r/r_0$.

In our case, if $R_{k,l}^j$ and $R_{k,l}^{j'}$ are separated such that $\xi \in R_{k,l}^j$ and $\zeta \in R_{k,l}^{j'}$ satisfy (3.3), a simple change of variables gives us

$$\|e^{it\Box}f_{k,l}^j e^{it\Box}f_{k,l}^{j'}\|_{L_t^{q/2}L_x^{r/2}} \leq C2^{(k+l)(2/r+2/q-2+2/p)}\|\hat{f}_{k,l}^j\|_{L^p}\|\hat{f}_{k,l}^{j'}\|_{L^p} \tag{3.6}$$

where $f_{k,l}^j$ is the inverse Fourier transform of $\hat{f}_{k,l}^j = \hat{f}\chi_{R_{k,l}^j}$ supported in a rectangle $R_{k,l}^j$.

Lemma 3.5. *Let* (q, r) *be an admissible pair with* $2 < q \leq 4 \leq r$. *Then there is a constant* $C = C(q, r)$ *such that*

$$\|e^{it\Box}f\|_{L_t^qL_x^r} \leq \|\hat{f}\|_{X_{p^*}^{q,q}}$$

for some p^* *with* $12/7 < p^* < 2$.

Proof. We use the notation and terminology in [12] to decompose $\mathbb{R}^2$. Let $R_{k,l}^j$ be a rectangle of dimension $2^{-k} \times 2^{-l}$ as in Definition 3.1. We consider the rectangles $R_{k-1,l-1}^{j_1}$, $R_{k-1,l}^{j_2}$ and $R_{k,l-1}^{j_3}$ containing $R_{k,l}^j$ as the mother, father, and stepfather, respectively. If $R_{k,l}^j$ and $R_{k,l}^{j'}$ have adjacent mothers, but their fathers and stepfathers are not adjacent, we use the notation $R_{k,l}^j \sim R_{k,l}^{j'}$ or simply $j \sim j'$. Then

$$\|e^{it\Box}f\|_{L_t^qL_x^r}^2 = \|e^{it\Box}fe^{it\Box}f\|_{L_t^{q/2}L_x^{r/2}} = \Big\|\Big\|\sum_{k,l}\sum_{j\sim j'}e^{it\Box}f_{k,l}^je^{it\Box}f_{k,l}^{j'}\Big\|_{L_x^{r/2}}\Big\|_{L_t^{q/2}}.$$

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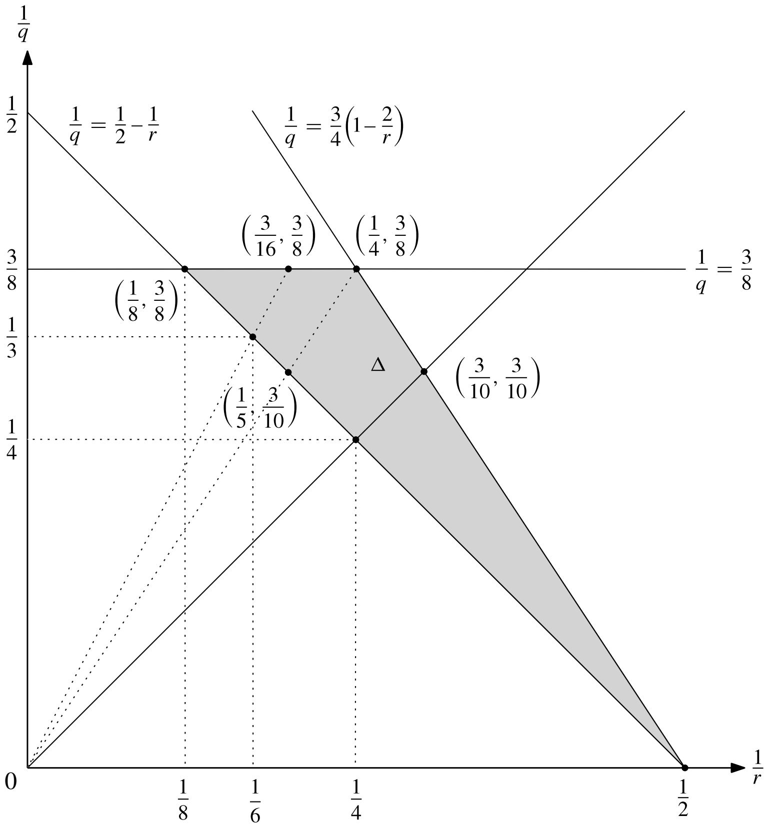

FIGURE 1. For any given admissible pair (q_0, r_0) with $1/q_0 < 3/8$, there is a pair $(q_1, r_1) \in \Delta$. So we can determine p in (3.5) using the ratio of q_0 to q_1.

Now let us assume for the moment that

$$\Big\| \sum_{k,l} \sum_{j\sim j'} e^{it\Box} f^j_{k,l} e^{it\Box} f^{j'}_{k,l} \Big\|_{L^{r/2}_x} \le C\Big(\sum_{k,l} \sum_{j\sim j'} \| e^{it\Box} f^j_{k,l} e^{it\Box} f^{j'}_{k,l} \|^{q/2}_{L^{r/2}_x} \Big)^{2/q}. \tag{3.7}$$

Then, using the fact that (q, r) is an admissible pair, together with (3.6) and the Cauchy–Schwarz inequality, we can say that

$$\begin{aligned}
\Big\| \sum_{k,l} \sum_{j\sim j'} e^{it\Box} f^j_{k,l} e^{it\Box} f^{j'}_{k,l} \Big\|_{L^{q/2}_t L^{r/2}_x} &\le C \Big\| \Big(\sum_{k,l} \sum_{j\sim j'} \| e^{it\Box} f^j_{k,l} e^{it\Box} f^{j'}_{k,l} \|^{q/2}_{L^{r/2}_x} \Big)^{2/q} \Big\|_{L^{q/2}_t} \\
&\le C\Big(\sum_{k,l} \sum_{j\sim j'} \| e^{it\Box} f^j_{k,l} e^{it\Box} f^{j'}_{k,l} \|^{q/2}_{L^{q/2}_t L^{r/2}_x} \Big)^{2/q} \\
&\le C\Big(\sum_{k,l} \sum_{j\sim j'} (2^{(k+l)(2/p^*-1)} \| \hat{f}^j_{k,l} \|_{L^{p^*}} \| \hat{f}^{j'}_{k,l} \|_{L^{p^*}})^{q/2} \Big)^{2/q}
\end{aligned}$$

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$$
\begin{aligned}
&\le C\Big(\sum_{k,l} 2^{(k+l)(2/p^*-1)(q/2)} \sum_j \|\hat{f}^j_{k,l}\|^q_{L^{p^*}}\Big)^{2/q}\\
&= C\Big(\sum_{k,l}\sum_j (2^{(k+l)(1/p^*-1/2)}\|\hat{f}^j_{k,l}\|_{L^{p^*}})^q\Big)^{2/q}
\end{aligned}
$$

for some $12/7 < p^* < 2$ determined by an admissible pair (q, r) by (3.5).

Thus,

$$
\|e^{it\square} f\|_{L^q_t L^r_x} \le C\Big(\sum_{k,l}\sum_j (2^{(k+l)(1/p^*-1/2)}\|\hat{f}^j_{k,l}\|_{L^{p^*}})^q\Big)^{1/q} = C\|\hat{f}\|_{X^{q,q}_{p^*}}.
$$

We now turn to the proof of (3.7) for an admissible pair (q, r). For each t, the support of the Fourier transform of $e^{it\square}\hat{f}^j_{k,l}e^{it\square}\hat{f}^{j'}_{k,l}$ in x is contained in $R^j_{k,l} + R^{j'}_{k,l}$, which is a subset of $\tilde{R}^j_{k,l} = \{(m_1, m_2) \in \mathbb{R}^2 : |m_1 - (j_1 + 3)2^{-k+1}| \le C2^{-k}, |m_2 - (j_2 + 3)2^{-l+1}| \le C2^{-l}\}$. It is easy to verify that $\sum_{k,l}\sum_{j\sim j'} \chi_{\tilde{R}^j_{k,l}}$ is bounded and also that $2\tilde{R}^j_{k,l}$ are almost disjoint. Let us denote by $2R$ the rectangle with the same center as R and side lengths twice those of R. Since (q, r) is admissible and $q \le r$, we have $q/2 = (r/2)' = \min(r/2, (r/2)')$. Therefore our claim will follow from the following estimate.

Lemma 3.6 [16, Lemma 6.1]. *Let R_k be a collection of rectangles in frequency space such that the dilates $2R_k$ are almost disjoint, and suppose that f_k are a collection of functions whose Fourier transforms are supported on R_k. Then for all $1 \le p \le \infty$,*

$$
\Big\|\sum_k f_k\Big\|_p \lesssim \Big(\sum_k \|f_k\|^{p^*}_p\Big)^{1/p^*},
$$

where $p^* = \min(p, p')$. □

We are now ready to prove the decomposition lemma for the initial datum.

Lemma 3.7. *Suppose that $f \in L^2(\mathbb{R}^2)$, $0 < \varepsilon \le \|e^{it\square} f\|_{L^q_t L^r_x}$ and (q, r) is an admissible pair. Then there exist a natural number $N = N(\|f\|_{L^2}, \varepsilon)$ and a finite sequence of functions $\{f_n\}_{1\le n\le N}$ such that $\hat{f}_n$ is supported in a rectangle R_n, $|\hat{f}_n| \le A|R_n|^{-1/2}$ for some constant A, and*

$$
\Big\|e^{it\square} f - \sum_{n=1}^N e^{it\square} f_n\Big\|_{L^q_t L^r_x(\mathbb{R}^3)} < \varepsilon.
$$

Proof. By Lemmas 3.2 and 3.5, there exist $p < 2$ and a rectangle R_1 such that

$$
\varepsilon \le \|e^{it\square} f\|_{L^q_t L^r_x} \le C\Big(|R_1|^{p/2-1}\int_{R_1} |\hat{f}|^p\Big)^{(1/p)(1-\theta)} \|f\|^\theta_{L^2}
$$

for some $\theta \in (0, 1)$.

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It follows that

$$\int_{R_1} |\hat{f}|^p \geq (\varepsilon \|f\|_{L^2}^{-\theta})^{p/(1-\theta)} |R_1|^{1-p/2} =: c|R_1|^{1-p/2}.$$

Let $\lambda = (2c^{-1}\|f\|_{L^2}^2)^{1/(2-p)}|R_1|^{-1/2}$. By Plancherel's theorem,

$$\int_{R_1 \cap \{|\hat{f}|>\lambda\}} |\hat{f}|^p = \int_{R_1 \cap \{|\hat{f}|>\lambda\}} |\hat{f}|^{p-2}|\hat{f}|^2 \leq \lambda^{p-2}\|f\|_{L^2}^2.$$

On the other hand,

$$\int_{R_1 \cap \{|\hat{f}|\leq\lambda\}} |\hat{f}|^p = \int_{R_1} |\hat{f}|^p - \int_{R_1 \cap \{|\hat{f}|>\lambda\}} |\hat{f}|^p \geq c|R_1|^{1-p/2} - \lambda^{p-2}\|f\|_{L^2}^2. \tag{3.8}$$

By Hölder's inequality,

$$\int_{R_1 \cap \{|\hat{f}|\leq\lambda\}} |\hat{f}|^p \leq \Big(\int_{R_1 \cap \{|\hat{f}|\leq\lambda\}} |\hat{f}|^2 \Big)^{p/2} |R_1|^{1-p/2},$$

and hence, by (3.8),

$$\Big(\frac{c}{2}\Big)^{2/p} \leq \int_{R_1 \cap \{|\hat{f}|\leq\lambda\}} |\hat{f}|^2.$$

Define f_1 and f^1 by $\hat{f}_1 = \hat{f}\chi_{R_1 \cap \{|\hat{f}|\leq\lambda\}}$ and $\hat{f}^1 = \hat{f} - \hat{f}_1$. Then $\hat{f}_1$ is supported in $|R_1|$ and $|\hat{f}_1| \leq \lambda = A|R_1|^{-1/2}$, where

$$A = (2c^{-1}\|f\|_{L^2}^2)^{1/(2-p)} = (2(\varepsilon\|f\|_{L^2}^{-\theta})^{-p/(1-\theta)}\|f\|_{L^2}^2)^{1/(2-p)} = (2\varepsilon^{-p/(1-\theta)}\|f\|_{L^2}^{2+\theta p/(1-\theta)})^{1/(2-p)}.$$

If $\|e^{it\Box}f^1\|_{L_t^q L_x^r} \geq \varepsilon$, we repeat the above procedure with f^1, a rectangle R_2 and $\lambda_1 = (2c^{-1}\|f^1\|_{L^2}^2)^{1/(2-p)}|R_2|^{-1/2}$ in place of f, R_1 and λ. Continuing in this way we get a sequence of functions $\hat{f}_{k-1} = \hat{f}_k + \hat{f}^k$ where $\hat{f}_k$ is supported in a rectangle R_k, and

$$|\hat{f}_k| \leq (2c^{-1}\|\hat{f}^{k-1}\|_{L^2}^2)^{1/(2-p)}|R_k|^{-1/2} \leq (2\varepsilon^{-p/(1-\theta)}\|f\|_{L^2}^{2+\theta p/(1-\theta)})^{1/(2-p)}|R_k|^{-1/2} = A|R_k|^{-1/2}.$$

Furthermore,

$$\int |\hat{f}_k|^2 \geq \Big(\frac{(\varepsilon\|\hat{f}^{k-1}\|_{L^2}^{-\theta})^{p/(1-\theta)}}{2} \Big)^{2/p} \geq \Big(\frac{(\varepsilon\|f\|_{L^2}^{-\theta})^{p/(1-\theta)}}{2} \Big)^{2/p} = \Big(\frac{c}{2}\Big)^{2/p}.$$

Since the R_k are pairwise disjoint by construction,

$$\|\hat{f}\|_{L^2}^2 = \|\hat{f}_1\|_{L^2}^2 + \|\hat{f} - \hat{f}_1\|_{L^2}^2 = \|\hat{f}_1\|_{L^2}^2 + \|\hat{f}_2\|_{L^2}^2 + \|\hat{f} - \hat{f}_1 - \hat{f}_2\|_{L^2}^2$$

and

$$\Big\|\hat{f} - \sum_{j=1}^n \hat{f}_j\Big\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2 - \sum_{j=1}^n \|\hat{f}_j\|_{L^2}^2 \leq \|\hat{f}\|_{L^2}^2 - n\Big(\frac{c}{2}\Big)^{2/p}. \tag{3.9}$$

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So the Strichartz estimate in Lemma 2.1 and (3.9) imply that

$$\Big\|e^{it\Box}f-\sum_{i=1}^{n}e^{it\Box}f_j\Big\|_{L_t^qL_x^r}^2\le\Big\|\hat{f}-\sum_{j=1}^{n}\hat{f}_j\Big\|_{L^2}^2\le\Big\|\hat{f}\Big\|_{L^2}^2-n\Big(\frac{c}{2}\Big)^{2/p}.$$

As a result, there exists a number N such that

$$\Big\|e^{it\Box}f-\sum_{i=1}^{N}e^{it\Box}f_j\Big\|_{L_t^qL_x^r}<\varepsilon. \qquad \Box$$

The next observation will be useful for proving Lemma 3.9.

LEMMA 3.8. *Let $2<q\le r\le\infty$. Suppose that $\hat{f}$ is supported in the unit square. For any (q,r) satisfying $2/q+3/r<3/2$, $2/q+2/r<2-2/q$ and $2/q>2(1/2-1/r)$, there exists a constant $C=C(q,r)$ such that*

$$\|e^{it\Box}f\|_{L_t^qL_x^r}\le C\|\hat{f}\|_{L^\infty}.$$

PROOF. We may assume that $\|\hat{f}\|_{L^\infty}=1$. It suffices to show that $\|e^{it\Box}f\|_{L_t^qL_x^r}\le C$ for some constant C. Let $r^*=\min(r/2,(r/2)')$. Then

$$\begin{aligned}\Big\|\sum_{j\sim j'}e^{it\Box}f_{k,l}^{j}e^{it\Box}f_{k,l}^{j'}\Big\|_{L_t^{q/2}L_x^{r/2}}&\le\tilde{C}\Big\|\Big(\sum_{j\sim j'}\|e^{it\Box}f_{k,l}^{j}e^{it\Box}f_{k,l}^{j'}\|_{L_x^{r/2}}^{r^*}\Big)^{1/r^*}\Big\|_{L_t^{q/2}}\\&\le\tilde{C}\Big(\sum_{j\sim j'}\|e^{it\Box}f_{k,l}^{j}e^{it\Box}f_{k,l}^{j'}\|_{L_t^{q/2}L_x^{r/2}}^{q/2}\Big)^{2/q}\\&\le\tilde{C}\Big(\sum_{j\sim j'}(2^{(k+l)(2/r+2/q-1)}\|\hat{f}_{k,l}^{j}\|_{L^2}\|\hat{f}_{k,l}^{j'}\|_{L^2})^{q/2}\Big)^{2/q}\\&=\tilde{C}2^{(k+l)(2/r+2/q-1)}\Big(\sum_{j\sim j'}\|\hat{f}_{k,l}^{j}\|_{L^2}^{q/2}\|\hat{f}_{k,l}^{j'}\|_{L^2}^{q/2}\Big)^{2/q}\\&\le\tilde{C}2^{(k+l)(2/r+2/q-1)}\Big(\sum_{j}\|\hat{f}_{k,l}^{j}\|_{L^2}^{q}\Big)^{2/q}.\end{aligned}$$

It then follows that

$$\begin{aligned}\|e^{it\Box}f\|_{L_t^qL_x^r}^2&\le\sum_{k,l}\Big\|\sum_{j\sim j'}e^{it\Box}f_{k,l}^{j}e^{it\Box}f_{k,l}^{j'}\Big\|_{L_t^{q/2}L_x^{r/2}}\\&=\tilde{C}\sum_{k+l\ge0}2^{(k+l)(2/r+2/q-1)}\Big(\sum_{j}\|\hat{f}_{k,l}^{j}\|_{L^2}^{q}\Big)^{2/q}\\&\quad+\tilde{C}\sum_{k+l<0}2^{(k+l)(2/r+2/q-1)}\Big(\sum_{j}\|\hat{f}_{k,l}^{j}\|_{L^2}^{q}\Big)^{2/q}\\&=:I+II.\end{aligned}$$

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First, by Hölder's inequality,

$$\Big(\sum_j \|\hat{f}^j_{k,l}\|^q_{L^2}\Big)^{2/q} \le \Big(\sum_j \|\hat{f}^j_{k,l}\|^q_{L^q}\Big)^{2/q} 2^{(k+l)(2/q-1)} \le 2^{(k+l)(2/q-1)}\|\hat{f}\|^2_{L^q}.$$

Hence,

$$I \le \tilde{C} \sum_{k+l\ge 0} 2^{(k+l)(4/q+2/r-2)}\|\hat{f}\|^2_{L^q} \le C_1\|\hat{f}\|^2_{L^q} \le C_1.$$

The last inequality follows from the fact that $\hat{f}$ is supported in the unit square and $\|\hat{f}\|_{L^\infty} = 1$.

On the other hand, since $2 < q$ and $2/r + 2/q - 1 > 0$,

$$II \le \tilde{C} \sum_{k+l<0} 2^{(k+l)(2/r+2/q-1)} \sum_j \|\hat{f}^j_{k,l}\|^2_{L^2} = \tilde{C} \sum_{k+l<0} 2^{(k+l)(2/r+2/q-1)}\|\hat{f}\|^2_{L^2} \le C_2.$$

Combining these two estimates, we can conclude that there exists a constant $C = C(q, r)$ such that $\|e^{it\Box} f\|_{L^q_t L^r_x} \le C$. □

From the following lemma, we could find the mass concentrating region.

Lemma 3.9. *Let (q, r) be an admissible pair. Suppose that $f \in L^2(\mathbb{R}^2)$ and its Fourier transform $\hat{f}$ is supported in a rectangle R with center $\zeta = (\zeta_1, \zeta_2)$ and also that $|\hat{f}| \le A|R|^{-1/2}$ for some constant $A > 0$. Let $\varepsilon > 0$ be given. Then there exists a finite sequence of sets $\{Q_n\}_{1\le n\le N(A,\|f\|_{L^2},\varepsilon)}$ defined by $Q_n = \{(t, x) \in \mathbb{R}\times\mathbb{R}^2 : (x_1 - 2\pi t\zeta_2, x_2 - 2\pi t\zeta_1) \in R_n, t \in I_n\}$, where R_n is a rectangle of measure $|R|^{-1}$ and I_n is an interval of length $|R|^{-1}$, such that*

$$\|e^{it\Box} f\|_{L^q_t L^r_x(\mathbb{R}^3\setminus \cup Q_n)} < \varepsilon.$$

Proof. Suppose that a rectangle R has dimensions $2a \times 2b$ and center ζ. Then $|\hat{f}| \le A(ab)^{-1/2}$. Now, we make use of a change of variables after a translation $\xi \mapsto \xi + \zeta$ to get a function supported in the unit square:

$$\begin{aligned}
|e^{it\Box} f(x)| &= \Big|\int \hat{f}(\xi)e^{2\pi i(x\cdot\xi - 2\pi t\xi_1\xi_2)}\,d\xi\Big| \\
&= \Big|\int_{|\xi_1|\le a,|\xi_2|\le b} \hat{f}(\xi+\zeta)e^{2\pi i(x\cdot(\xi+\zeta)-2\pi t(\xi_1+\zeta_1)(\xi_2+\zeta_2))}\,d\xi\Big| \\
&= \Big|\int_{|\xi_1|\le a,|\xi_2|\le b} \hat{f}(\xi+\zeta)e^{2\pi i((x_1-2\pi t\zeta_2, x_2-2\pi t\zeta_1)\cdot(\xi_1,\xi_2)-2\pi t\xi_1\xi_2)}\,d\xi\Big| \\
&= \Big|\int_{|\bar{\xi}_1|\le 1,|\bar{\xi}_2|\le 1} \hat{f}((a\bar{\xi}_1, b\bar{\xi}_2)+\zeta) \\
&\qquad \times e^{2\pi i((x_1-2\pi t\zeta_2, x_2-2\pi t\zeta_1)\cdot(a\bar{\xi}_1, b\bar{\xi}_2)-2\pi tab\bar{\xi}_1\bar{\xi}_2)}(ab)\,d\bar{\xi}_1\,d\bar{\xi}_2\Big| \\
&= (ab)^{1/2}|e^{it'\Box} f'(x')|
\end{aligned}$$

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where

$$\hat{f}'(\xi_1, \xi_2) = (ab)^{1/2}\hat{f}((a\xi_1, b\xi_2) + \zeta), \quad t' = abt$$

and

$$x' = (a(x_1 - 2\pi\zeta_2), b(x_2 - 2\pi t\zeta_1)).$$

Note that $\hat{f}'$ is supported in the unit square and that $|\hat{f}'| \le (ab)^{1/2}|\hat{f}| \le A$. By Lemma 3.8, for any admissible pair (q, r), we can find a pair $(\bar{q}, \bar{r})$ such that $\bar{q} < q$, $\bar{r} < r$ and $\bar{r}/\bar{q} = r/q$, and there is a constant C such that $\|e^{it'\square} f'\|_{L^{\bar{q}}_{t'} L^{\bar{r}}_{x'}} \le C\|\hat{f}'\|_{L^\infty}$.

Let $E \subset \mathbb{R} \times \mathbb{R}^2$ be the set defined by $\{(t', x') \in \mathbb{R} \times \mathbb{R}^2 : |e^{it'\square} f'(x')| < \lambda\}$ for some λ. Then

$$\begin{aligned}\|e^{it'\square} f'\|^q_{L^q_{t'} L^r_{x'}(E)} &= \int_{\mathbb{R}} \bigg(\int_{E_{t'}} |e^{it'\square} f'(x')|^{\bar{r}+r-\bar{r}}\, dx' \bigg)^{q/r} dt' \\ &\le \lambda^{(r-\bar{r})q/r} \|e^{it'\square} f'\|^{\bar{q}}_{L^{\bar{q}}_{t'} L^{\bar{r}}_{x'}} \le C\lambda^{(r-\bar{r})q/r}\|\hat{f}'\|^{\bar{q}}_{L^\infty} \le C\lambda^{(r-\bar{r})q/r} A^{\bar{q}},\end{aligned}$$

where $E_t = \{x \in \mathbb{R}^2 : (t, x) \in E\}$. For a given ε, if we choose

$$\lambda_0 \le \min\{2^{-1}(C^{-1} A^{-\bar{q}} \varepsilon^{q^2})^{r/q(r-\bar{r})}, \tfrac{1}{4}A\}$$

sufficiently small, we have $\|e^{it'\square} f'\|_{L^q_{t'} L^r_{x'}(\tilde{E})} \le \varepsilon^q$ where $\tilde{E} = \{(t', x') : |e^{it'\square} f'(x')| < 2\lambda_0\}$.

Since $\hat{f}'$ is supported in the unit square and $|\hat{f}'| \le A$, it follows that $|e^{it'\square} f'(x') - e^{it''\square} f'(x'')| \le cA(|x' - x''| + |t' - t''|)$ for some constant $c > 1$. If $|x' - x''| \le \lambda_0/2cA$ and $|t' - t''| \le \lambda_0/2cA$, then $|e^{it'\square} f'(x')| < \lambda_0$ implies that $|e^{it''\square} f'(x'')| < 2\lambda_0$.

So, for some index set S, we can choose a family of sets $(P_r)_{r\in S} = (J_r, K_r)_{r\in S} \subset \mathbb{R} \times \mathbb{R}^2$ such that, for $(t', x') \in \{|e^{it'\square} f'(x')| \ge 2\lambda_0\}$, K_r is a square of center x' with $|K_r| = (\lambda_0/cA)^2 \le 1/16$ and $J_r \subset \mathbb{R}$ is a closed interval of center t' with $|J_r| = \lambda_0/cA \le 1/4$. Also, $(P_r)_{r\in S}$ satisfies the following: for $(r, s) \in S \times S$ with $r \ne s$, $\mathrm{Int}(P_r) \cap \mathrm{Int}(P_s) = \emptyset$ and

$$\{|e^{it'\square} f'(x')| \ge 2\lambda_0\} \subset \bigcup_{r\in S} P_r \subset \{|e^{it'\square} f'(x')| \ge \lambda_0\}$$

where $\mathrm{Int}(P_r)$ is the interior of P_r.

Let N be the cardinality of S. Then, by the Strichartz inequality, N is bounded. In fact,

$$\begin{aligned}N\Big(\frac{\lambda_0}{cA}\Big)^3 = \Big|\bigcup_{r\in S} P_r\Big| &\le |\{|e^{it'\square} f'(x')| \ge \lambda_0\}| \\ &\le \lambda_0^{-4} \|e^{it'\square} f'\|^4_{L^4(\mathbb{R}^3)} \le \lambda_0^{-4}\|f'\|^4_{L^2} = \lambda_0^{-4}\|f\|^4_{L^2}.\end{aligned}$$

Since $\{|e^{it'\square} f'(x')| \ge 2\lambda_0\}$ is covered by $\{P_n\}_{1\le n\le N}$,

$$\int_{\mathbb{R}} \bigg(\int_{\bar{P}_t} |e^{it'\square} f'(x')|^r\, dx' \bigg)^{q/r} dt' < \varepsilon^q$$

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where

$$\bar{P}_t = \Big\{x' \in \mathbb{R}^2 : (t', x') \in \mathbb{R}^3 \Big\backslash \bigcup_{n=1}^{N} P_n\Big\}.$$

For each $1 \le n \le N$, let Q_n be the set

$$\Big\{(t, x) : \Big|x_1 - 2\pi t\zeta_2 - \frac{x_1^n}{a}\Big| < \frac{1}{4a}, \Big|x_2 - 2\pi t\zeta_1 - \frac{x_2^n}{b}\Big| < \frac{1}{4b}, \Big|t - \frac{t^n}{ab}\Big| < \frac{1}{8ab}\Big\}$$

where $(t^n; x_1^n, x_2^n) = (t^n; x^n)$ denotes the center of P_n. Let $\bar{Q}_t = \{x' \in \mathbb{R}^2 : (t', x') \in \mathbb{R}^3 \backslash \bigcup Q_n\}$. Then

$$\begin{aligned}
&\|e^{it\Box} f\|^q_{L_t^q L_x^r(\mathbb{R}^3 \backslash \cup Q_n)} \\
&= (ab)^{q/2} \int_{\mathbb{R}} \Big(\int_{\bar{Q}_t} |e^{it'\Box} f'(x')|^r \, dx\Big)^{q/r} \, dt \\
&= (ab)^{q/2} \int_{\mathbb{R}} \Big(\int_{\bar{Q}_t} |e^{iabt\Box} f'(a(x_1 - 2\pi t\zeta_2), b(x_2 - 2\pi t\zeta_1))|^r \, dx\Big)^{q/r} \, dt \\
&\le (ab)^{q/2} \int_{\mathbb{R}} \Big(\int_{\bar{P}_t} |e^{i\bar{t}\Box} f'(\bar{x}_1, \bar{x}_2)|^r \frac{1}{ab} \, d\bar{x}\Big)^{q/r} \frac{1}{ab} \, d\bar{t} \\
&= (ab)^{q/2 - (q/r+1)} \|e^{i\bar{t}\Box} f'\|^q_{L_t^q L_x^r(\mathbb{R}^3 \backslash \cup P_n)} < (ab)^{q/2 - (q/r+1)} \varepsilon^q.
\end{aligned}$$

Therefore, we may conclude that

$$\|e^{it\Box} f\|_{L_t^q L_x^r(\mathbb{R}^3 \backslash \cup Q_n)} < (ab)^{1/2 - (1/r+1/q)} \varepsilon = \varepsilon. \qquad \Box$$

4. Mass concentration phenomenon

The following result implies Theorem 1.1, as was observed in Remarks 1.2 and 1.3.

THEOREM 4.1. *Suppose that $u = u(t, x)$ is a solution to*

$$\begin{cases} iu_t + \Box u + \gamma |u|^2 u = 0 \\ u(0, x) = u_0(x) \in L^2(\mathbb{R}^2) \end{cases}$$

for some $\gamma \in \mathbb{R} \backslash \{0\}$. Let (q, r) be an admissible pair with $q \le r \le 6$. Suppose that the solution satisfies $\|u\|_{L_t^q L_x^r([0,t)\times\mathbb{R}^2)} < \infty$ for $0 < t < T_{\max}$ and that $\|u\|_{L_t^q L_x^r([0,T_{\max})\times\mathbb{R}^2)} = \infty$. Then

$$\limsup_{t \nearrow T_{\max}} \sup_{\substack{\text{a rectangle } R \\ |R| \le (T_{\max} - t)}} \Big(\int_R |u(t, x)|^2 \, dx\Big)^{1/2} > \varepsilon$$

where ε is a constant depending only on γ and $\|u_0\|_{L^2(\mathbb{R}^2)}$.

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PROOF. For a small fixed $\eta > 0$, and for all times $T_0 < T_{\max}$, there exists $T_1 < T_{\max}$ such that $\|u\|_{L_t^q L_x^r((T_0,T_1)\times\mathbb{R}^2)} = \eta$. By Duhamel's principle, for $t \in (T_0, T_{\max})$,

$$u_t(x) = e^{i(t-T_0)\Box} u_{T_0}(x) + i\gamma \int_{T_0}^{t} e^{i(t-s)\Box} |u(s)|^2 u(s)\, ds.$$

Step 1. Controlling the inhomogeneous part.

For any $t \in (T_0, T_1)$, let us set $F(u) = i\gamma \int_{T_0}^{t} e^{i(t-s)\Box} |u(s)|^2 u(s)\, ds$. It follows that

$$\|F(u)\|_{L_t^q L_x^r((T_0,T_1)\times\mathbb{R}^2)} \le |\gamma| C \| |u|^2 u\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} = |\gamma| C \|u\|^3_{L_t^q L_x^r} = |\gamma| C \eta^3$$

by (2.3) and Remark 1.3.

Hence, if we choose η small enough such that

$$\eta \le (3^q 2((|\gamma| C)^2 + 1))^{-1/4} \le (1 + |\gamma| C)^{-1/2}, \tag{4.1}$$

it follows that

$$\|e^{i(t-T_0)\Box} u_{T_0}\|_{L_t^q L_x^r((T_0,T_1)\times\mathbb{R}^2)} \ge \eta - |\gamma| C \eta^3 \ge \eta^3.$$

Step 2. Decomposing the initial data.

We start with

$$\begin{aligned}\eta^q &= \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^r\, dx \Big)^{q/r} dt \\ &\le 3^q \Big(I + II + \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^2 \Big| \sum_{n=1}^{N_0} e^{i(t-T_0)\Box} f_n \Big|^{r-2} dx \Big)^{q/r} dt \Big)\end{aligned}$$

where

$$I = \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^2 |u(x,t) - e^{i(t-T_0)\Box} u_{T_0}|^{r-2}\, dx \Big)^{q/r} dt$$

$$II = \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^2 \Big| e^{i(t-T_0)\Box} u_{T_0} - \sum_{n=1}^{N_0} e^{i(t-T_0)\Box} f_n \Big|^{r-2} dx \Big)^{q/r} dt$$

and $\{f_n\}_{n=1}^{N_0}$ is as in the proof of Lemma 3.7 below.

Using Hölder's inequality with $r/2$ and $r/(r-2)$, we estimate

$$\begin{aligned} I &= \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^2 |F(u)|^{r-2}\, dx \Big)^{q/r} dt \\ &\le \int_{T_0}^{T_1} \Big(\Big(\int_{\mathbb{R}^2} |u|^r\, dx \Big)^{2/r} \Big(\int_{\mathbb{R}^2} |F(u)|^r\, dx \Big)^{(r-2)/r} \Big)^{q/r} dt \\ &= \int_{T_0}^{T_1} \Big(\int_{\mathbb{R}^2} |u|^r\, dx \Big)^{2q/r^2} \Big(\int_{\mathbb{R}^2} |F(u)|^r\, dx \Big)^{q(r-2)/r^2} dt \end{aligned}$$

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$$\leq \Big(\int_{T_0}^{T_1}\Big(\int_{\mathbb{R}^2}|u|^r\,dx\Big)^{q/r}\,dt\Big)^{2/r}\Big(\int_{T_0}^{T_1}\Big(\int_{\mathbb{R}^2}|F(u)|^r\,dx\Big)^{q/r}\,dt\Big)^{(r-2)/r}$$
$$= \|u\|_{L_t^qL_x^r}^{2q/r}\|F(u)\|_{L_t^qL_x^r}^{q(r-2)/r} \leq \eta^{2q/r}(|\gamma|C\eta^3)^{q(r-2)/r} = (|\gamma|C)^2\eta^{q+4}$$

because of the fact that $3q-4q/r = q+4$.

Similarly, by Lemma 3.7,

$$II \leq \|u\|_{L_t^qL_x^r}^{2q/r}\Big\|e^{i(t-T_0)\Box}u_{T_0} - \sum_{n=1}^{N_0} e^{i(t-T_0)\Box}f_n\Big\|_{L_t^qL_x^r}^{q(r-2)/r} \leq \eta^{2q/r}(\eta^3)^{q(r-2)/r} = \eta^{q+4}.$$

Therefore, by (4.1),

$$\int_{T_0}^{T_1}\Big(\int_{\mathbb{R}^2}|u|^2\Big|\sum_{n=1}^{N_0} e^{i(t-T_0)\Box}f_n\Big|^{r-2}\,dx\Big)^{q/r}\,dt \geq \frac{\eta^q}{3^q2}.$$

Then there exists an integer n_0 between 1 and N_0 and a function $\hat{f}_0 = \hat{f}_{n_0}$ such that for some $\varepsilon_0 > 0$,

$$\int_{T_0}^{T_1}\Big(\int_{\mathbb{R}^2}|u|^2|e^{i(t-T_0)\Box}f_0|^{r-2}\,dx\Big)^{q/r}\,dt \geq \varepsilon_0$$

where $\hat{f}_0$ is supported in R and $|\hat{f}_0| \leq A|R|^{-1/2}$ from Lemma 3.7.

Step 3. Figuring out the concentration region.

From Lemma 3.9, we can show that there exist an integer N_1 and a set of rectangles $\{Q_n\}_{1\leq n\leq N_1}$, where $Q_n = \{(t,x)\in\mathbb{R}^3 : (x_1-2\pi t\zeta_2, x_2-2\pi t\zeta_1)\in R_n, t\in I_n\}$, R_n is a rectangle of measure $|R|^{-1}$, and I_n is an interval of length $|R|^{-1}$ such that

$$\|e^{i(t-T_0)\Box}f_0\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^2\setminus\bigcup_{n=1}^{N_1}Q_n)} < \Big(\frac{\varepsilon_0}{2\eta^{2q/r}}\Big)^{r/q(r-2)}.$$

By Hölder's inequality with $2/r + (r-2)/r = 1$, on

$$\tilde{Q}_t = \Big\{x\in\mathbb{R}^2 : (t,x)\in((T_0,T_1)\times\mathbb{R}^2)\Big\backslash\bigcup_{n=1}^{N_1}Q_n\Big\},$$

we have

$$\int_{T_0}^{T_1}\Big(\int_{\tilde{Q}_t}|u|^2|e^{i(t-T_0)\Box}f_0|^{r-2}\,dx\Big)^{q/r}\,dt$$
$$\leq \|u\|_{L_t^qL_x^r}^{2q/r}\|e^{i(t-T_0)\Box}f_0\|_{L_t^qL_x^r(((T_0,T_1)\times\mathbb{R}^2)\setminus\bigcup_{n=1}^{N_1}Q_n)}^{q(r-2)/r}$$
$$< \eta^{2q/r}\Big(\frac{\varepsilon_0}{2\eta^{2q/r}}\Big) = \frac{\varepsilon_0}{2}.$$

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It follows that, on $((T_0, T_1)\times\mathbb{R}^2)\cap(\bigcup_{n=1}^{N_1} Q_n)$,

$$\int_{T_0}^{T_1}\Big(\int_{\tilde{Q}^t}|u|^2|e^{i(t-T_0)\square}f_0|^{r-2}\,dx\Big)^{q/r}\,dt\ge\frac{\varepsilon_0}{2}$$

where

$$\tilde{Q}^t=\Big\{x\in\mathbb{R}^2:(t,x)\in((T_0,T_1)\times\mathbb{R}^2)\cap\Big(\bigcup_{n=1}^{N_1}Q_n\Big)\Big\}.$$

Hence there exists a rectangle $Q_0=R_0\times I_0\in\{Q_n\}_{n=1}^{N_1}$ such that

$$\int_{(T_0,T_1)\cap I_0}\Big(\int_{Q_0^t}|u(x,t)|^2|e^{i(t-T_0)\square}f_0|^{r-2}\,dx\Big)^{q/r}\,dt\ge\frac{\varepsilon_0}{2N_1}=:\varepsilon_1$$

where $Q_0^t=\{x\in\mathbb{R}^2:(x_1-2\pi t\zeta_2,x_2-2\pi t\zeta_1)\in R_0,\,t\in I_0\}$, R_0 is a rectangle of measure $|R|^{-1}$ and I_0 is an interval of length $|R|^{-1}$.

Step 4. Determining the size of windows.

Since $|\hat{f}_0|\le A|R|^{-1/2}$ and $\hat{f}_0$ is supported in R,

$$|e^{i(t-T_0)\square}f_0|\le\int_R|\hat{f}|\le|R|A|R|^{-1/2}=A|R|^{1/2},$$

and

$$\begin{aligned}\varepsilon_1&\le\int_{(T_0,T_1)\cap I_0}\Big(\int_{Q_0^t}|u(t,x)|^2|e^{i(t-T_0)\square}f_0|^{r-2}\,dx\Big)^{q/r}\,dt\\&\le(A|R|^{1/2})^{q(r-2)/r}\int_{(T_0,T_1)\cap I_0}\Big(\int_{Q_0^t}|u(t,x)|^2\,dx\Big)^{q/r}\,dt\\&\le(A|R|^{1/2})^{q(r-2)/r}(T_1-T_0)\|u_0\|_{L^2(\mathbb{R}^2)}^{2q/r}.\end{aligned}$$

Thus

$$T_1-T_0\ge\frac{\varepsilon_1}{(A|R|^{1/2})^{q(r-2)/r}\|u_0\|_{L^2(\mathbb{R}^2)}^{2q/r}}=:\theta.$$

We can observe that

$$\int_{T_1-\frac{1}{2}\theta}^{T_1}\Big(\int_{Q_0^t}|u(t,x)|^2|e^{i(t-T_0)\square}f_0|^{r-2}\,dx\Big)^{q/r}\,dt\le\frac{1}{2}\theta(A|R|^{1/2})^{q(r-2)/r}\|u_0\|_{L^2}^{2q/r}=\frac{\varepsilon_1}{2}.$$

So

$$\begin{aligned}\frac{\varepsilon_1}{2}&\le\int_{(T_0,T_1-\frac{1}{2}\theta)\cap I_0}\Big(\int_{Q_0^t}|u|^2|e^{i(t-T_0)\square}f_0|^{r-2}\Big)^{q/r}\,dt\\&\le|I_0|\sup_{t\in(T_0,T_1-\frac{1}{2}\theta)}\Big(\int_{Q_0^t}|u|^2|e^{i(t-T_0)\square}f_0|^{r-2}\Big)^{q/r}\\&\le|R|^{-1}(A|R|^{1/2})^{q(r-2)/r}\Big(\sup_{t\in(T_0,T_1-\frac{1}{2}\theta)}\int_{Q_0^t}|u|^2\,dx\Big)^{q/r}.\end{aligned}$$

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Then we can say that

$$\sup_{t\in(T_0,T_1-\frac{1}{2}\theta)}\Big(\int_{Q_0^t}|u|^2\,dx\Big)^{q/r}\geq\frac{\varepsilon_1}{2A^{q(r-2)/r}},$$

or

$$\sup_{t\in(T_0,T_1-\frac{1}{2}\theta)}\int_{Q_0^t}|u|^2\,dx\geq C\Big(\frac{\varepsilon_1}{2}\Big)^{r/q}$$

where $C = 1/A^{r-2}$.

Therefore, for all $T_0 < T_{\max}$, there exist $t_0 \in (T_0, T_1 - 1/2\theta)$ and a rectangle $Q_0^{t_0}$ such that

$$\int_{Q_0^{t_0}}|u(t_0,x)|^2\,dx>\frac{C}{4}\Big(\frac{\varepsilon_1}{2}\Big)^{r/q}.$$

Note that

$$t_0\leq T_{\max}-\frac{1}{2}\theta=T_{\max}-\frac{\varepsilon_2}{|R|^{q(r-2)/2r}}$$

where $\varepsilon_2 = \varepsilon_1(2A^{q(r-2)/r}\|u_0\|_{L^2}^{2q/r})^{-1}$.

Because (q, r) is an admissible pair, $q(r-2)/2r = q/2 - q/r = 1$. Then

$$|Q_0^{t_0}|=\frac{1}{|R|}\leq\frac{1}{\varepsilon_2}(T_{\max}-t_0).$$

Dividing $Q_0^{t_0}$ into $m = \lceil 1/\varepsilon_2\rceil$ rectangles, there exists a rectangle R' such that $|R'| \leq T_{\max} - t_0$. Therefore,

$$\int_{R'}|u(t_0,x)|^2\,dx>\frac{C}{4m}\Big(\frac{\varepsilon_1}{2}\Big)^{r/q}=:\varepsilon_3. \tag{4.2}$$

Step 5. Conclusion.

We consider a sequence $\{T_n\}$ such that $0 = T_1 < T_2 < \cdots < T_n < T_{n+1} < \cdots < T_{\max}$ and $\|u\|_{L_t^qL_x^r((T_n,T_{n+1})\times\mathbb{R}^2)} = \eta$. For each interval (T_n, T_{n+1}), there exist $t_n \in (T_n, T_{n+1})$ such that

$$\sup_{\substack{\text{a rectangle } R\\ |R|\leq(T_{\max}-t_n)}}\Big(\int_R|u(t_n,x)|^2\,dx\Big)^{1/2}>\sqrt{\varepsilon_3}$$

by (4.2). Thus we get a sequence $\{t_n\}$ of time such that $t_n \to T_{\max}$ as $n \to \infty$. This gives the conclusion of the theorem with $\varepsilon = \sqrt{\varepsilon_3}$. □

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SEHEON HAM, Department of Mathematics,
Pohang University of Science and Technology, Pohang 790-784, Korea
e-mail: beatles8@postech.ac.kr

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