1

The Fundamentals of Hydrodynamics

Everything flows and nothing abides; everything gives way and nothing stays fixed.

Heraclitus (c. 535–c. 475 BCE)

1.1 Definition of a fluid

THE PHYSICS of hydrodynamics (HD), namely conservation of mass, conservation of energy, and Newton's second law, are all concepts familiar to first-year undergraduate students, though the mathematics to solve the relevant equations is not. Consider an *ensemble of particles* within some volume V, and let these particles interact with each other via elastic collisions. We can let V remain fixed (in which case we allow the particles to collide elastically with the walls of the container too), or we can let V increase or decrease as the particles move apart or come together; it does not matter. If the mass, total energy, and momentum of the ensemble of particles are M, E_T, and $\vec{S}$ respectively, then we have:

$$\frac{dM}{dt} = 0, \qquad \text{conservation of mass;} \tag{1.1}$$

$$\frac{dE_\mathrm{T}}{dt} = \sum \mathcal{P}_\mathrm{app}, \qquad \text{conservation of total energy;} \tag{1.2}$$

$$\frac{d\vec{S}}{dt} = \sum \vec{F}_\mathrm{ext}, \qquad \text{Newton's second law.} \tag{1.3}$$

Here, $\sum \mathcal{P}_\mathrm{app}$ is the rate at which work is done (power) by all forces *applied* to the ensemble of particles, and $\sum \vec{F}_\mathrm{ext}$ are all forces *external* to and acting on the ensemble of particles. Note that the applied forces – normally just collisions from neighbouring ensembles of particles – are typically a subset of the external forces, which include collisions from neighbouring particles *plus* forces arising from gravity, magnetism, radiation, *etc.* This is because in addition to the thermal and kinetic energies, the *total energy*, E_T, includes gravitational, magnetic, radiative, and possibly other energies as well.

It is how we model the collisional forces from neighbouring ensembles of particles that defines both what constitutes a fluid and how Eq. (1.1)–(1.3) are further developed. Consider a small cube with volume $\Delta V = (\Delta l)^3$ as shown in Fig. 1.1*a*.

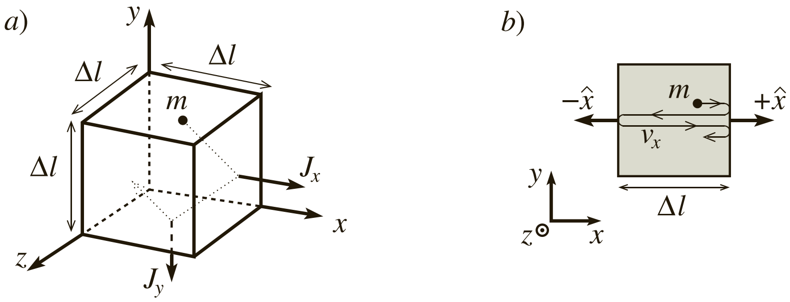


Figure 1.1. *a*) A single particle bounces elastically from the walls of a cube of edge length, Δl, imparting impulses J_x, J_y, *etc.* *b*) An x–y cut through the cube in panel *a* showing one particle whose motion is entirely in the x-direction.

Let the walls of the cube be perfectly reflecting and let there be just one particle inside the cube moving at some speed v in an arbitrary direction.

When the particle collides with the wall, both the particle and cube suffer a change in momentum in a direction normal to the surface of the cube. Moments later, the particle collides with a different wall, and the particle and cube suffer changes in momentum in a direction normal to that wall. A change in momentum is an impulse, J, which when multiplied by the time over which the collision occurs, Δt, constitutes the average force. Thus formally, the "pressure", p, the collision exerts on the wall of the box is this average force divided by the area of the wall:

$$p \sim \frac{J\Delta t}{(\Delta l)^2}.$$

In this scenario, the "pressure" is highly variable in time, and by no means could the "pressure" be construed as isotropic. At a given time, the "pressure" one wall feels will have nothing to do with the "pressures" felt by the other walls.

However, by arbitrarily increasing the number of particles, $\mathcal{N}$, inside our small volume, ΔV, the number of collisions with a given wall, n, occurring in a time Δt will be the same at each wall to within some arbitrarily small variance, Δn. Put another way, averaged over Δt, particle collisions exert the same "pressure" on each wall to within a variance made as small as we please by making $\mathcal{N}$ as large as we please. Thus, we have rendered the particle "pressure" inside the cube *isotropic* because each wall now feels the same force.

There is a contrived exception to this picture. If all the particles were to be placed initially on the mid-plane of the cube and all were launched with the same speed towards one wall of the cube, then it is only with this and the opposite wall that particles would ever collide, and they would do so in a highly ordered, periodic fashion. The remaining four walls would, in principle, never feel any collisions, and thus the "pressure" in the cube would not be isotropic even with $\mathcal{N}$ chosen arbitrarily large. Such a well-ordered and well-directed ensemble of particles is said to be *streaming* and, as $\mathcal{N}$ is made larger, it becomes increasingly difficult in practice to maintain streaming motion. Small perturbations will eventually cause one particle to collide with another which in turn collide with others, and the ensuing chain reaction quickly reduces the streaming motion to chaos. Isotropic "pressure" (the same "pressure" measured on each of the six walls) is once again the result.

We can now state the key criterion for an ensemble of particles to be treated as a fluid. If there is a sufficient number of particles inside our box (volume element) of dimension Δl so that the motion of particles within the volume element can always be considered isotropic, then the effect of the collisions of particles against the walls of the volume element (which may be rigid walls, or "soft" walls of neighbouring ensembles of particles) is to exert an isotropic "pressure" against all walls. Since isotropy is maintained by particle–particle collisions within the volume element, we may "mathematise" this criterion as,

$$\delta l \ll \Delta l < \mathcal{L}, \tag{1.4}$$

where δl is the mean free path (collision length) of the particles, Δl is the length of one side of our cubic volume element containing an arbitrarily large number of particles, and $\mathcal{L}$ is the smallest length scale of interest in our physical problem. If Ineq. (1.4) holds, we say the ensemble of particles behaves as a *fluid* or a *continuum*. This assumption is an important one; it allows us to treat the applied forces resulting from collisions – which otherwise could be *extremely* difficult to deal with – in a very simple way, namely as an isotropic "pressure".

1.2 A quick review of kinetic theory

To now, I have been enclosing the word *pressure* in quotation marks. This is because I haven't yet made the logical connection between particle collisions (and more specifically, the momentum transferred by particle collisions) and what we commonly think of as *pressure*, such as the *barometric* pressure of the air. So, before we examine how Eq. (1.1)–(1.3) become the equations of hydrodynamics (HD) under the assumption that the ensemble of particles behaves as a fluid (when Ineq. 1.4 is valid), let us review how the "pressure" and the "temperature" of a fluid relate to properties of the ensemble of particles. These ideas form the basis of *kinetic theory*, often exposed to students for the first time in a first-year physics course.[1]

Consider a cube whose edges of length Δl are aligned with the x-, y-, and z-axes of a Cartesian coordinate system, as depicted in Fig. 1.1. Returning to our example in the previous section, suppose a single point particle of mass m moves inside the cube with velocity $v_x \hat{x}$ and collides with the wall whose normal is $+\hat{x}$. If collisions are all elastic, then the particle reflects from the wall with a velocity $-v_x \hat{x}$ and thus suffers a change in momentum of $\Delta S_x = -2mv_x$. Conservation of momentum then demands that an impulse of $+2mv_x$ be imparted against the wall. At a time $\Delta t = 2\Delta l / v_x$ later, the same particle again collides with the wall, imparting another impulse of $+2mv_x$ against it. Thus, the rate at which momentum is delivered to the

---

[1] For example, Halliday, Resnick, & Walker (2003).

wall by a single particle is given by,

$$\frac{\Delta S_x}{\Delta t} = \frac{2mv_x}{2\Delta l/v_x} = \frac{mv_x^2}{\Delta l} = \langle F\rangle,$$

where $\langle F\rangle$ is the average force felt by the wall. Thus, the average pressure exerted by this one particle, defined as force per unit area, is given by,

$$\langle p\rangle = \frac{\langle F\rangle}{(\Delta l)^2} = \frac{mv_x^2}{V},$$

where $V = (\Delta l)^3$ is the volume of the cube. For $\mathcal{N}$ particles, we simply add over all particles:

$$p \equiv \sum_{i=1}^{\mathcal{N}}\langle p_i\rangle = \sum_{i=1}^{\mathcal{N}}\frac{mv_{x,i}^2}{V} = \frac{m}{V}\sum_{i=1}^{\mathcal{N}}v_{x,i}^2 = \frac{m\mathcal{N}}{V}\langle v_x^2\rangle, \tag{1.5}$$

where each point particle is assumed to have the same mass, m, and where $\langle v_x^2\rangle = \sum v_{x,i}^2/\mathcal{N}$ is the arithmetic mean of the squares of the particle velocities.

For any given particle, $v^2 = v_x^2 + v_y^2 + v_z^2$ and, for large $\mathcal{N}$, one would expect $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$ since one Cartesian direction shouldn't be favoured over another. Thus,

$$\langle v^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle = 3\langle v_x^2\rangle, \tag{1.6}$$

and Eq. (1.5) becomes,

$$p = \frac{\mathcal{N}mv_{\rm rms}^2}{3V}, \tag{1.7}$$

where,

$$v_{\rm rms} \equiv \sqrt{\langle v^2\rangle},$$

is the *root-mean-square* (rms) speed of the particles in the volume V. Comparing Eq. (1.7) with the *ideal gas law*:

$$p = \frac{\mathcal{N}k_{\rm B}T}{V}, \tag{1.8}$$

(where $k_{\rm B} = 1.3807 \times 10^{-23}\ {\rm J\,K^{-1}}$ is the Boltzmann constant) yields:

$$T = \frac{mv_{\rm rms}^2}{3k_{\rm B}} \quad\Rightarrow\quad \frac{3}{2}k_{\rm B}T = \frac{1}{2}mv_{\rm rms}^2 = \langle K\rangle, \tag{1.9}$$

where $\langle K\rangle$ is the average kinetic energy per point particle. Thus, while the *pressure*, p, is a measure of the rate at which momentum is transferred from the particles of the fluid (gas) to, for example, the diaphragm of the measuring device (barometer), the *temperature* (or more precisely $3k_{\rm B}T/2$) is a measure of the average kinetic energy of the particles.

The randomly directed kinetic energy of a system of $\mathcal{N}$ particles is called its *internal energy*, E, and, for the point particles under discussion, is given by,

$$E = \mathcal{N}\langle K\rangle = \frac{3}{2}\mathcal{N}k_{\rm B}T.$$

The factor $3/2$ is significant and warrants comment. A point particle, as may be found exclusively in a monatomic gas, has three *degrees of freedom* of motion, namely *translation* in each of the three Cartesian directions (Fig. 1.2, left).

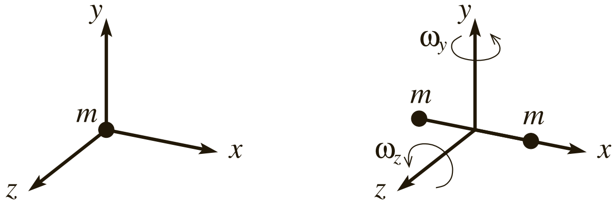


Figure 1.2. A point particle (left) has three degrees of freedom for movement, while a "dumb-bell" (right) has five.

From Eq. (1.6), we have $\langle v_i^2 \rangle = \langle v^2 \rangle / 3$ for $i = x, y, z$, and thus to each (translational) degree of freedom we can associate an internal energy $E_i = \mathcal{N} k_\mathrm{B} T / 2$, where $E = E_x + E_y + E_z = 3\, E_i$.

Now, a diatomic molecule (essentially two point masses connected by a massless rod) has the same three translational degrees of freedom as a monatomic particle *plus* two *rotational* degrees of freedom, namely rotation about each of the two principle axes orthogonal to its own axis (the x-axis in Fig. 1.2, right), for a total of five degrees of freedom.[2] Note that spinning about the x-axis itself does not constitute a degree of freedom as the moment of inertia about this axis is essentially zero. Because of the *principle of equipartition*,[3] each degree of freedom stores the same amount of kinetic energy, and the internal energy of a diatomic gas must be,

$$E = \frac{5}{2} \mathcal{N} k_\mathrm{B} T.$$

Thus, in general, we write,

$$E = \frac{1}{\gamma - 1} \mathcal{N} k_\mathrm{B} T, \tag{1.10}$$

where $\gamma = 5/3$ for a monatomic gas, $\gamma = 7/5$ for a diatomic gas, and $4/3 \le \gamma < 7/5$ for molecules more complex than diatomic.[4] One can show that $\gamma = C_P / C_V$, the ratio of specific heats of the gas, and that for an adiabatic gas (where heat is neither lost nor gained from the system), $p \propto \rho^\gamma$, where ρ is the *mass density* of the gas.

Dividing Eq. (1.10) by the volume of the sample and using Eq. (1.8) gives an expression for the *internal energy density*, e:

$$e = \frac{E}{V} = \frac{1}{\gamma - 1} \frac{\mathcal{N} k_\mathrm{B} T}{V} = \frac{p}{\gamma - 1}.$$

Thus, an alternate form of the ideal gas law, and the form most frequently used in

---

[2] In principle, there are also two vibrational degrees of freedom which, at "ordinary temperatures", statistical mechanics tells us are insignificant.

[3] Left to their own devices, systems will distribute the available energy equally among all possible ways energy can be stored. Thus, for a large number of diatomic molecules randomly colliding with each other and the walls of their container, one would not expect $m \langle v_x^2 \rangle$ to differ significantly from $m \langle v_y^2 \rangle$ or $m \langle v_z^2 \rangle$ any more than it should differ from $I_y \langle \omega_y^2 \rangle$ or $I_z \langle \omega_z^2 \rangle$, where I_y and I_z are the moments of inertia about the y- and z-axes respectively.

[4] Polyatomic molecules are significantly more complex than diatomic molecules, and the full power of statistical mechanics along with a tensor treatment of its moment of inertia are required to explain the value of γ for any individual molecule.

hydrodynamics, is,

$$\boxed{p = (\gamma - 1)\, e,} \tag{1.11}$$

which states that the rate at which momentum is transferred via collisions is proportional to the average kinetic energy density (*i.e.*, per unit volume) of the random particle motion.

Possibly the second most frequently used form of the ideal gas law in hydrodynamics is,

$$p = \frac{\rho k_\mathrm{B} T}{m}, \tag{1.12}$$

which follows directly from Eq. (1.8) noting that $\rho = \mathcal{N}m/V$. Finally, from Eq. (1.9) [and replacing the '3' with $2/(\gamma - 1)$], we find:

$$v_\mathrm{rms} = \sqrt{\frac{2k_\mathrm{B}T}{(\gamma-1)\, m}} = \sqrt{\frac{2p}{(\gamma-1)\,\rho}}. \tag{1.13}$$

Thus, the rms speed goes as the square root of the temperature. We shall encounter another characteristic speed of the gas proportional to the square root of the temperature in §2.1.1, namely the sound speed, c_s. Indeed, c_s and v_rms arise from essentially the same physics, as will be explained when the sound speed is properly introduced.

1.3 The equations of ideal hydrodynamics

In hydrodynamics, the adjective *ideal* means that internal dissipative forces such as viscosity are ignored. A fluid without (with) viscosity is said to be *inviscid* (*viscid*). In this chapter, our discussion is exclusively restricted to inviscid flow. Viscid flow is more the realm of terrestrial HD (though there are important applications for astrophysical fluids as well), and is covered in some depth in Chap. 8.

We begin our discussion by defining the adjectives *extensive* and *intensive*. Variables such as mass, volume, and energy which are *proportional* to the amount of substance being measured are *extensive* quantities, while mass density (often just referred to as density), energy density, and temperature are *independent* of the amount of substance being studied and are examples of *intensive* quantities.

To give a precise relationship between extensive and intensive quantities, consider a small sample of substance with volume ΔV. For every extensive quantity, $Q(V,t)$, of that sample, we can define a corresponding intensive quantity, $q(\vec{r},t)$, such that,

$$q(\vec{r},t) = \lim_{\Delta V \to 0} \frac{\Delta Q(V,t)}{\Delta V} = \frac{\partial Q(V,t)}{\partial V}. \tag{1.14}$$

This is a *microscopic* description of the system; q may well change from point to point. A *macroscopic* description of the system can be obtained by integrating Eq. (1.14) over a finite volume, V, to recover Q:

$$Q(V,t) = \int_V q(\vec{r},t)\, dV. \tag{1.15}$$

Note that Eq. (1.15) requires that q be an *integrable* function of the coordinates over the volume V, and thus q can be discontinuous and have poles of order less than unity. On the other hand, Eq. (1.14) requires that Q be a *differentiable* function of V, and thus it must be both continuous and free from any poles of any order. Evidently, differentiability is a more restrictive requirement than integrability, and this observation will have important consequences as we develop the theory further.

We're now ready to introduce and prove a theorem that provides a particularly simple way to derive the equations of hydrodynamics from the conservation laws of Eq. (1.1)–(1.3).

Theorem 1.1. Theorem of hydrodynamics.[5] *If the time dependence of an extensive quantity, Q, is given by:*

$$\frac{dQ}{dt} = \Sigma, \tag{1.16}$$

where Σ represents the possibly time-dependent "source terms" (reasons for Q not being "conserved"), then the evolution equation for the corresponding intensive quantity, $q(\vec{r},t)$, is given by,

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\vec{v}) = \sigma, \tag{1.17}$$

where $\vec{v} = d\vec{r}/dt$, $Q = \int_V q\,dV$, $\Sigma = \int_V \sigma dV$, and where the product $q\vec{v}$ must be a differentiable *function of the coordinates.*

Proof:

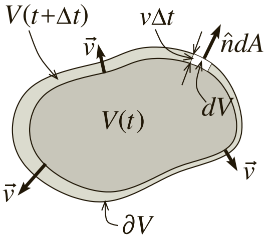
$$\frac{dQ}{dt} = \Sigma \quad \Rightarrow \quad \frac{d}{dt}\int_V q\,dV = \int_V \sigma dV,$$

where, in general, the volume element $V = V(t)$ also varies in time. Thus, using the standard definition of the derivative,

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} q\,dV &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} q(\vec{r},t+\Delta t)dV - \int_{V(t)} q(\vec{r},t)dV\right] \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{V(t+\Delta t)-V(t)} q(\vec{r},t+\Delta t)dV\right. \\
&\qquad\qquad \left. + \int_{V(t)} q(\vec{r},t+\Delta t)dV - \int_{V(t)} q(\vec{r},t)dV\right] \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\Delta V} q(\vec{r},t+\Delta t)dV \\
&\qquad\qquad + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{V(t)}\big[q(\vec{r},t+\Delta t) - q(\vec{r},t)\big]dV,
\end{aligned}$$

---

[5] This theorem is a variant of *Reynolds' transport theorem*, a volume-integral application of the Leibniz formula for the derivative of an integral.



where, as shown in the inset, performing the volume integral over the difference in volumes, $\Delta V = V(t+\Delta t) - V(t)$, is the same as integrating over the closed surface, ∂V, using a volume differential given by $dV = (\vec v \Delta t)\cdot(\hat n dA)$. Thus,

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} q\,dV &= \lim_{\Delta t\to 0}\frac{1}{\cancel{\Delta t}}\oint_{\partial V} q(\vec r, t{+}\Delta t)(\vec v\cancel{\Delta t})\cdot(\hat n dA) \\
&\qquad + \int_{V(t)}\lim_{\Delta t\to 0}\frac{q(\vec r, t{+}\Delta t) - q(\vec r, t)}{\Delta t}dV \\
&= \oint_{\partial V} q(\vec r, t)\vec v\cdot\hat n dA + \int_{V(t)}\frac{\partial q(\vec r, t)}{\partial t}dV \\
&= \int_{V(t)}\nabla\cdot\big(q(\vec r, t)\vec v\big)dV + \int_{V(t)}\frac{\partial q(\vec r, t)}{\partial t}dV \quad \text{(Gauss; Eq. A.30)} \\
&= \int_{V(t)}\bigg(\frac{\partial q(\vec r, t)}{\partial t} + \nabla\cdot\big(q(\vec r, t)\vec v\big)\bigg)dV = \int_{V(t)}\sigma(\vec r, t)dV \\
&\Rightarrow \quad \int_V\bigg(\frac{\partial q}{\partial t} + \nabla\cdot(q\vec v) - \sigma\bigg)dV = 0.
\end{aligned}$$

As this is true for any V, the integrand must be zero, proving the theorem. □

Note that q is not the conserved quantity, Q is (at least to within a known source term, Σ). However, since Q is the volume-integral of q, we'll refer to q as a *volume-conserved quantity.*

The quantity $q\vec v \equiv \vec f_Q$ is the *advective flux density* of Q whose units are those of Q times $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$; this will require a little unpacking. The *flux*,[6] $\mathcal{F}_Q$, of a vector field, $\vec f_Q$, is a measure of how much $\vec f_Q$ "passes through" a given surface area with arbitrary normal, $\hat n$. Mathematically,

$$\mathcal{F}_Q = \oint_S \vec f_Q\cdot\hat n\,dA \qquad \text{or} \qquad \mathcal{F}_Q = \int_\Sigma \vec f_Q\cdot\hat n\,dA, \tag{1.18}$$

depending on whether the surface is closed (S) or open (Σ) respectively. Thus, the units of $\vec f_Q$ are those of $\mathcal{F}_Q$ per unit area, and $\vec f_Q$ can also be interpreted as a *flux density* of $\mathcal{F}_Q$. And so, $\mathcal{F}_Q$ is the flux of $\vec f_Q$ while $\vec f_Q$ is the flux *density* of $\mathcal{F}_Q$.

An *advective* flux density is more specific to fluid dynamics and refers to some quantity, Q, being *advected* (*i.e.*, transported) by the flow across a surface at a certain rate. Thus, while $\vec f_Q$ is the *flux density* of $\mathcal{F}_Q$ with units of $\mathcal{F}_Q$ per unit area, $\vec f_Q = q\vec v$ is also the *advective* flux density of Q – the volume integral of q – with units of Q per unit area per unit time. It is the "per unit time" part that triggers the adjective *advective.*

Evidently, we have four different types of "fluxes" to keep straight (flux, flux

---

[6] From the Latin *fluxus* or "flow", this term was introduced to physics by Sir Isaac Newton.

density, advective flux, advective flux density) and the literature seems to blur all four; often you'll find any or all of these terms used interchangeably. In this text, while I maintain the distinction between *flux* and *flux density*, I've chosen to drop the adjective *advective* to simplify the language a bit, relying instead on context. If a particular flux/flux density has a "per unit time" aspect to it, it is an *advective* flux/flux density; otherwise just flux/flux density.

Last point before getting to the equations of HD: Eq. (1.16) is an *integral* equation (Q and Σ both being volume integrals of intensive quantities, q and σ), and thus represents a *global* statement (valid over a finite sample of the fluid) on the conservation of the extensive quantity, Q. On the other hand, Eq. (1.17) is a *differential* equation (often referred to as the *differential form* of Eq. 1.16) and thus represents a *local* statement (valid at a point) on the conservation of Q, involving the corresponding intensive quantity, q. *Global and local forms of an equation are not identical.* Because differential equations require the functions to be differentiable, solutions of the differential form of the equations can be more restrictive than those of the integral form where functions need only be integrable. More on this in §1.5.

Example 1.1. Let $Q = M$, the mass of the sample of fluid. Find the evolution equation for the corresponding intensive quantity, $q = \rho$ (mass density).

Solution: From Eq. (1.1), $\Sigma = 0 \Rightarrow \sigma = 0$, and Theorem 1.1 requires that:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \; = \; 0.} \tag{1.19}$$

This is the *continuity equation*; the first equation of HD. □

Example 1.2. Let $Q = E_\mathrm{T}$, the total energy of the fluid sample[7] of mass M:

$$E_\mathrm{T} \; = \; E + \frac{1}{2} M v^2 + M\phi,$$

where E is the internal (thermal) energy and ϕ is the gravitational potential. Find the evolution equation for the corresponding intensive quantity, the *total energy density*, namely,

$$e_\mathrm{T} \; = \; e + \frac{1}{2}\rho v^2 + \rho\phi, \tag{1.20}$$

where once again, e is the internal energy density, whose units $\mathrm{J\,m^{-3}} = \mathrm{N\,m^{-2}}$ are the same as those for pressure, as expected from Eq. (1.11).

Solution: From Eq. (1.2), $\Sigma = \mathcal{P}_\mathrm{app} \Rightarrow \sigma = \wp_\mathrm{app}$, the *applied power density* interpreted as the rate at which work is done on a unit volume of the fluid sample by all *applied* forces. Thus, Theorem 1.1 implies:

$$\frac{\partial e_\mathrm{T}}{\partial t} + \nabla \cdot (e_\mathrm{T} \vec{v}) \; = \; \wp_\mathrm{app}. \tag{1.21}$$

---

[7] When we introduce magnetism in Chap. 4, we'll add a magnetic term to E_T.

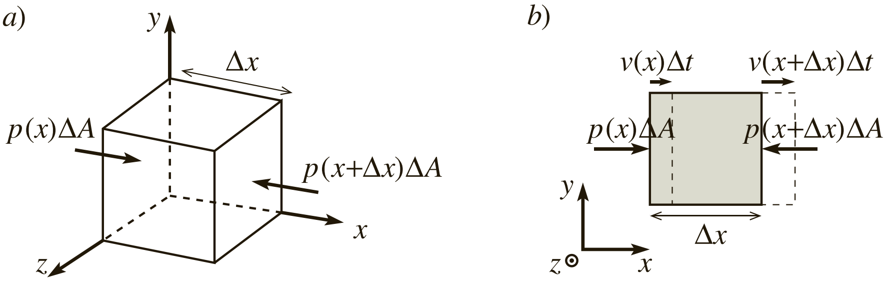


Figure 1.3. *a*) A cube of edge length Δx with external pressure forces acting on the x-faces indicated. *b*) An x–y cut through the cube in panel (*a*) showing both the pressure forces on and motion of the x-faces.

As discussed in §1.1, applied forces are collisions of external particles with the fluid sample. Thus, the applied power is the rate at which work is done on the fluid sample by the external fluid as the former expands or contracts within the latter.

To find an expression for the applied power, $\mathcal{P}_{\rm app}$, consider a small cube of fluid with dimension Δx in the x-direction and cross-sectional area $\Delta A = \Delta V/\Delta x$ (Fig. 1.3). The pressure force exerted on the left face of the cube is $F(x) = +p(x)\Delta A$ and, in time Δt, the left face is displaced by $v_x(x)\Delta t$. Thus, the work done on the left face by the external fluid is $\Delta W_{\rm L} = +p(x)\,v_x(x)\Delta t\,\Delta A$. Similarly, the work done on the right face is $\Delta W_{\rm R} = -p(x+\Delta x)\,v_x(x+\Delta x)\Delta t\,\Delta A$ [since $p(x+\Delta x)$ and $v(x+\Delta x)$ are oppositely directed; Fig. 1.3*b*], and the net work done on the fluid cube is:

$$\begin{aligned}\Delta W \;=\; \Delta W_{\rm L} + \Delta W_{\rm R} \;&=\; p(x)\,v_x(x)\Delta t\,\Delta A - p(x+\Delta x)\,v_x(x+\Delta x)\Delta t\,\Delta A\\ &=\; p(x)\,\delta V(x) - p(x+\Delta x)\,\delta V(x+\Delta x),\end{aligned}$$

where $\delta V(x)$ [$\delta V(x+\Delta x)$] is the small volume change on the left [right] face of the cubic sample of volume ΔV by virtue of the motion of the left [right] face. Because of its form, this work is frequently referred to as the "pdV term".

Dividing ΔW by Δt gives us the applied power,

$$\mathcal{P}_{\rm app} \;=\; \frac{\Delta W}{\Delta t} \;=\; -\Delta A\,\Delta x\,\frac{p(x+\Delta x)\,v_x(x+\Delta x) - p(x)\,v_x(x)}{\Delta x} \;=\; -\Delta V\,\frac{\Delta(pv_x)}{\Delta x},$$

and thus the applied power density is given by:

$$p_{\rm app} \;=\; \frac{\mathcal{P}_{\rm app}}{\Delta V} \;=\; -\frac{\Delta(pv_x)}{\Delta x}.$$

Taking into account similar terms in the y- and z-directions, and letting $\Delta \to \partial$, we have:

$$p_{\rm app} \;=\; -\nabla\cdot(p\,\vec{v}). \qquad (1.22)$$

Substituting Eq. (1.22) into Eq. (1.21) yields:

$$\frac{\partial e_{\rm T}}{\partial t} + \nabla\cdot(e_{\rm T}\vec{v}) \;=\; -\nabla\cdot(p\,\vec{v}),$$

$$\Rightarrow \quad \boxed{\frac{\partial e_{\rm T}}{\partial t} + \nabla \cdot \big((e_{\rm T} + p)\, \vec{v} \big) \;=\; 0,} \tag{1.23}$$

the *total energy equation* and the second equation of ideal HD. □

Example 1.3. Let $Q = \vec{S}$, the *total momentum* of the fluid sample. Find the evolution equation for the corresponding intensive quantity, $q = \vec{s} = \rho\vec{v}$ (the *momentum density*).

Solution: From Eq. (1.3), $\Sigma = \sum \vec{F}_{\rm ext} \Rightarrow \sigma = \sum \vec{f}_{\rm ext}$, the *external force densities.* Thus, Theorem 1.1 requires that:

$$\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\,\vec{v}) \;=\; \sum \vec{f}_{\rm ext}, \tag{1.24}$$

where the Cartesian representation of the divergence term is:

$$\nabla \cdot (\vec{s}\,\vec{v}) \;=\; \Big(\nabla \cdot (s_x \vec{v}), \nabla \cdot (s_y \vec{v}), \nabla \cdot (s_z \vec{v}) \Big).$$

(See §A.4 for other orthogonal coordinate systems.)

For now, we will limit the external force densities to terms arising from pressure gradients and gravity. In Chap. 4, we'll add the Lorentz force, in Chap. 8 viscous stress, and in Chap. 10, drag forces exerted between ions and neutral particles. Starting with the pressure gradient, consider once again the small cube of fluid with edge length Δx and face area ΔA in Fig. 1.3*a*. If the pressure at the left and right sides of the cube are respectively $p(x)$ and $p(x + \Delta x)$, then the net pressure force acting on the cube in the x-direction is given by:

$$F(x + \Delta x) + F(x) \;=\; -p(x + \Delta x)\,\Delta A + p(x)\,\Delta A \;=\; -\frac{\Delta p}{\Delta x}\,\Delta A\,\Delta x \;=\; -\frac{\Delta p}{\Delta x}\,\Delta V.$$

Thus, the pressure force density in the x-direction is:

$$f_x \;=\; \frac{\Delta F_x}{\Delta V} \;=\; -\frac{\Delta p}{\Delta x} \;\rightarrow\; -\frac{\partial p}{\partial x} \quad \text{as } \Delta x \rightarrow 0.$$

Accounting for all three components,

$$\vec{f}_p \;=\; -\nabla p. \tag{1.25}$$

The gravitational force density, $\vec{f}_\phi$, is even simpler to derive. If the fluid sample has mass ΔM, then the gravitational force on ΔM is $-\Delta M\,\nabla\phi$, where ϕ is the local gravitational potential arising from all external masses, including other regions of fluid and distant or embedded point masses (*e.g.*, stars). Thus, $\vec{f}_\phi$ is given by:

$$\vec{f}_\phi \;=\; -\frac{\Delta M\,\nabla\phi}{\Delta V} \;\rightarrow\; -\,\rho\,\nabla\phi \quad \text{as } \Delta V \rightarrow 0. \tag{1.26}$$

Substituting both Eq. (1.25) and (1.26) into Eq. (1.24) yields the *momentum equation*, the third and final equation of ideal HD:

$$\boxed{\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\,\vec{v}) \;=\; -\nabla p - \rho\,\nabla\phi.} \qquad \square \tag{1.27}$$

Summary of §1.3: Equations (1.19), (1.23), and (1.27) constitute two scalar equations and one vector equation which, when combined with Eq. (1.11), (1.20), and $\vec{s} = \rho\vec{v}$ (the *constitutive equations*), provide a closed system of equations for the fluid *flow variables*, namely the volume-conserved quantities ρ, $\vec{s}$, and $e_{\rm T}$. This suite of equations comprises our first set of equations of ideal hydrodynamics:

Equation Set 1:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) &= 0; && \text{continuity}\\
\frac{\partial e_{\rm T}}{\partial t} + \nabla\cdot\big((e_{\rm T}+p)\vec{v}\big) &= 0; && \text{total energy equation}\\
\frac{\partial\vec{s}}{\partial t} + \nabla\cdot(\vec{s}\,\vec{v}) &= -\nabla p - \rho\,\nabla\phi; && \text{momentum equation}\\
e_{\rm T} &= e + \frac{1}{2}\rho v^2 + \rho\phi; && \text{constitutive equation 1}\\
p &= (\gamma-1)e; && \text{constitutive equation 2}\\
\vec{s} &= \rho\vec{v}. && \text{constitutive equation 3}
\end{aligned}$$

The gravitational potential, ϕ, is computed by adding up all the potentials of the contributing point masses, and/or by computing the self-gravitational potential of the gas from the density distribution from Poisson's equation:

$$\nabla^2\phi = 4\pi G\rho. \tag{1.28}$$

As a PDE, Poisson's equation is qualitatively different from the equations of hydrodynamics. It has no time derivative, spatial derivatives are second order, and Poisson's equation is an example of an *elliptical* PDE rather than the *hyperbolic* PDEs of HD (App. C). Analytical methods for solving Poisson's equation can be found in any intermediate or advanced text on electrodynamics (*e.g.*, Paris & Hurd, 1969; Lorrain & Corson, 1970; Jackson, 1975 to suggest a few), while numerical treatments can be found in widely available resources such as *Numerical Recipes* (Press *et al.*, 1992). We shall not address such methods in this text.

1.4 The internal energy density

Equation (1.23) governs the evolution of the total energy density, $e_{\rm T}$. We can eliminate the need for the first constitutive equation by finding an evolution equation for the internal energy density, e, alone, and our approach shall be via thermodynamics.

The combined first and second law of thermodynamics is:

$$T dS = dE + p\,dV, \tag{1.29}$$

where the only new variable being introduced is S, the total *entropy* of the fluid sample.[8] If the mass of the sample, $M = \rho V$, is fixed, then:

$$dV = -\frac{M}{\rho^2}\,d\rho,$$

and Eq. (1.29) becomes:

$$TdS = dE - \frac{Mp}{\rho^2}\,d\rho. \tag{1.30}$$

Define $s \equiv S/M$ and $\varepsilon \equiv E/M$ to be the *specific* entropy and *specific* internal energy respectively. As the term *density* is used to connote *per unit volume*, so the term *specific* is used to connote *per unit mass*. Note, for example, that the specific internal energy and internal energy density are related by $e = \rho\varepsilon$, and while $e \propto p$, the pressure (Eq. 1.11), $\varepsilon \propto T$, the temperature.

With these definitions, Eq. (1.30) becomes,

$$\frac{d\varepsilon}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt} = T\frac{ds}{dt}, \tag{1.31}$$

where I've divided through by the differential dt to obtain an expression relating time derivatives.

Now, because $\varepsilon = e/\rho$, we have:

$$\frac{d\varepsilon}{dt} = \frac{1}{\rho}\frac{de}{dt} - \frac{e}{\rho^2}\frac{d\rho}{dt}.$$

Further, from continuity (Eq. 1.19) and use of the chain rule for partial derivatives, we have:

$$\frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = \frac{d\rho}{dt} + \rho\nabla\cdot\vec{v} = 0 \quad\Rightarrow\quad \frac{d\rho}{dt} = -\rho\,\nabla\cdot\vec{v}.$$

Substituting these into Eq. (1.31) yields:

$$\frac{1}{\rho}\frac{de}{dt} + \frac{p+e}{\rho}\nabla\cdot\vec{v} = T\frac{ds}{dt}. \tag{1.32}$$

Another invocation of the chain rule gives us:

$$\frac{de}{dt} = \frac{\partial e}{\partial t} + \vec{v}\cdot\nabla e,$$

and thus, upon multiplying through by ρ, Eq. (1.32) becomes:

$$\frac{\partial e}{\partial t} + \vec{v}\cdot\nabla e + e\,\nabla\cdot\vec{v} = -p\,\nabla\cdot\vec{v} + \rho T\frac{ds}{dt}$$

$$\Rightarrow\quad \frac{\partial e}{\partial t} + \nabla\cdot(e\,\vec{v}) = -p\,\nabla\cdot\vec{v} + p\,\frac{m}{k_{\rm B}}\frac{ds}{dt},$$

where the ideal gas law (1.12) has been used to replace ρT with $pm/k_{\rm B}$. Finally, by defining the *unitless entropy per particle*, $\mathcal{S} \equiv ms/k_{\rm B}$, we obtain:

$$\frac{\partial e}{\partial t} + \nabla\cdot(e\,\vec{v}) = -p\left(\nabla\cdot\vec{v} - \frac{d\mathcal{S}}{dt}\right). \tag{1.33}$$

---

[8] Unavoidably, S is an over-used symbol. It has already been defined and indeed is used throughout the text as the closed surface of integration. Here, it is being used to represent the total entropy of the fluid (an extensive quantity), while the vector $\vec{S}$ represents the total momentum.

For an adiabatic process, the entropy per particle remains constant, and we arrive at our final form for the evolution equation for internal energy density:

$$\boxed{\frac{\partial e}{\partial t} + \nabla \cdot (e\,\vec{v}) \;=\; -p\,\nabla \cdot \vec{v}.} \tag{1.34}$$

Note that Eq. (1.34) can replace Eq. (1.20) and (1.23), thus giving rise to a somewhat simpler set of hydrodynamical equations:

Equation Set 2:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{v}) \;&=\; 0; && \text{continuity}\\
\frac{\partial e}{\partial t} + \nabla \cdot (e\vec{v}) \;&=\; -p\nabla \cdot \vec{v}; && \text{internal energy equation}\\
\frac{\partial \vec{s}}{\partial t} + \nabla \cdot (\vec{s}\,\vec{v}) \;&=\; -\nabla p - \rho\,\nabla\phi; && \text{momentum equation}\\
p \;&=\; (\gamma - 1)e; && \text{constitutive equation 2}\\
\vec{s} \;&=\; \rho\vec{v}. && \text{constitutive equation 3}
\end{aligned}$$

1.5 Primitive, integral, and conservative form

For a so-called *barotropic* gas (where p is a function of ρ only; both adiabatic and isothermal gases are examples of *barotropes*), it is left to Problem 1.5 to derive the so-called *pressure equation*:

$$\boxed{\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p \;=\; -\rho\,\frac{dp}{d\rho}\,\nabla \cdot \vec{v}.} \tag{1.35}$$

It is further left to Problem 1.2 to show that the continuity equation, (Eq. 1.19), and the momentum equation, (Eq. 1.27), combine to yield an evolution equation for the velocity:

$$\boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} \;=\; -\frac{1}{\rho}\nabla p - \nabla\phi,} \tag{1.36}$$

where, in Cartesian coordinates, we have:

$$(\vec{v} \cdot \nabla)\,\vec{v} \;=\; (\vec{v} \cdot \nabla v_x, \vec{v} \cdot \nabla v_y, \vec{v} \cdot \nabla v_z).$$

(See §A.4 for other orthogonal coordinate systems.) Equation (1.36) is known as *Euler's equation* named for Leonhard Euler (1707–1783), the Swiss mathematician and physicist often described as *the* most prolific mathematician of all time.[9]

---

[9] `www.wikipedia.org/wiki/Leonhard_Euler`

Collecting Eq. (1.35) and (1.36) with the continuity equation, (Eq. 1.19), gives us our third set of HD equations:

Equation Set 3:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0; \qquad \text{continuity}$$

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = -\rho \frac{dp}{d\rho} \nabla \cdot \vec{v}; \qquad \text{pressure equation}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p - \nabla \phi. \qquad \text{Euler's equation}$$

These three equations form a closed set; no constitutive equations are necessary. Equation Set 3 is said to be in *primitive form* because it governs the time evolution of the three so-called *primitive variables* ρ, p, and $\vec{v}$.

Finally, one can write down the equations of ideal HD in *integral form* by performing volume integrals on each term of Eq. Set 1, this time setting $\phi = 0$. This yields a set of *integro-differential* equations highly reminiscent of the fundamental conservation laws (Eq. 1.1–1.3) upon which our current discussion is based. Accordingly, I have designated these as *Equation Set 0*:

Equation Set 0: Integral Equations of Ideal HD

$$\frac{\partial M}{\partial t} + \oint_S \rho \vec{v} \cdot \hat{n}\, d\sigma = 0; \tag{1.37}$$

$$\frac{\partial E_{\mathrm{T}}}{\partial t} + \oint_S (e_{\mathrm{T}} + p) \vec{v} \cdot \hat{n}\, d\sigma = 0; \tag{1.38}$$

$$\frac{\partial \vec{s}}{\partial t} + \oint_S \rho (\vec{v}\vec{v}) \cdot \hat{n}\, d\sigma = -\oint_S p \hat{n}\, d\sigma, \tag{1.39}$$

with constitutive equations:

$$M = \int_V \rho\, dV; \quad E_{\mathrm{T}} = \int_V e_{\mathrm{T}}\, dV; \quad e_{\mathrm{T}} = \frac{p}{\gamma - 1} + \frac{\rho v^2}{2}; \quad \text{and} \quad \vec{s} = \int_V \rho \vec{v}\, dV,$$

and where S is the surface (not entropy!) enclosing the volume element V. Because all spatial derivatives in Eq. Set 1 are either perfect divergences or perfect gradients, their volume integrals can be replaced with surface integrals by the use of Gauss' theorem (Eq. A.30 and A.31). *Note that the same cannot be done with the internal energy equation in Eq. Set 2 and the pressure and Euler's equations in Eq. Set 3 because of the imperfect divergences and gradients in these equations.*

The integral form in Eq. Set 0 completely exposes the three conservation laws upon which fluid dynamics is based. For each equation, the time rate of change of the extensive quantity within a given volume element, V, however large or small is determined completely by the (advective) *flux density* of that quantity (integrand of the surface integral) passing through the closed surface, S.[10] Since Eq. Set 0 follows so directly from Eq. Set 1, Eq. Set 1 is said to be in *conservative form*.

The distinction between conservative and primitive forms is more than semantic. To be valid, the primitive equations require that the primitive variables p, $\vec{v}$, and thus ρ be individually differentiable – and therefore continuous – everywhere. This necessarily precludes discontinuities in ρ, p, and $\vec{v}$ and thus the primitive equations are valid only for *smooth flow*. On the other hand, the conservative equations only require that the functions (flux densities) $\rho\vec{v}$, $\vec{v}(e_\mathrm{T} + p)$, and $\rho\vec{v}\vec{v} + p\,\mathsf{I}$[11] be continuous,[12] and not necessarily the primitive variables individually. Thus, the conservative equations in terms of the conservative variables (ρ, e_T, $\vec{s}$) can, in principle, admit solutions with discontinuities in ρ, p, and $\vec{v}$ (*i.e.*, *discontinuous flow*) so long as these discontinuities combine to yield continuous flux densities. We shall exploit this observation when we write down the *Rankine–Hugoniot jump conditions* in §2.2.3, and then again for MHD in §5.3.

Of course, in addition to discontinuous solutions, the conservative equations also admit all smooth solutions admitted by the primitive equations, and thus the conservative set of equations is the more general of the two. Still, there are times when use of the primitive equations is far more convenient, as we shall see when we discuss the all-important Riemann problem in Chap. 3 and 6.

Problem Set 1

1.1 On a cold winter afternoon, you enter your winter cabin (which has not been heated for weeks) freezing cold. You light a roaring fire in the hearth and after an hour, the cabin is warm enough to take off your winter clothing.

a) Does the air in your cabin contain more, less, or the same total internal energy, E, now that it is warm than when it was cold? Explain.

b) If you conclude that the air contains less or the same internal energy after being heated as before, where does all the energy from the fire go?

1.2 Derive Euler's equation (Eq. 1.36 in the text) from the continuity and momentum equations (Eq. 1.19 and 1.27). Your proof should be valid for all coordinate systems, not just Cartesian.

Hint: Vector identity (A.21) from App. A should be particularly helpful.

---

[10] Note that in this picture, the pressure p contributes to the flux densities of both $\vec{S}$ and E_T.

[11] $\vec{v}\vec{v}$ is the *dyadic product* of $\vec{v}$ with itself creating a rank 2 tensor (matrix; see Eq. A.16), while I is the "identity tensor", which you can think of as the identity matrix.

[12] To see how one arrives at the conclusion that the momentum equation, (1.27), only requires that $\rho\vec{v}\vec{v} + p\,\mathsf{I}$ be continuous, it is instructive to note that formally, $\nabla p = \nabla \cdot (p\,\mathsf{I})$. Try it!

1.3 Derive the internal energy equation for an adiabatic gas (Eq. 1.34 in the text) from the hydrodynamical equations alone by substituting the definition for e_T (Eq. 1.20) into the total energy equation (Eq. 1.23) and then by using the continuity equation (Eq. 1.19) and Euler's equation (Eq. 1.36) to simplify.

Hint: The gravitational potential, ϕ, solves Poisson's equation (Eq. 1.28) and, as such, has no explicit time dependence. Thus, you can set $\partial\phi/\partial t = 0$.

1.4*

a) Equation (1.34) in the text is the evolution equation for the internal energy of an adiabatic gas. Show that the analogous equation for an *isothermal* gas is:

$$\frac{\partial e}{\partial t} + \nabla \cdot (e\,\vec{v}) = 0. \tag{1.40}$$

Physically, what do you suppose is happening in an isothermal gas to maintain its isothermality?

b) For an adiabatic gas, we argued that the unitless entropy per particle, $\mathcal{S}$, remains constant in time. Find $d\mathcal{S}/dt$ for an isothermal gas.

c) We can model a real gas by an equation of state of the form $p = \kappa\rho^n$ where, in principle, both κ and the power-law index n could vary from point to point. For an adiabatic gas, $n = \gamma$, while for an isothermal gas, $n = 1$ (why?). Argue that for a real gas, $1 < n < \gamma$ and thus the isothermal and adiabatic conditions represent limits in between which a given real gas should be found.

1.5* A *barotropic* equation of state is one where the pressure depends only on the density, that is $p = p(\rho)$.

a) Starting with the internal energy density equation for an adiabatic gas, Eq. (1.34) in the text, show that:

$$\frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p = -\gamma p\,\nabla \cdot \vec{v}. \tag{1.41}$$

b) Starting with the continuity equation and assuming a barotropic equation of state, derive the "pressure equation", Eq. (1.35).

c) Show that for an adiabatic gas where $p \propto \rho^\gamma$, Eq. (1.35) reduces to Eq. (1.41).

1.6 The *vorticity* is defined as $\vec{\omega} = \nabla \times \vec{v}$, and is a measure of fluid *circulation.*

a) Starting from either Eq. (1.27) or (1.36) in the text and assuming the fluid to be barotropic (as defined in Problem 1.5), show that the evolution equation for the vorticity is given by:

$$\frac{\partial\vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}). \tag{1.42}$$

Hint: Vector identity (A.15) in App. A *might* be of help.

b) If the fluid is not barotropic [*e.g.*, $p = p(\rho, e)$], show that Eq. (1.42) is still valid if the fluid is *incompressible*, that is where the density may be taken as constant in both space and time and thus the continuity equation (Eq. 1.19) reduces to $\nabla \cdot \vec{v} = 0$.

1.7 Define the *circulation*, Γ, of a fluid about a closed loop, C, to be:

$$\Gamma = \oint_C \vec{v} \cdot d\vec{l}.$$

By inspection, Γ is non-zero only if there is net circulation about the loop, whence its name.

a) Show that:

$$\Gamma = \int_\Sigma \vec{\omega} \cdot d\vec{\sigma}, \tag{1.43}$$

where Σ is the open surface enclosed by the closed loop, C, and $\vec{\omega} = \nabla \times \vec{v}$ is the vorticity defined in Problem 1.6. This should be a one-liner. Thus, the circulation, Γ, can also be interpreted as the "vorticity flux" passing through a closed loop.

b) Prove that for a barotropic (Problem 1.5) or incompressible (ρ = constant) fluid,

$$\frac{d\Gamma}{dt} = 0.$$

This is *Kelvin's circulation theorem*, and asserts that vorticity flux is a conserved quantity for inviscid barotropic flow.

Hint: Start with Eq. (1.43) and examine $d\Gamma/dt$, noting that the surface over which the integral is performed, Σ, is also time-dependent; this must somehow be taken into account in taking the time derivative. If this doesn't seem like a familiar problem, review the proof of the theorem of hydrodynamics (Theorem 1.1 in the text). Finally, you should come to a point where Eq. (1.42) from Problem 1.6 would be useful; feel free to use it!

Discussion: As we shall see in Chap. 4, lines of magnetic induction and vortex lines share many properties since both $\vec{\omega}$ and $\vec{B}$ are solenoidal ($\nabla \cdot \vec{\omega} = \nabla \cdot \vec{B} = 0$), and both are governed by an "induction equation" (*cf.* Eq. 1.42 and 4.4). Given that magnetic flux is a conserved quantity, it should then come as no surprise that vorticity flux is also conserved.

An immediate consequence of Kelvin's circulation theorem is that if a barotropic or incompressible fluid starts off with zero vorticity (and thus zero circulation everywhere), it must develop in a such way to maintain zero vorticity. If it didn't, then one could find a patch of area through which $\Gamma \neq 0$, violating Kelvin's theorem. Note that dissipative encounters with walls or introduction of viscosity (numerical or physical) into the fluid, which are not present in Euler's equation used to prove Kelvin's theorem, could cause an initially irrotational fluid to develop vorticity. Otherwise, Kelvin's theorem essentially states that for an inviscid fluid, "once irrotational, always irrotational".

If one can establish that $\vec{\omega} = \nabla \times \vec{v} = 0$ for all time, then the velocity field can be expressed as the gradient of a scalar; $\vec{v} = \nabla\psi$. Such a *velocity potential* can be useful, particularly for incompressible fluids where $\nabla \cdot \vec{v} = 0$ since this means the velocity potential will satisfy Laplace's equation, $\nabla^2\psi = 0$. In this case, all the mathematics used in problems in electrostatics and, in particular, in *potential theory* can be brought to bear on solving Laplace's equation instead of the much more difficult Euler's equation.