A CHARACTERIZATION OF DETERMINANTS OVER TOPOLOGICAL FIELDS

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0. Introduction. Let K be a topological field. On introducing the vector-space topology on the $n \times n$ matrices over K, it becomes clear that the determinant map φ enjoys the following properties:

(A) φ is a continuous surjective homomorphism from $GL_n(K)$ to K^*,
(B) $\varphi(\mu\mathbf{a}) = \mu^n\varphi(\mathbf{a})$, for each non-zero μ in K, and all $\mathbf{a}$ in $GL_n(K)$.

It is known **[1]** that, when, in (A), the requirement that φ be a continuous surjection is replaced by the requirement that $\varphi(\mathbf{a})$ be a polynomial in the elements of $\mathbf{a}$, then condition (B) is enough to characterise φ as the determinant; the purpose of this paper is to examine the extent to which conditions (A) and (B) suffice to imply that φ is the determinant map. More precisely, let us say that K is *n-determinant characterised* if (A) and (B) imply that φ is the determinant. We shall, in fact, find all solutions φ of (A) and (B) for a wide class of fields K, namely, all K whose completions L are local fields, in the sense of **[2]**; the latter are precisely the locally compact, non-discrete fields. The basic result, on which all else depends, is

THEOREM 1. *Let K be any topological field, and suppose that* (A) *and* (B) *are satisfied by φ. Then φ is* 1 *on $SL_n(K)$, and induces a continuous epimorphism φ^*: $GL_n(K)/SL_n(K) \to K^*$; in the same way, the determinant induces a continuous isomorphism α: $K^* \to GL_n(K)/SL_n(K)$. The composite map $\varphi^* \circ \alpha$ is a continuous surjective endomorphism of K^* which also fixes the nth powers pointwise.*

This theorem is proved in § 1; in § 2, we derive two straightforward corollaries of Theorem 1, which we state as lemmas.

LEMMA 2.1. *Any field K for which L is $\mathbb{R}$ or $\mathbb{C}$ is n-determinant characterised for each natural number n.*

LEMMA 2.2. *Let K be any topological field, and suppose that n is coprime to t, the order of the torsion subgroup T of K^*. Then K is n-determinant characterised.*

The main complication arises when $(n, t) > 1$. The result for dense subfields K of local fields L is that the general solution of (A) and (B) is the determinant, multiplied by roots of unity associated in a natural way with the norm residue symbols of certain extensions. The precise statements are given in § 3; an added bonus is another proof of Lemma 2.1, as a detail of a general result on local fields.

1. Proof of Theorem 1.. We consider first the effect of restricting φ to the subgroup $SL_n(K)$; the centre Z of $SL_n(K)$ consists of all matrices $\alpha\mathbf{1}$, where $\alpha^n = 1$. In view of condition (B), the kernel of φ contains Z. Thus φ induces a homomorphism φ^{**}: $PSL_n(K) \to K^*$. The kernel of φ cannot be trivial for $n > 1$, as the domain is not abelian; hence, as $PSL_n(K)$ is simple, ker φ^{**} is the whole group. It follows that the kernel of φ contains $SL_n(K)$; since the latter is a closed normal subgroup of $GL_n(K)$, φ induces a continuous epimorphism $\varphi^*: GL_n(K)/SL_n(K) \to K^*$. The determinant maps $GL_n(K)$ onto K^*, with kernel $SL_n(K)$, so that there is an induced continuous isomorphism $\alpha: K^* \cong GL_n(K)/SL_n(K)$. Then the composite map $f = \varphi^* \circ \alpha$ is a continuous surjective endomorphism $K^* \to K^*$. Since both the determinant and φ satisfy (B), it follows that $f(\mu^n) = \mu^n$ for all μ in K^*.

2. Since everything in $\mathbb{C}$ is an nth power, Lemma 2.1 is now immediate when $L = \mathbb{C}$. The same is true when $L = \mathbb{R}$ and n is odd. When n is even, we note that f fixes at least the positive reals, and is also surjective; hence it must be the identity. Lemma 2.2 is readily proved by observing that, for all μ in K^*, $f(\mu)/\mu$ must be an nth root of unity, hence 1, since n is coprime to t.

3. The twisting factors. We now stipulate that K be discrete, with locally compact completion L. Thus L is one of the following:

(i) the real field $\mathbb{R}$;

(ii) the complex field $\mathbb{C}$;

(iii) the completion of an algebraic number field with respect to an archimedean valuation and thus a finite extension of a p-adic completion $\mathbb{Q}_p$ of the rational field $\mathbb{Q}$;

(iv) the completion of a finite algebraic function field in one variable over a finite constant field and thus a field of formal power series $F((T))$, where $F = GF(q)$.

Let us write $s = (n, t)$; then f, of Theorem 1, fixes nth powers if and only if it fixes sth powers. From the equation $f(\mu^s) = \mu^s$, we deduce that $f(\mu) = \mu \,.\, \varepsilon^{\sigma(\mu)}$, where $\sigma: K^* \to \mathbb{Z}_s$ is a continuous homomorphism; writing $\sigma(K^*) \cong \mathbb{Z}_r$, where r divides s, we may assume that ε is a primitive rth root of 1. All relevant maps extend uniquely to the completion L; hence we can say that $L^*/\ker\sigma \cong \mathbb{Z}_r$, so that $\ker\sigma$ is a closed normal subgroup of finite index in L^*. Hence there exists a unique cyclic field extension F/L such that $\ker\sigma = N_{F/L}(F^*)$, and, further,

$$\mathbb{Z}_r \cong L^*/\ker\sigma = L^*/N_{F/L}(F^*) \cong \operatorname{Gal} F/L. \tag{3.1}$$

(See [3], especially pp. 153–154 and 159–161.) The right-hand isomorphism in (3.1) is induced by the norm-residue map $\alpha \mapsto (\alpha, F/L)$. If we choose a specific isomorphism $h: \operatorname{Gal} F/L \to \mathbb{Z}_r$, the norm-residue symbol induces a homomorphism $L^* \to L^*$ by $\alpha \mapsto \varepsilon^{h((\alpha, F/L))}$. As h differs from σ by at most an automorphism of $\mathbb{Z}_r$, we may suppose ε chosen to make $f(\mu) = \mu \,.\, \varepsilon^{h((\alpha, F/L))}$. The condition that f be surjective is satisfied if and only if $1 + h((\varepsilon F/L))$ is coprime to r, that is, if and only if $(\varepsilon, F/L) = \gamma\delta^{-1}$, where both γ and δ generate $\operatorname{Gal} F/L$. This can give rise to non-trivial f, as when, e.g., r is odd, $\varepsilon^{1/r} \notin L$, and $F = L(\varepsilon^{1/r})$. Here ε is a norm, so that $(\varepsilon, F/L) =$ identity.

Since $\mathbb{C}^*$ contains no proper closed subgroups of finite index (for $\mathbb{C}^*$ is homeomorphic to $\mathbb{R} \oplus \mathbb{R}/\mathbb{Z}$), we obtain another proof that $\mathbb{C}$ is n-determinant characterised for all n. Further, $\mathbb{R}^*$ has only the positive reals as a norm subgroup, corresponding to the extension field $\mathbb{C}$, and then the corresponding f is $x \mapsto x \,.\, (\operatorname{sgn} x)^m$, for some integer m. Surjectivity implies that m is even; so we have Lemma 2.1 again. The formal power-series and p-adic cases admit pathological solutions φ as above because, essentially, they are far from being algebraically closed.

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