MATHEMATICAL NOTES

A Review of Elementary Mathematics and Science.

A Problem in Probability.

The following is a generalisation of the well known probability question, viz., "n letters are each addressed to one of n houses, in how many ways can they all be misdelivered, one to each house?" Or in the original form, "n cards numbered 1 to n are drawn from a bag one by one, what is the probability that the order in which a card is drawn will in no case coincide with the number on the card?"

Laplace generalised the problem in one direction, but the following is a double generalisation permitting a line of attack which appears to be new.

If we have α different groups each of n similar objects and β dissimilar objects ungrouped, if, also, we have α places each marked to correspond to one of the α groups and γ places unmarked, in how many different ways can the $\alpha+\gamma$ places receive each one of the $n\alpha+\beta$ objects without any of the objects being assigned to a place with a corresponding mark?

The problem may be visualized more clearly from the following scheme:

α groups of n	β objects ungrouped
A, B, C, D..............	P, Q, R, S............
$a, b, c, d,$..................	x, y, z..................
α corresponding places	γ places unmarked.

Let $f(\alpha, \beta, \gamma, n)$ be the solution.

It will be seen that $f(\alpha-1, \beta+n, \gamma+1, n)$ is the solution of an analogous problem for which the scheme is

$\alpha-1$ groups of n	$\beta+n$ objects ungrouped
A, B, C, D..................	P, Q, R, S.................
a, b, c, d......................	x, y, z......................
$\alpha-1$ corresponding places	$\gamma+1$ places unmarked.

In the latter problem make a group of n of the $\beta+n$ ungrouped objects and consider these with reference to the place x.

(1) x may be filled up from one of these—in n different ways. The allocation is then completed in $f(\alpha-1, \beta+n-1, \gamma, n)$ ways.

(2) x may be avoided completely by this group of n. This places the group in the same relation to x in particular and to the other places in general as one of the $\alpha-1$ groups, say A, is placed in regard to a and the remaining places, *i.e.* the allocation is made in $f(\alpha, \beta, \gamma, n)$ ways.

Hence we have the fundamental relation

$$f(\alpha-1, \beta+n, \gamma+1, n) = nf(\alpha-1, \beta+n-1, \gamma, n) + f(\alpha, \beta, \gamma, n) \ldots \text{I.}$$

Put $\alpha=1$, then

$$f(1, \beta, \gamma, n) = f(0, \beta+n, \gamma+1, n) - nf(0, \beta+n-1, \gamma, n)$$
$$= {}^{\beta+n}P_{\gamma+1} - n\,{}^{\beta+n-1}P_{\gamma}$$

in the ordinary notation for permutations.

Put $\alpha=2$,

$$f(2, \beta, \gamma, n) = f(1, \beta+n, \gamma+1, n) - nf(1, \beta+n-1, \gamma, n)$$
$$= {}^{\beta+2n}P_{\gamma+2} - 2n\,{}^{\beta+2n-1}P_{\gamma+1} + n^2\,{}^{\beta+2n-2}P_{\gamma}$$

making use of the result just obtained.

This leads to the general assumption

$$f(\alpha, \beta, \gamma, n) = \sum_{k=0}^{a} (-n)^k\, {}^{a}C_k\, {}^{\beta+an-k}P_{\gamma+a-k}$$

It is only necessary to show that this satisfies the fundamental relation I. A few transformations easily establish that relation I. is satisfied.

Since

$$ {}^{\beta+an-k}P_{\gamma+a-k} = \frac{{}^{na+\beta}P_{a+\gamma}}{{}^{na+\beta}C_k \,\underline{|k}}.$$

The solution is

$$f(\alpha, \beta, \gamma, n) = {}^{na+\beta}P_{a+\gamma} \sum_{k=0}^{\alpha} \frac{(-n)^k}{\underline{|k}} \frac{{}^{a}C_k}{{}^{na+\beta}C_k}$$

or since ${}^{n\alpha+\beta}P_{\alpha+\gamma}$ gives the unrestricted number of ways of filling up the places, the probability that no object is in a place correspondingly marked in a fortuitous distribution is

$$\sum_{k=0}^{\alpha} \frac{(-n)^k}{\underline{|k}} \cdot \frac{{}^{\alpha}C_k}{{}^{n\alpha+\beta}C_k}.$$

The particular result for the problem enunciated at the beginning is got by putting $n=1$, $\beta=\gamma=0$

$$i.e. \quad \sum_{k=0}^{\alpha} \frac{(-1)^k}{\underline{|k}}$$

or $\underline{|\alpha} \sum_{k=0}^{\alpha} \frac{(-1)^k}{\underline{|k}}$ according to the mode of statement of the problem.

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Analytical Note on Lines Forming a Harmonic Pencil.

The following is a simple proof of the theorem that the concurrent lines whose equations are

$$a_1 x + b_1 y + c_1 = 0 \quad \text{.......(1)}$$

$$a_2 x + b_2 y + c_2 = 0 \quad \text{.......(2)}$$

$$a_1 x + b_1 y + c_1 = k\,(a_2 x + b_2 y + c_2) \quad \text{.......(3)}$$

$$a_1 x + b_1 y + c_1 = -k\,(a_2 x + b_2 y + c_2) \quad \text{.......(4)}$$

form a harmonic pencil.

Let a line through the origin parallel to the line (2) intersect (4) in $A\ (x_1, y_1)$, and (3) in $B\ (x_2, y_2)$.

The pencil is harmonic if the mid-point of AB, $C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, lies on (1).

Since OAB is parallel to (2) we have

$$a_2 x_1 + b_2 y_1 = a_2 x_2 + b_2 y_2 = 0.$$

Hence since $A\ (x_1, y_1)$ lies on (4),

$$a_1 x_1 + b_1 y_1 + c_1 = -k\,c_2,$$

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