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RAMIFICATION GROUPS OF ABELIAN LOCAL FIELD EXTENSIONS

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1. Introduction. Let k be a local field; that is, a complete discrete-valued field having a perfect residue class field. If L is a finite Galois extension of k, then L is also a local field. Let G denote the Galois group $G_{L|k}$. Then the nth ramification group G_n is defined by

$$G_n = \{\sigma \in G\colon \sigma a - a \in P_L^{n+1} \text{ for all } a \in O_L\}, \qquad n \in \mathbf{Z},\ n \geqq 0,$$

where O_L denotes the ring of integers of L, and P_L is the prime ideal of O_L. The ramification groups form a descending chain of invariant subgroups of G:

$$(1) \qquad G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots \supseteq G_s = 1.$$

In this paper, an attempt is made to characterize (in terms of the arithmetic of k) the ramification filters (1) obtained from abelian extensions $L|k$.

For real x, $x \geqq 0$, let $\varphi(x) = \varphi_{L|k}(x)$ denote the function given by

$$\varphi(x) = \sum_{i=1}^{n} \frac{1}{(G_0:G_i)} + \frac{x-n}{(G_0:G_{n+1})},$$

where n is the integer satisfying $n \leqq x < n+1$. For real x, $n-1 < x \leqq n$, we define $G_x = G_n$, and we define the xth ramification group (in the upper numbering) by

$$G^x = G_{\varphi^{-1}(x)}, \qquad x \text{ real},\ x \geqq 0.$$

In this way we obtain a filtration

$$(2) \qquad G \supseteq G^0 \supseteq G^1 \supseteq G^2 \supseteq \ldots \supseteq G^t = 1.$$

By the important theorem of Hasse and Arf [**3**; **1**; **13**, pp. 101–104], $G_n \supset G_{n+1} \Rightarrow \varphi(n)$ is an integer. Because of this theorem, the function φ and the filter (1) can be recovered from (2). Thus, it is enough to characterize the filters (2) obtained from abelian extensions $L|k$.

If $k \subseteq L \subseteq N$, and if $L|k$ and $N|k$ are finite Galois extensions, then the natural restriction $G_{N|k} \to G_{L|k}$ carries $G_{N|k}{}^x$ onto $G_{L|k}{}^x$ for all real $x \geqq 0$; see [**4**; **2**]. In view of this result, if $M|k$ is any (possibly infinite) Galois extension, we define the xth ramification group (upper numbering) by inverse limits:

$$G_{M|k}{}^x = \operatorname{inv}\lim_L (G_{L|k}{}^x), \qquad x \text{ real},\ x \geqq 0,$$

where L runs through all finite Galois extensions of k in M.

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In particular, let $A_k = G_{k_a|k}$, where k_a denotes the maximal abelian extension of k. Thus A_k has a ramification filter

$$A_k \supseteq A_k^0 \supseteq A_k^1 \supseteq A_k^2 \supseteq \dots . \tag{3}$$

The finite abelian extensions L of k are in one-to-one correspondence with the open subgroups U of A_k. If L corresponds to U, then $U = G_{k_a|L}$, $G_{L|k} \cong A_k/U$, and $G_{L|k}^x \cong A_k^x U/U$. In this way, the filtrations (2) coming from abelian extensions $L|k$ can all be obtained from (3). Thus, the original problem reduces to the problem of characterizing the ramification filter (3) as a topological filtered group.

In Theorem 1, we examine the filtration (3) in the case that the residue class field $\bar{k}$ is algebraically closed. This result is a direct application of Serre's local class field theory [**12**]. Theorems 1, 2, and 3 prepare the way for Theorem 4 in which we examine the filtration (3) in the general case. Theorem 5 shows how the properties of (3) given in Theorem 4 actually characterize this filtration, provided the homology group $H_1(g, S_K[p])$ is zero.

In Theorem 6, we examine the ramification filter of an arbitrary finite abelian extension $L|k$; in Theorem 7 we show that, provided $H_1(g, S_K[p]) = 0$, the properties given in Theorem 6 characterize the ramification filters of finite abelian extensions of k. In this regard, the interested reader should consult [**8**], where a somewhat weaker solution is obtained, but for non-abelian extensions; also see [**6**, Appendix 2].

Theorem 8 (together with the remark following it) gives various interpretations of the condition $H_1(g, S_K[p]) = 0$; also see [**7**].

2. Preliminary concepts and terminology.

(a) *Cohomology and homology of profinite groups*. Let G be a profinite group, and let A be a topological G-module. The topological group A^G (A_G) is defined to be the largest submodule (quotient module) of A which is fixed by G. If A is a discrete G-module satisfying

$$A = \text{dir lim}_U (A^U)$$

(where U runs through all open subgroups of G), then the discrete cohomology groups

$$H^q(G, A), \qquad q \geqq 0,$$

may be defined as in [**5**] or [**14**]. Dually, if A is a compact G-module satisfying

$$A = \text{inv lim}_U (A_U),$$

then the compact homology groups

$$H_q(G, A), \qquad q \geqq 0,$$

may be simply defined by Pontryagin duality [**10**]:

$$H_q(G, A) = \chi H^q(G, \chi(A)).$$

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(b) *Fields.* If k is any field, then k_a will denote the maximal abelian extension of k, and A_k will denote the Galois group $G_{k_a|k}$. k_+ will denote the additive group of k, and $k^\times = k - \{0\}$, the multiplicative group. If n is a positive integer, we let $S_k[n]$ denote the group of nth roots of unity in k. If n_1 is a multiple of n_2, then there is a canonical "index" mapping $S_k[n_1] \to S_k[n_2]$ given by $x \to x^i$, where $i = (S_k[n_1]: S_k[n_2])$. We define S_k to be the inverse limit of the groups $S_k[n]$ under the above mappings. If $L|k$ is a Galois extension with Galois group G, then the groups $S_L[n]$ and S_L are compact G-modules, and one may verify that

$$(S_L[n])_G \cong S_k[n] \qquad \text{and} \qquad (S_L)_G \cong S_k.$$

(c) *Local fields.* In this paper, a local field is defined as a complete discrete-valued field with perfect residue class field. If k is a local field, then $\bar{k}$ will denote the residue class field of k, and p will denote the characteristic of $\bar{k}$. We define $e = v(p)$, where v denotes the normalized valuation on k. Thus $e = e_k$ satisfies $0 < e \leqq \infty$. $f = f_k$ will denote the function defined by

$$f(n) = \min\{np, n + e\}, \qquad n \in \mathbf{Z}, n > 0.$$

3. The algebraically closed case. The ramification structure in this case is given by Serre [**12**]. The results we will need are stated in the following theorem.

THEOREM 1. *Let K be a local field whose residue class field $\bar{K}$ is algebraically closed. Let $A_K = A_K^0 \supseteq A_K^1 \supseteq A_K^2 \supseteq \ldots$ denote the filter of ramification subgroups of A_K. Then we have the following:*

(i) *$A_K/A_K^1 \cong S_{\bar{K}}$ (canonically);*

(ii) *If $p = 0$, then $A_K^1 = 0$.*

If $p \neq 0$, and $n \geqq 1$, then

(iii) *$A_K^n/A_K^{n+1} \cong \chi(\bar{K}_+)$, the character group of $\bar{K}_+$;*

(iv) *The mapping $\sigma \to \sigma^p$ carries A_K^n into $A_K^{f(n)}$.*

Let

$$\bar{p}_n: A_K^n/A_K^{n+1} \to A_K^{f(n)}/A_K^{f(n)+1}$$

denote the homomorphism derived from (iv). *Then*

(v) *$\bar{p}_n$ is bijective if $n \neq e/(p-1)$;*

(vi) *If $n = e/(p-1)$, we have the exact sequence*

$$0 \to A_K^n/A_K^{n+1} \xrightarrow{\bar{p}_n} A_K^{f(n)}/A_K^{f(n)+1} \to S_K[p] \to 0.$$

Proof. Let U_K^n, $n \geqq 0$, denote the higher unit groups of K, and let π_i, $i \geqq 0$, denote the homotopy functors. By [**12**], $A_K^n \cong \pi_1(U_K^n)$ for all $n \geqq 0$. Recall [**11**] that if we apply homotopy to an exact sequence of pro-algebraic groups

$$0 \to G' \to G \to G'' \to 0$$

we obtain a 6-term exact homotopy sequence

$$0 \to \pi_1(G') \to \pi_1(G) \to \pi_1(G'') \to \pi_0(G') \to \pi_0(G) \to \pi_0(G'') \to 0.$$

If we consider the 6-term homotopy sequence corresponding to the sequence

$$0 \to U_K^{n+1} \to U_K^n \to U_K^n/U_K^{n+1} \to 0,$$

and recall that $\pi_0(U_K^{n+1}) = 0$, we see that

$$A_K^n/A_K^{n+1} \cong \pi_1(U_K^n/U_K^{n+1}), \qquad n \geqq 0.$$

Now U_K^0/U_K^1 is isomorphic to the multiplicative group $\bar{K}^\times$ in a canonical way, and so $A_K/A_K^1 \cong \pi_1(\bar{K}^\times) \cong S_{\bar{K}}$ (canonically). If $n \geqq 1$, then $U_K^n/U_K^{n+1} \cong \bar{K}_+$, and so $A_K^n/A_K^{n+1} \cong \pi_1(\bar{K}_+)$. If $p = 0$, then $\pi_1(\bar{K}_+) = 0$. Otherwise $\pi_1(\bar{K}_+) \cong \chi(\bar{K}_+)$ canonically. Thus we have proved (i), (ii), and (iii).

If $n \geqq 1$, then the higher unit groups U_K^n satisfy:

(iv)′ $(U_K^n)^p \subseteq U_K^{f(n)}$.

Let $\bar{p}_n$: $U_K^n/U_K^{n+1} \to U_K^{f(n)}/U_K^{f(n)+1}$ denote the homomorphism derived from (iv)′. Then

(v)′ $\bar{p}_n$ is bijective if $n \neq e/(p-1)$;

(vi)′ If $n = e/(p-1)$, we have the exact sequence

$$0 \to S_K[p] \to U_K^n/U_K^{n+1} \to U_K^{f(n)}/U_K^{f(n)+1} \to 0.$$

(See Serre [**12**, § 1.7] for all these results.)

(iv) and (v) follow immediately on applying π_1 to the results (iv)′ and (v)′. Now suppose that $n = e/(p-1)$. Taking the 6-term sequence corresponding to (vi)′, and noting that $\pi_1(S_K[p]) = 0$, $\pi_0(S_K[p]) = S_K[p]$, and $\pi_0(U_K^n/U_K^{n+1}) = 0$, we obtain the exact sequence of (vi).

Remark 1. Since n takes only integral values, condition (vi) will be vacuous if $e/(p-1)$ is not an integer. It is known that $e/(p-1)$ is an integer if and only if $S_K[p] \neq 0$; see [**12**, § 1.7].

Remark 2. The mappings of the previous Theorem 1 may be given explicitly as follows.

(1) $A_K/A_K^1 \cong S_{\bar{K}}$. Let n be a positive integer prime to p, and let π be a prime of K. If $\sigma \in A_K$, then $\sigma\sqrt[n]{\pi}/\sqrt[n]{\pi} \in S_K[n] = S_{\bar{K}}[n]$. The mapping $A_K/A_K^1 \to S_{\bar{K}}$ may be defined by $\bar{\sigma} \to (\sigma\sqrt[n]{\pi}/\sqrt[n]{\pi})_n$ where n runs through all positive integers prime to p. This mapping is actually independent of the choice of π.

(2) $A_K^n/A_K^{n+1} \cong \chi(\bar{K}_+)$, $n \geqq 1$. Let $L|K$ be a finite abelian extension. Then we have an exact sequence

$$0 \to G_{L|K}^n/G_{L|K}^{n+1} \to U_L^{\psi(n)}/U_L^{\psi(n)+1} \to U_K^n/U_K^{n+1} \to 0;$$

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see [**12** or **13**]. Choosing uniformizing elements in L and K, this sequence reduces to

$$0 \to G_{L|K}{}^{n}/G_{L|K}{}^{n+1} \to \bar{K}_{+} \xrightarrow{f} \bar{K}_{+} \to 0,$$

where f is an additive polynomial. Let $\chi \in \chi(G_{L|K}{}^{n}/G_{L|K}{}^{n+1})$. From the theory of additive polynomials [**9**], there exists a unique additive polynomial g and a unique element $u \in \bar{K}$ such that the diagram

$$\begin{array}{ccccccc} 0 \to G_{L|K}{}^{n}/G_{L|K}{}^{n+1} & \to & \bar{K}_{+} & \xrightarrow{f} & \bar{K}_{+} & \to & 0 \\ \downarrow \chi & & \downarrow g & & \downarrow u & & \\ 0 \to \mathbf{Z}/p\mathbf{Z} & \to & \bar{K}_{+} & \xrightarrow{\mathscr{P}} & \bar{K}_{+} & \to & 0 \end{array}$$

commutes. (Here $\mathscr{P}$ denotes the additive polynomial $x \to x^{p} - x$, and $u: \bar{K}_{+} \to \bar{K}_{+}$ denotes the scalar multiplication $x \to ux$.) In this way, we obtain an injective homomorphism $\chi(G_{L|K}{}^{n}/G_{L|K}{}^{n+1}) \to \bar{K}_{+}$ given by $\chi \to u$. Proceeding to the inverse limit, we obtain an injective homomorphism $\chi(A_{K}{}^{n}/A_{K}{}^{n+1}) \to \bar{K}_{+}$ which is, in fact, an isomorphism, by [**12**]. Dualizing yields the required isomorphism.

(3) Assume that $s = ep/(p-1) \in \mathbf{Z}$. Then the mapping $A_{K}{}^{s}/A_{K}{}^{s+1} \to S_{K}[p]$ may be given by $\bar{\sigma} \to \sigma\sqrt[p]{\pi}/\sqrt[p]{\pi}$, where π is a prime of K. This mapping is independent of the choice of π.

4. The general case. Now let k be an arbitrary local field. We wish to study the ramification filter

$$A_{k} \supseteq A_{k}{}^{0} \supseteq A_{k}{}^{1} \supseteq A_{k}{}^{2} \supseteq \ldots .$$

To utilize the results of Theorem 1, we let K denote the maximal unramified extension of k. Thus K is a discrete-valued field with an algebraically closed residue class field. Although K is not complete, it is Henselian; thus the ramification groups $A_{K}{}^{n}$, $n \geqq 0$, may be identified with the ramification groups $A_{\tilde{K}}{}^{n}$, $n \geqq 0$, where $\tilde{K}$ denotes the completion of K. Thus, the results of Theorem 1 apply to A_{K}.

Let $g = G_{K|k}$; then g acts on the groups $A_{K}{}^{n}$, $n \geqq 0$, through inner automorphism:

$$\sigma \to \tilde{\tau}\sigma\tilde{\tau}^{-1} \quad \text{for all } \sigma \in A_{K}{}^{n} \text{ and } \tau \in g.$$

(Here, $\tilde{\tau}$ denotes any extension of τ to K_{a}.) In this way, the groups $A_{K}{}^{n}$, $A_{K}{}^{n}/A_{K}{}^{n+1}$, $n \geqq 0$, become compact g-modules. g also acts on the groups $S_{\bar{K}}$, $\chi(\bar{K}_{+})$, and $S_{K}[p]$ in the natural way, and one may verify that *the mappings given in Theorem* 1 *are g-module homomorphisms.* (For Theorem 1 (iii), one should be more precise and say that the isomorphism $A_{K}{}^{n}/A_{K}{}^{n+1} \cong \chi(\bar{K}_{+})$ will be a g-module isomorphism provided that the prime used to define the isomorphism $U_{K}{}^{n}/U_{K}{}^{n+1} \cong \bar{K}_{+}$ is a prime from k.)

The natural restriction $A_K \to A_k$ is a g-module homomorphism, and since g operates trivially on A_k, we obtain a derived homomorphism $(A_K)_g \to A_k$.

THEOREM 2. *The sequence*

$$0 \to (A_K)_g \to A_k \to A_{\bar{k}} \to 0$$

is split-exact.

Proof. At this point we introduce a notation which will also be used later: If G is a profinite group and l is a prime integer, then $G(l)$ will denote the maximal pro-l-factor of G. In particular, if G is abelian, then the natural mapping $G \to \prod_l G(l)$ will be an isomorphism.

To prove Theorem 2, it is enough to show that, for each prime l, the sequence

$$0 \to (A_K)_g(l) \to A_k(l) \to A_{\bar{k}}(l) \to 0 \tag{4}$$

is split-exact. We note immediately that $(A_K)_g(l) = (A_K(l))_g$. Let $H = G_{K_a|k}$. Then we have the exact sequence

$$0 \to A_K \to H \to g \to 0.$$

Applying the dualized form of the 5-term exact sequence [**5**, p. 160], we obtain

$$\to H_2(g, \mathbf{Z}_l) \to H_1(A_K, \mathbf{Z}_l)_g \to H_1(H, \mathbf{Z}_l) \to H_1(g, \mathbf{Z}_l) \to 0.$$

Since $H_1(G, \mathbf{Z}_l)$ is the maximal abelian pro-l-factor group of G, this reduces to

$$\to H_2(g, \mathbf{Z}_l) \to (A_K(l))_g \to A_k(l) \to A_{\bar{k}}(l) \to 0. \tag{5}$$

In case $l = p$, g has cohomological p-dimension not greater than one [**5**, p. 203], and so $H_2(g, \mathbf{Z}_p) = 0$. Further, $A_{\bar{k}}(p)$ is a free abelian pro-p-group; thus the mapping $A_k(p) \to A_{\bar{k}}(p)$ splits. If $l \neq p$, then $A_K{}^1(l) = 0$, and hence $A_K(l) = (A_K/A_K{}^1)(l) \cong S_{\bar{K}}(l)$; thus $(A_K(l))_g \cong (S_{\bar{K}}(l))_g = (S_{\bar{K}})_g(l)$. Thus (5) takes the form

$$S_{\bar{k}}(l) \xrightarrow{\gamma} A_k(l) \to A_{\bar{k}}(l) \to 0. \tag{6}$$

Let π be a prime of k, and suppose that $\bar{k}$ contains a primitive l^nth root of unity. Then $k(l^n\!\!\sqrt{\pi})$ is cyclic of degree l^n over k, and $\alpha_n\colon \sigma \to \sigma \sqrt[l^n]{\pi}/\sqrt[l^n]{\pi}$ defines a homomorphism from A_k onto $S_{\bar{k}}[l^n]$. In this way, we obtain a homomorphism $\alpha\colon A_k(l) \to S_{\bar{k}}(l)$. One checks immediately that $\alpha\gamma = 1$; thus (6) (and hence (4)) is split-exact.

THEOREM 3. $(A_K{}^n)_g \cong A_k{}^n$ *and* $(A_K{}^n/A_K{}^{n+1})_g \cong A_k{}^n/A_k{}^{n+1}$ *for all* $n \geqq 0$.

Proof. By the previous theorem we have $(A_K{}^0)_g \cong A_k{}^0$. If $p = 0$, then $A_K{}^n = A_k{}^n = 0$ for $n \geqq 1$, and the result is trivial. Assume that $p \neq 0$ and that we have already proved $(A_K{}^n)_g \cong A_k{}^n$. Then from the exact sequence

$$0 \to A_K{}^{n+1} \to A_K{}^n \to A_K{}^n/A_K{}^{n+1} \to 0$$

we obtain the homology sequence

$$H_1(g, A_K{}^n/A_K{}^{n+1}) \xrightarrow{\delta_n} (A_K{}^{n+1})_g \to A_k{}^n \to (A_K{}^n/A_K{}^{n+1})_g \to 0.$$

If $n > 0$, then $A_K^n/A_K^{n+1} \cong \chi(\bar{K}_+)$; thus $H_1(g, A_K^n/A_K^{n+1}) = 0$ by additive Galois cohomology. On the other hand, $(A_K^0/A_K^1)(p) = 0$; hence also $H_1(g, A_K^0/A_K^1)(p) = 0$. But $(A_K^1)_g$ is a pro-p-group. Thus δ_0 must be trivial. Hence for all $n \geqq 0$, δ_n is trivial, and so we have the exact sequence

$$0 \to (A_K^{n+1})_g \to A_k^n \to (A_K^n/A_K^{n+1})_g \to 0. \tag{7}$$

Since the image of $(A_K^{n+1})_g$ in A_k^n is A_k^{n+1} (by ramification theory), we have $(A_K^{n+1})_g \cong A_k^{n+1}$. Comparing (7) with the exact sequence

$$0 \to A_k^{n+1} \to A_k^n \to A_k^n/A_k^{n+1} \to 0,$$

we see that $(A_K^n/A_K^{n+1})_g \cong A_k^n/A_k^{n+1}$. Thus, by induction, the result is true for all $n \geqq 0$.

THEOREM 4. *Let k be a local field. Then the ramification filter*

$$A_k \supseteq A_k^0 \supseteq A_k^1 \supseteq A_k^2 \supseteq \ldots$$

satisfies the following:

(i) *A_k is a profinite abelian group, A_k^n is a closed subgroup of A_k for all $n \geqq 0$, and $\bigcap_{n=0}^{\infty} A_k^n = 0$;*

(ii) *$A_k/A_k^0 \cong A_{\bar{k}}$ (topologically), and the exact sequence*

$$0 \to A_k^0 \to A_k \to A_{\bar{k}} \to 0$$

splits by a topological homomorphism;

(iii) *A_k^0/A_k^1 is topologically isomorphic to $S_{\bar{k}}$;*

(iv) *If $p = 0$, then $A_k^1 = 0$.*

If $p \neq 0$, and if $n \geqq 1$, then

(v) *A_k^n/A_k^{n+1} is topologically isomorphic to $\chi(\bar{k}_+)$;*

(vi) *The mapping $\sigma \to \sigma^p$ maps A_k^n into $A_k^{f(n)}$.*

Let $\bar{p}_n$: $A_k^n/A_k^{n+1} \to A_k^{f(n)}/A_k^{f(n)+1}$ denote the homomorphism derived from (vi)*; then:*

(vii) *$\bar{p}_n$ is bijective if $n \neq e/(p-1)$;*

(viii) *If $n = e/(p-1)$, then we have the exact sequence*

$$0 \to H_1(g, S_K[p]) \to A_k^n/A_k^{n+1} \xrightarrow{\bar{p}_n} A_k^{f(n)}/A_k^{f(n)+1} \to S_k[p] \to 0.$$

Proof. (i) is well-known, and (ii) follows immediately from Theorem 2. To prove (iii) and (v), note that $A_k^n/A_k^{n+1} \cong (A_K^n/A_K^{n+1})_g$, by Theorem 3. If $n = 0$, then $A_K^n/A_K^{n+1} \cong S_{\bar{K}}$ by Theorem 1, and since $(S_{\bar{K}})_g = S_{\bar{k}}$, (iii) follows. If $n \geqq 1$, then $A_K^n/A_K^{n+1} \cong \chi(\bar{K}_+)$ by Theorem 1. Also, $(\chi(\bar{K}_+))_g = \chi(\bar{K}^g) = \chi(\bar{k}_+)$. Thus (v) follows. (vi) is immediate from Theorem 1 together with the surjectivity of the homomorphism $A_K^n \to A_k^n$. (vii) follows from Theorem 1 together with the isomorphism $A_k^n/A_k^{n+1} \cong (A_K^n/A_K^{n+1})_g$. To prove (viii), consider the exact sequence of Theorem 1 (vi). Applying homology and Theorem 3, this yields the exact sequence

$$\to H_1(g, A_K^{f(n)}/A_K^{f(n)+1}) \to H_1(g, S_K[p]) \to A_k^n/A_k^{n+1} \\ \to A_k^{f(n)}/A_k^{f(n)+1} \to S_k[p] \to 0.$$

Since $A_K^{f(n)}/A_K^{f(n)+1} \cong \chi(\bar{K}_+)$, the group $H_1(g, A_K^{f(n)}/A_K^{f(n)+1}) = 0$. This yields (viii).

THEOREM 5. *Suppose that* $H_1(g, S_K[p]) = 0$, *or that* $p = 0$. *Then properties* (i)–(viii) *of Theorem* 4 *completely characterize* A_k *as a topological filtered group.* (*That is, if* $A \supseteq A^0 \supseteq A^1 \supseteq A^2 \supseteq \ldots$ *is another topological filtered group satisfying* (i)–(viii), *then* A *is topologically and filter-isomorphic to* A_k.)

Proof. If $p = 0$, then $A_k \cong A_{\bar{k}} \times A_k^0 \cong A_{\bar{k}} \times S_{\bar{k}}$, and our proof is complete. If $p \neq 0$, then let I denote the set of integers i satisfying $0 < i < ep/(p-1)$, $(p, i) = 1$. Choose topological generators $\bar{x}_j$, $j \in J$ for $\chi(\bar{k})$ so that $\chi(\bar{k}) = \prod_{j \in J} \langle \bar{x}_j \rangle$ (direct product). Thus J is the dimension of $\bar{k}$ as a vector space over $\mathbf{Z}/p\mathbf{Z}$. If $i \in I$, we have $A_k^i/A_k^{i+1} \cong \chi(\bar{k})$; thus there is a continuous homomorphism β_i: $A_k^i \to \chi(\bar{k})$. Choose a continuous function φ_i: $\chi(\bar{k}) \to A_k^i$ such that $\beta_i \varphi_i = 1$ (see [**5**, p. 166]), and define $x_{ij} = \varphi_i(\bar{x}_j)$ for all $j \in J$. Let $X = \{x_{ij} : i \in I, j \in J\}$. (If $S_k[p] \neq 1$, let $s = ep/(p-1)$; we enlarge X to include an additional element $x_s \in A_k^s$ such that the image of x_s under the canonical mapping $A_k^s \to S_k[p]$ generates $S_k[p]$.) The set X converges to zero as in [**5**, p. 198]. The surjectivity properties of the mappings $\bar{p}_n$ assures us that X generates A_k^1 topologically. Further, the injectivity of the $\bar{p}_n$ (since $H_1(g, S_K[p]) = 0$), assures us that X is a set of free generators for A_k^1. Thus $A_k^1 = \prod_{x \in X} \langle x \rangle$ (direct product), where $\langle x \rangle \cong Z_p$ denotes the closed subgroup of A_k^1 generated by x.

Define $X^n = \{x_{ij}^{p^{n(i)}} : i \in I, j \in J\}$, where $n(i)$ is the minimal integer such that $n \leqq f^{n(i)}(i)$. (If $S_k[p] \neq 1$, we adjoin to X^n the element $x_s^{p^{n(s)}}$, where $n(s)$ is the minimal integer such that $n \leqq f^{n(s)}(s)$.) One sees immediately that $A_k^n = \prod_{y \in X^n} \langle y \rangle$ (direct product). These remarks show that the filter $A_k^1 \supseteq A_k^2 \supseteq \ldots$ is completely characterized by properties (v)–(viii) of Theorem 4.

On the other hand, since A_k^1 is a pro-p-group whereas A_k^0/A_k^1 is prime to p, we see that the sequence

$$0 \to A_k^1 \to A_k^0 \to A_k^0/A_k^1 \to 0$$

splits. Taking this together with property (ii), we see that $A_k \cong A_k/A_k^0 \times A_k^0/A_k^1 \times A_k^1 \cong A_{\bar{k}} \times S_{\bar{k}} \times A_k^1$. This completes the proof.

5. Applications to finite abelian extensions.

THEOREM 6. *Let* $L|k$ *be any finite abelian extension, and let* G *denote the Galois group* $G_{L|k}$. *Then the filter of ramification subgroups*

$$G \supseteq G^0 \supseteq G^1 \supseteq G^2 \supseteq \ldots \supseteq G^r = 1$$

has the following properties:

(i) *There is a continuous homomorphism* φ: $A_{\bar{k}} \to G$ *such that the derived homomorphism* $\bar{\varphi}$: $A_{\bar{k}} \to G/G^0$ *is surjective;*

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(ii) *G^0/G^1 is cyclic; the number $m = (G^0:G^1)$ being such that $\bar{k}$ contains a primitive mth root of unity;*

(iii) *If $p = 0$, then $G^1 = 1$.*

If $p \neq 0$, and if $n \geqq 1$, then

(iv) *G^n/G^{n+1} is an elementary p-group whose rank is not greater than the dimension of the vector space $\bar{k}$ over $\mathbf{Z}/p\mathbf{Z}$;*

(v) $(G^n)^p \subseteq G^{f(n)}$.

Let $\bar{p}_n$: $G^n/G^{n+1} \to G^{f(n)}/G^{f(n)+1}$ denote the homomorphism derived from (v). *Then*

(vi) *$\bar{p}_n$ is surjective if $n \neq e/(p-1)$;*

(vii) *If $n = e/(p-1)$, then the cokernel of $\bar{p}_n$ is isomorphic to a subgroup of $S_k[p]$.*

Proof. The natural restriction homomorphism $A_k \to G$ carries A_k^n onto G^n for all $n \geqq 0$. Thus Theorem 6 follows immediately from Theorem 4.

THEOREM 7. *Suppose that either $H_1(g, S_K[p]) = 0$ or $p = 0$ and that*

$$G \supseteq G^0 \supseteq G^1 \supseteq G^2 \supseteq \ldots \supseteq G^r = 0$$

is any finite abelian filtered group which satisfies conditions (i)–(vii) *of Theorem* 6. *Then there exists a finite abelian extension $L|k$ and an isomorphism γ: $G_{L|k} \to G$ such that $\gamma(G_{L|k}^n) = G^n$ for all $n \geqq 0$.*

Proof. It is enough to construct a continuous homomorphism ψ: $A_k \to G$ (onto) such that $\psi(A_k^n) = G^n$ for all $n \geqq 0$. (For if such ψ is given, we can choose L to be the fixed field of the kernel of ψ.) By Theorem 4 (ii), $A_k \cong A_k^0 \times A_{\bar{k}}$. Thus it is enough to construct ψ_0: $A_k^0 \to G^0$ such that $\psi_0(A_k^n) = G^n$ for all $n \geqq 0$. For if such ψ_0 is given, then combining with φ given by (i), we can define ψ: $A_k \to G$ by $\psi(\alpha, \beta) = \psi_0(\alpha)\varphi(\beta)$. Similar considerations show that we can reduce the problem another stage: It is enough to construct a continuous homomorphism ψ_1: $A_k^1 \to G^1$ such that $\psi_1(A_k^n) = G^n$ for all $n \geqq 1$.

If $p = 0$, then $A_k^1 = G^1 = 1$, and our proof is complete. Otherwise, we define a subset $Y \subseteq G$ analogous to the X defined in the proof of Theorem 5. We define Y to consist of the elements y_{ij}, $i \in I$, $j \in J$, where $y_{ij} \in G^i$ for all $j \in J$, and such that the cosets $\bar{y}_{ij} \in G^i/G^{i+1}$, $j \in J$, generate G^i/G^{i+1}. (If $S_k[p] \neq 1$, we include an additional element y_s such that $y_s \in G^s$ and $\bar{y}_s$ generates $G^{ep/(p-1)}/(G^{e/(p-1)})^p$.) The surjectivity properties of the mappings $\bar{p}_n$: $G^n/G^{n+1} \to G^{f(n)}/G^{f(n)+1}$ assures us that Y generates G; and if Y^n is defined analogously to X^n, we see that Y^n generates G^n. The natural mapping $X \to Y$ yields a continuous homomorphism ψ_1: $A_k^1 \to G^1$ (since A_k^1 is a free abelian pro-p-group on X). Since X^n maps onto Y^n, we see that $\psi_1(A_k^n) = G^n$, $n \geqq 1$. Thus, the proof is complete.

Remark 3. Theorem 7 holds even if $H_1(g, S_K[p]) \neq 0$, provided we deal only with groups G satisfying $G^{ep/(p-1)} = 1$.

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Remark 4. In applying Theorem 7 or Remark 3, condition (i) of Theorem 6 is certainly the least pleasing since, among all the conditions, it is non-arithmetic. A rather drastic cure would be to restrict our attention to totally ramified extensions: then condition (i) becomes vacuous (this is certainly permissible when $\bar{k}$ is algebraically closed). In a similar vein, if we restrict our attention to p-extensions, then (by additive Kummer Theory), condition (i) may be replaced by:

(i)′ The rank of the p-group G/G^0 is not greater than the dimension of $\bar{k}/\mathscr{P}(\bar{k})$ over $\mathbf{Z}/p\mathbf{Z}$.

An important special case is when $\bar{k}$ is quasi-finite [**13**]. In this case, $g \cong \text{inv lim}_n \mathbf{Z}/n\mathbf{Z}$, and condition (i) may be replaced by the simple condition:

(i)″ G/G^0 is cyclic.

Example. Let $L|k$ be a cyclic extension, and let $i_1 < i_2 < \ldots < i_r$ be the set of (upper) jumps of $L|k$ which are larger than zero. Define I as before, namely, I consists of all positive integers less than $ep/(p-1)$ which are not divisible by p (if $S_k[p] \neq 1$, then we enlarge I to include $ep/(p-1)$). Then by straightforward computation we see that: Conditions (v), (vi), and (vii) of Theorem 6 (or 7) are equivalent to

(v)′ $i_1 \in I$, and

(vi)′ if $n \geqq 1$, then either $i_{n+1} \in I$ and $i_{n+1} > f(i_n)$, or $i_{n+1} = f(i_n)$.

In particular, if $i_n \geqq e/(p-1)$, then $i_{n+1} = i_n + e$. Thus the ramification eventually "stabilizes" if $e < \infty$, and it may even stabilize immediately as in the case $e = 1$.

6. The condition $H_1(g, S_K[p]) = 0$.

Let $G_k = G_{k_s|k}$, where k_s denotes the maximal separable extension of k. In view of [**7**], we can now prove the following interesting result.

THEOREM 8. *Suppose that $p \neq 0$. Then the following statements are equivalent:*

(i) *$A_k(p)$ is a free abelian pro-p-group;*

(ii) *$A_k{}^1$ is a free abelian pro-p-group;*

(iii) *$H_1(g, S_K[p]) = 0$;*

(iv) *$G_k(p)$ is a free pro-p-group.*

Proof. Taking p-factors of Theorem 4 (ii), we obtain

$$0 \to A_k{}^1 \to A_k(p) \to A_{\bar{k}}(p) \to 0. \tag{8}$$

Since $A_{\bar{k}}(p)$ is a free abelian pro-p-group, (8) splits, and we obtain $A_k(p) \cong A_k{}^1 \times A_{\bar{k}}(p)$. Thus the torsion part of $A_k(p)$ is the same as that of $A_k{}^1$. Hence, the equivalence of (i) and (ii).

To prove the equivalence of (ii) and (iii), we note that, if $H_1(g, S_K[p]) = 0$, then (ii) follows from the proof of Theorem 5. Conversely, if $H_1(g, S_K[p]) \neq 0$, then by Theorem 4 (viii), there exists $\sigma \in A_k{}^{e/(p-1)} - A_k{}^{e/(p-1)+1}$ such that $\sigma^p \in A_k{}^{ep/(p-1)+1}$. But since $\bar{p}_n$: $A_k{}^n/A_k{}^{n+1} \to A_k{}^{n+e}/A_k{}^{n+e+1}$ is surjective for all $n > e/(p-1)$, we deduce that $A_k{}^{ep/(p-1)+1} = (A_k{}^{e/(p-1)+1})^p$. Thus there is an

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element $\tau \in A_k^{e/(p-1)+1}$ such that $\tau^p = \sigma^p$. Thus $\sigma\tau^{-1}$ is a non-trivial torsion element of A_k^1, and so A_k^1 is not a free abelian pro-p-group.

Finally, we note that the equivalence of (iii) and (iv) is a direct consequence of the results in [**7**].

Remark 5. A concrete interpretation of the group $H_1(g, S_K[p])$ is given in [**7**] and in [**6**, p. 101]. Specifically, we have

(i) If $e/(p-1)$ is not an integer (i.e. $e/(p-1)$ is rational or ∞), then $H_1(g, S_K[p]) = 0$;

(ii) If $e/(p-1)$ is an integer, then $H_1(g, S_K[p])$ corresponds to a certain class $\mathscr{C}$ of extension fields of degree p over $\bar{k}$, and $H_1(g, S_K[p]) = 0$ if and only if $\mathscr{C} = \emptyset$. If $S_k[p] \neq 1$, then $\mathscr{C}$ is precisely the class of cyclic extensions of degree p over $\bar{k}$. If $S_k[p] = 1$, then $\mathscr{C}$ consists of certain non-Galois extensions. An important corollary is: *If $\bar{k}$ is quasi-finite and if $S_k[p] = 1$, then $H_1(g, S_K[p]) = 0$.*

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