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TRANSLATES OF SEQUENCES IN SETS OF POSITIVE MEASURE

BY
D. BORWEIN AND S. Z. DITOR

Given a measurable set A of real numbers with measure $mA > 0$, and a sequence $\{d_n\}$ of real numbers converging to zero, is there always an x such that $x + d_n \in A$ for all n sufficiently large?

The answer to this question, which was posed to the authors by P. Erdös, is NO. The actual situation can be described as follows.

THEOREM 1. (i) *If A is a measurable set with $mA > 0$ and $\{d_n\}$ is a sequence converging to zero, then, for almost all $x \in A$, $x + d_n \in A$ for infinitely many n.*

(ii) *There is a measurable set A with $mA > 0$ and a monotonic sequence $\{d_n\}$ of positive numbers converging to zero such that, for all x, $x + d_n \notin A$ for infinitely many n.*

Proof. (i) Suppose without loss in generality that A is compact. Let $E_k = \bigcup_{n=k}^{\infty} (A - d_n)$ and let $E = \bigcap_{k=1}^{\infty} E_k$. Then $x \in E$ if and only if $x + d_n \in A$ for infinitely many n. Since A is closed, $E \subset A$. Further, $mE_k \geq m(A - d_k) = mA$, and, since $mE_1 < \infty$ and $E_k \supset E_{k+1}$, $mE_k \to mE$. Hence $mA = mE$. The desired conclusion follows.

(ii) Let $B_1 = \{1\}$, $B_2 = \{2, 3\}$, $B_3 = \{4, 5, 6\}$, and, in general, $B_n = \{N+1-n, N+2-n, \ldots, N\}$ where $N = n(n+1)/2$. Let A_n be the set of numbers x in $[0, 1]$ admitting a dyadic expansion

$$x = \sum_{i=1}^{\infty} \frac{\xi_i}{2^i}, \qquad \xi_i \in \{0, 1\}$$

such that, for every $j \in \{1, 2, \ldots, n\}$, $\xi_i = 0$ for some $i \in B_j$. Let $A = \bigcap_{k=1}^{\infty} A_k$. Then $mA = \prod_{j=1}^{\infty} (1 - 2^{-j}) > 0$. The probability argument for this assertion is clear. The assertion can, however, be established directly as follows.

For each subset σ of $\{1, 2, \ldots, N\}$ which meets every B_j with $j \in \{1, 2, \ldots n\}$, let $x_\sigma = \sum_{i=1}^{N} \xi_i / 2^i$, where $\xi_i = 0$ if $i \in \sigma$, $\xi_i = 1$ otherwise. There are $\prod_{j=1}^{n} (2^j - 1)$ such subsets σ, and $A_n = \bigcup_\sigma [x_\sigma, x_\sigma + 2^{-N}]$. Hence $mA_n = 2^{-N} \prod_{j=1}^{n} (2^j - 1) = \prod_{j=1}^{n} (1 - 2^{-j})$, and, since $mA_1 < \infty$ and $A_n \supset A_{n+1}$, $mA_n \to mA$.

Further, since each A_n is closed, so also is A.

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Now let D_n be the set of positive numbers x admitting a dyadic expansion

$$x=\sum_{i\in B_n}\frac{\xi_i}{2^i},\qquad \xi_i\in\{0,1\}.$$

Then $\bigcup_{n=1}^{\infty} D_n$ can be enumerated so as to form a monotonic sequence $\{d_k\}$ of positive numbers converging to zero.

Suppose $x\in A$ and n_0 is any positive integer. Then there is a $d_n\in D_{n_0}$ such that $x+d_n=\sum_{i=1}^{\infty}\xi_i/2^i$ where every $\xi_i\in\{0,1\}$ and $\xi_i=1$ for all $i\in B_{n_0}$. Hence $x+d_n\notin A$ for infinitely many n.

On the other hand if $x\notin A$, then, since A is closed, $x+d_n\notin A$ for all n sufficiently large.

The following theorem is a generalization of part (i) of Theorem 1. The proof given below is similar to, but probably of necessity somewhat less elementary than, the proof of part (i) of Theorem 1.

THEOREM 2. *If A is a measurable set with $mA>0$ and $\{d_{1,n}\}$, $\{d_{2,n}\},\ldots,\{d_{r,n}\}$ are r sequences each converging to zero, then, for almost all $x\in A$, $x+d_{1,n}$, $x+d_{2,n},\ldots,$ $x+d_{r,n}$ are all in A for infinitely many n.*

Proof. Suppose without loss in generality that A is bounded. Let $d_{0,n}\equiv 0$, let $E_k=\bigcup_{n=k}^{\infty}\bigcap_{i=0}^{r}(A-d_{i,n})$ and let $E=\bigcap_{k=1}^{\infty}E_k$. Then $E\subset A$ and $x\in E$ if and only if $x+d_{1,n}$, $x+d_{2,n},\ldots,x+d_{r,n}$ are all in A for infinitely many n. Now

$$m(A\sim(A-h))=\int_{-\infty}^{\infty}\chi_A(t)(\chi_A(t)-\chi_A(t+h))\,dt$$

$$\le\int_{-\infty}^{\infty}|\chi_A(t)-\chi_A(t+h)|\,dt\to 0\quad\text{as}\quad h\to 0,$$

by a standard result. It follows that

$$m\Big(A\sim\bigcap_{i=0}^{r}(A-d_{i,k})\Big)=m\bigcup_{i=0}^{r}(A\sim(A-d_{i,k}))\to 0\quad\text{as}\quad k\to\infty,$$

and hence that

$$mE_k\ge m\bigcap_{i=0}^{r}(A-d_{i,k})\to mA.$$

Further, since $mE_1<\infty$ and $E_k\supset E_{k+1}$, $mE_k\to mE$. Thus $mA=mE$, and the theorem is established.

NOTE: Taking $d_{j,n}=jd_n$ in Theorem 2, we get as a special case the known result that a measurable set of positive measure contains finite arithmetic progressions of arbitrary length.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA N6A 5B9

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