Canad. Math. Bull. Vol. **55** (4), 2012 pp. 821–829
http://dx.doi.org/10.4153/CMB-2011-133-2

New Examples of Non-Archimedean Banach Spaces and Applications

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Abstract. The study carried out in this paper about some new examples of Banach spaces, consisting of certain valued fields extensions, is a typical non-archimedean feature. We determine whether these extensions are of countable type, have t-orthogonal bases, or are reflexive. As an application we construct, for a class of base fields, a norm $\|\cdot\|$ on c_0, equivalent to the canonical supremum norm, without non-zero vectors that are $\|\cdot\|$-orthogonal and such that there is a multiplication on c_0 making $(c_0, \|\cdot\|)$ into a valued field.

1 Preliminaries and Basic Lemmas

Throughout this paper $K := (K, |\cdot|)$ is a non-archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\cdot| : K \to [0, \infty)$. For fundamentals on non-archimedean valued fields and their valued field extensions, see [1, 3]. Here we only fix some notations and recall some basic concepts which will be involved in the paper.

By $K[X]$ we mean the K-vector space of all polynomials with coefficients in K. Also, $K(X)$ denotes the (non-necessarily complete) field of rational functions over K with the non-archimedean valuation, which extends the valuation on K, defined by

$$\Big|\frac{\lambda_0 + \lambda_1 X + \cdots + \lambda_n X^n}{\mu_0 + \mu_1 X + \cdots + \mu_m X^m}\Big| := \frac{\max_{0\le i\le n} |\lambda_i|}{\max_{0\le j\le m} |\mu_j|},$$

where λ_i, μ_j are in K and not all μ_j equal to 0.

The set $G_K := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$ is a multiplicative group of positive real numbers, called the *value group* of K. We denote $|K| := G_K \cup \{0\}$.

The *closed unit ball* in K is $B_K := \{\lambda \in K : |\lambda| \le 1\}$. Similarly, the *open unit ball* in K is $B_K^- := \{\lambda \in K : |\lambda| < 1\}$. B_K is not only multiplicatively, but, due to the strong triangle inequality ($|\lambda + \mu| \le \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in K$), also additively closed. Thus, B_K is a commutative ring with identity. Further, B_K^- is easily seen to be an ideal in B_K and, since each $\lambda \in K$ with $|\lambda| = 1$ is invertible in B_K, even a maximal ideal. Thus, B_K/B_K^- is a field, called the *residue class field* of K and denoted by k. The canonical map $B_K \to k$ is written $\lambda \mapsto \overline{\lambda}$.

Received by the editors February 3, 2010.
Published electronically June 29, 2011.
The first author's research was partially supported by Ministerio de Ciencia e Innovación, MTM2010-20190-C02-02.
AMS subject classification: **46S10**, 12J25.
Keywords: non-archimedean Banach spaces, valued field extensions, spaces of countable type, orthogonal bases.

Note All the vector and Banach spaces considered in this paper are over K.

The new examples of non-archimedean Banach spaces treated in this paper are complete valued field extensions of K; we will focus on algebraically closed fields K (see the end of this section).

A *valued field extension* L of K is a non-archimedean valued field containing K as a subfield and such that the valuation of K is the restriction of the valuation of L (this last one is also denoted by $|\cdot|$).

A valued field extension L of K is called *immediate* if the value groups of K and L are the same and their residue class fields are naturally isomorphic, or equivalently, if for each $a \in L$, $a \neq 0$, $\inf\{|a-\lambda| : \lambda \in K\} < |a|$ ([2, Exercise 4.X and comments after Theorem 4.57]).

We call $(K, |\cdot|)$ *spherically complete* if it has no proper immediate valued field extensions, or equivalently, if each nested sequence of balls $B_1 \supset B_2 \supset \cdots$ in K has a non-empty intersection [2, Theorem 4.47].

Now let L_1 and L_2 be two spherically complete immediate valued field extensions of K. Then there is a bijective K-linear isometry $L_1 \to L_2$ that leaves K pointwise fixed, but we cannot always choose this map to be a field homomorphism [2, Theorem 4.59].

Despite this, we shall denote any spherically complete immediate valued field extension of K by $K^\vee$, and even call $K^\vee$ *the spherical completion* of K.

The field $\mathbb{Q}_p$ of p-adic numbers (where p is a prime number) is spherically complete (because it is locally compact, [3, Theorem 5.4]) and it is not algebraically closed ([3, Corollary 16.4]). The completion $\mathbb{C}_p$ of the algebraic closure of $\mathbb{Q}_p$ is algebraically closed [3, Corollary 17.2.(i)] and it is not spherically complete [3, Corollary 20.6]. The spherical completion of $\mathbb{C}_p$ is algebraically closed [2, Corollary 4.51] and clearly it is spherically complete.

To give an example of a non-algebraically closed and non-spherically complete field is a more delicate subject. Let $K := \mathbb{C}_p$. Let L be the completion of $K(X)$. Then L is the field of formal Laurent series in K constructed in Exercise 1.K of [2] for $\rho := 1$. It is easily seen that there is no element in L whose square is equal to X, so L is not algebraically closed. Also, as $\mathbb{C}_p$ is separable [3, Corollary 17.2.(iv)], then so is $K(X)$ and hence L is separable [3, Exercise 17.B]. Finally, since the valuation of L is dense, it follows that L is not spherically complete [3, Theorem 20.5], and we have the desired example.

Now let $E = (E, \|\cdot\|)$ be a (non-archimedean) Banach space. For fundamentals on non-archimedean Banach spaces we refer to [2]. Here we only fix some notations and recall some basic concepts that will be involved in the paper.

By $\|E\|$ we mean $\{\|x\| : x \in E\}$. For a set $X \subset E$, $\sharp X$ and $[X]$ are the cardinality and the linear hull of X, respectively; $\overline{X}$ denotes the closure of X with respect to the norm topology on E. For $X, Y \subset E$, $Y \setminus X := \{y \in E : y \in Y,\ y \notin X\}$. The *distance between two non-empty sets* $X, Y \subset E$ is $\operatorname{dist}(Y, X) := \inf\{\|y - x\| : y \in Y,\ x \in X\}$. For $a \in E$, instead of $\operatorname{dist}(\{a\}, X)$ we write $\operatorname{dist}(a, X)$.

By $L(E)$ we mean the Banach space of all continuous linear maps $E \to E$ and by E' the Banach space of all continuous linear maps $E \to K$. As usual $E'' := (E')'$ and E is called *reflexive* if the canonical map $E \to E''$ is a surjective isometry. E is said to be *of* *countable type* if it contains a countable set whose linear hull is dense in E.

Let I be a non-empty set, let $s := (s_i)_{i\in I} \in \mathbb{R}^I$ with $s_i > 0$ for all $i \in I$. The space $c_0(I,s) := \{(\lambda_i)_{i\in I} \in K^I : \lim_i |\lambda_i|\, s_i = 0\}$, equipped with the norm $\|(\lambda_i)_{i\in I}\| := \max_{i\in I} |\lambda_i|\, s_i$, is a Banach space, which is of countable type if and only if I is countable. When $s_i = 1$ for all $i \in I$, we write $c_0(I)$ instead of $c_0(I,s)$; if, additionally, $I = \mathbb{N}$, then $c_0(\mathbb{N})$ is the well-known space c_0 of all sequences in K tending to 0.

Two elements x, y of E are *orthogonal* to each other ($x \perp y$) if $\operatorname{dist}(x,[y]) = \|x\|$, or equivalently, if $\|\lambda x + \mu y\| = \max\{\|\lambda x\|, \|\mu y\|\}$ for all $\lambda, \mu \in K$. For two subspaces D_1, D_2 of E we put $D_1 \perp D_2$ if $x \perp y$ for all $x \in D_1$, $y \in D_2$. For $D_1 = [a]$, $a \in E$, instead of $[a] \perp D_2$, we write $a \perp D_2$ (observe that $a \perp D_2$ if and only if $\operatorname{dist}(a, D_2) = \|a\|$). We say that a subspace D_1 is *orthocomplemented* in E if there exists a subspace D_2 such that $D_1 \perp D_2$ and $D_1 \oplus D_2 = E$ (where $\oplus$ means *algebraic direct sum*), or equivalently, if there exists a continuous linear projection $Q\colon E \to D_1$ with $\|Q\| \leq 1$.

Let $t \in (0,1]$. A t-*orthogonal system* in E is a subset $X = \{e_i : i \in I\}$ of $E \setminus \{0\}$ such that if $i_1, \dots, i_n$ are distinct elements of I, then

$$(1.1) \qquad \|\lambda_{i_1} e_{i_1} + \cdots + \lambda_{i_n} e_{i_n}\| \geq t \max_{1\leq k\leq n} \|\lambda_{i_k} e_{i_k}\| \quad \text{for all } n \in \mathbb{N}, \lambda_{i_1}, \dots, \lambda_{i_n} \in K.$$

If in addition $\|e_i\| = 1$ for all $i \in I$, then X is called a t-*orthonormal system*.

A t-orthogonal system X is called a t-*orthogonal base* of E if in addition $\overline{[X]} = E$ (or equivalently, if every $x \in E$ can be written uniquely as $x = \sum_{i\in I} \lambda_{xi}\, e_i$, $\lambda_{xi} \in K$). All t-orthogonal bases in E have the same cardinality. When $t = 1$, we write "orthogonal" instead of "1-orthogonal" and in this case, by the strong triangle inequality for $\|\cdot\|$, we have that (1.1) is equivalent to

$$\|\lambda_{i_1} e_{i_1} + \cdots + \lambda_{i_n} e_{i_n}\| = \max_{1\leq k\leq n} \|\lambda_{i_k} e_{i_k}\| \quad \text{for all } n \in \mathbb{N}, \lambda_{i_1}, \dots, \lambda_{i_n} \in K.$$

Analogously we write "orthonormal" instead of "1-orthonormal".

Each Banach space with an orthogonal base $\{e_i : i \in I\}$ is isometrically isomorphic to $c_0(I,s)$, with $s_i := \|e_i\|$ for all i, in particular, isometrically isomorphic to $c_0(I)$ in the case in which the base is orthonormal.

1.1 New Examples of Banach Spaces

If L is a complete valued field extension of K, then the valuation on L makes it naturally into a Banach space. This fact generates a new class of examples of Banach spaces. For $a \in L \setminus K$, let $K(a)$ be the smallest subfield of L containing K and a and let $\overline{K(a)}$ be the closure of $K(a)$ in L. Clearly $\overline{K(a)}$ is a Banach space with the norm induced by the valuation of L.

The main result of this paper, Theorem 2.1, provides an answer when K is algebraically closed to the natural questions whether $\overline{K(a)}$ is of countable type, has a t-orthogonal base ($t \in (0,1]$) and, as a consequence, whether $\overline{K(a)}$ is reflexive (Corollary 3.3). An application of this theorem is given in Corollary 3.5.

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Note Unless explicitly stated otherwise, from now on we assume that K is algebraically closed and that L, a, $K(a)$, and $\overline{K(a)}$ are as described above.

The following lemmas will be used in the next section to prove Theorem 2.1.

Lemma 1.1 *Let $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ be elements of L such that $|\lambda_i - \mu_i| < |\mu_i|$ for each $i \in \{1, \dots, n\}$. Then*

$$\Big|\prod_{i=1}^{n} \lambda_i - \prod_{i=1}^{n} \mu_i\Big| < \Big|\prod_{i=1}^{n} \mu_i\Big|.$$

Proof The proof follows directly from the observation that $|\frac{\lambda_i}{\mu_i} - 1| < 1$ for each $i \in \{1, \dots, n\}$ and that $\{\lambda \in L : |\lambda - 1| < 1\}$ is a multiplicative group. ■

Lemma 1.2 *Let $a \perp K$, $|a| = 1$. Then for each polynomial*

$$P = \lambda_0 + \lambda_1 X + \cdots + \lambda_n X^n \in K[X]$$

and $\mu \in B_K$ we have $|P(\mu)| \leq \max_{0 \leq i \leq n} |\lambda_i| \leq |P(a)|$.

Proof Only the second inequality needs a proof. We may assume $\lambda_n \neq 0$. By algebraic closedness there are $\omega_1, \dots, \omega_n \in K$ such that $P = \lambda_n (X - \omega_1) \cdots (X - \omega_n)$ and by assumption we have $|a - \omega_i| \geq |a| = 1$ for all i, so that $|P(a)| \geq |\lambda_n|$. By the same token, we obtain $|\lambda_0 + \lambda_1 a + \cdots + \lambda_{n-1} a^{n-1}| \geq |\lambda_{n-1}|$, *i.e.*, $|\lambda_{n-1}| \leq |P(a) - \lambda_n a^n| \leq \max(|P(a)|, |\lambda_n|) = |P(a)|$, and we can proceed inductively. ■

2 Main Result

The main result of the paper is the following theorem, which provides an answer to the natural questions whether $\overline{K(a)}$ is of countable type and whether $\overline{K(a)}$ has a t-orthogonal base ($t \in (0, 1]$).

Theorem 2.1 *For the Banach space $\overline{K(a)}$ we have the following.*

(i) *If $\operatorname{dist}(a, K)$ is not attained, then $\overline{K(a)}$ is of countable type and has a t-orthogonal base for each $t \in (0, 1)$, but has no orthogonal base.*
(ii) *If $\operatorname{dist}(a, K)$ is attained, but not in $|K|$, then $\overline{K(a)}$ is of countable type and has an orthogonal base, but has no orthonormal base.*
(iii) *If $\operatorname{dist}(a, K)$ is attained and in $|K|$, then $\overline{K(a)}$ has an orthonormal base of cardinality $\sharp k$.*

Proof (i) First we show that $\overline{K(a)}$ is of countable type. For that we prove that the ring $K[a] := [1, a, a^2, \dots]$ is dense in $\overline{K(a)}$, which will be done in the next three steps.

(a) *For every $b \in K[a]$, $b \neq 0$ there is a $\mu \in K$ such that $|b - \mu| < |\mu|$.* To see this, let $b = \lambda_0 + \lambda_1 a + \cdots + \lambda_n a^n$; we may suppose $n \geq 1$, $\lambda_0, \lambda_1, \dots, \lambda_n \in K$, $\lambda_n \neq 0$. By algebraic closedness there exist $\omega_1, \dots, \omega_n \in K$ such that $b = \lambda_n (a - \omega_1) \cdots (a - \omega_n)$. Since $\operatorname{dist}(a, K) = \operatorname{dist}(a - \omega_i, K)$ is not attained, there are $\mu_1, \dots, \mu_n \in K$ such that $|a - \omega_i - \mu_i| < |\mu_i|$ for each $i \in \{1, \dots, n\}$. By Lemma 1.1 we have $|b - \mu| < |\mu|$, where $\mu := \lambda_n \mu_1 \cdots \mu_n$.

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(b) *For each $b \in K[a]$, $b \neq 0$ we have $b^{-1} \in \overline{K[a]}$.* In fact, by (a) there is a $\mu \in K$ such that $|b - \mu| < |\mu|$. Then $|\mu^{-1} b - 1| < 1$, so that

$$\mu b^{-1} = (1 - (1 - \mu^{-1} b))^{-1} = \sum_{n=0}^{\infty} (1 - \mu^{-1} b)^n \in \overline{K[a]}.$$

(c) *$K[a]$ is dense in $\overline{K(a)}$.* From (b) it follows that $K(a) \subset \overline{K[a]}$. This, together with the obvious inclusion $K[a] \subset \overline{K(a)}$, leads to $\overline{K[a]} = \overline{K(a)}$.

Secondly we show that $\overline{K(a)}$ has for each $t \in (0, 1)$ a t-orthogonal base. This follows from what we have just proved and [2, Theorems 3.15.(iii), 3.16.(ii)].

Finally we show that $\overline{K(a)}$ has no orthogonal base. Suppose it has; we derive a contradiction. This orthogonal base must be countable (because, as we proved before, $\overline{K(a)}$ is of countable type; so apply [2, Theorem 5.2]) and infinite (because, as K is algebraically closed, $K(a)$ and hence $\overline{K(a)}$ are infinite-dimensional vector spaces). Let us denote this orthogonal base by $\{e_1, e_2, \dots\}$. Then $\overline{K(a)}$ is isometrically isomorphic to $c_0(\mathbb{N}, s)$, with $s_n := \|e_n\|$ for all $n \in \mathbb{N}$. By [2, Lemma 4.35.(ii)], $[a]$ is orthocomplemented in $\overline{K(a)}$ and hence in $[1, a]$, which contradicts the hypothesis of (i).

(ii) Let $\lambda_0 \in K$ be such that $|a - \lambda_0| = \operatorname{dist}(a, K)$. Then $K(a - \lambda_0) = K(a)$ and $\operatorname{dist}(a - \lambda_0, K) = \operatorname{dist}(a, K)$, so we may replace a by $a - \lambda_0$; in other words, we may assume that $|a| \notin |K|$. It suffices to show that $\{a^n : n \in \mathbb{Z}\}$ is an orthogonal base of $\overline{K(a)}$.

First observe that $|a|^n \notin |K|$ for $n \in \mathbb{N}$. Indeed, if $|a|^n = |\lambda|$ for some $n \in \mathbb{N}$ and $\lambda \in K$, then by algebraic closedness there is a $\mu \in K$ with $\mu^n = \lambda$, so that $|a| = |\mu| \in |K|$, which is a contradiction. Next we prove orthogonality of $\{a^n : n \in \mathbb{Z}\}$. Let

$$x := \sum_{i=s}^{m} \lambda_i a^i,$$

where $\lambda_s, \dots, \lambda_m \in K$, not all 0. From what we have just proved it follows that $|\lambda_i a^i| \neq |\lambda_j a^j|$ for all $i, j \in \{s, \dots, m\}$ unless $i = j$ or $\lambda_i = \lambda_j = 0$. Then $|x| = \max_{s \le i \le m} |\lambda_i a^i|$, and orthogonality follows.

We proceed to show that $x^{-1} \in \overline{[a^n : n \in \mathbb{Z}]}$, where x is as above. There is a unique $j \in \{s, \dots, m\}$ with $|x| = |\lambda_j a^j|$. Then

$$(\lambda_j a^j)^{-1} x = 1 + (\lambda_j a^j)^{-1} \sum_{i \neq j} \lambda_i a^i = 1 + v,$$

where $v \in \overline{[a^n : n \in \mathbb{Z}]}$, $|v| < 1$. Thus

$$(\lambda_j a^j)\, x^{-1} = \sum_{n=0}^{\infty} (-v)^n \in \overline{[a^n : n \in \mathbb{Z}]},$$

implying $x^{-1} \in \overline{[a^n : n \in \mathbb{Z}]}$. Now continuity of the inverse map shows that $\overline{[a^n : n \in \mathbb{Z}]}$ is a field, hence must be equal to $\overline{K(a)}$. Then we obtain that $\{a^n : n \in \mathbb{Z}\}$ is an orthogonal base of $\overline{K(a)}$.

(iii) The proof here is somewhat more involved. Let $\lambda_0 \in K$ with $|a - \lambda_0| = \operatorname{dist}(a, K) \in |K|$. Then $K(a - \lambda_0) = K(a)$ and $\operatorname{dist}(a - \lambda_0, K) = \operatorname{dist}(a, K)$, so we may assume that $a \perp K$ and $|a| = 1$. Let $\sigma\colon k \to B_K$ be such that $\overline{\sigma(u)} = u$ for all $u \in k$. Let

$$S := \{a^s : s \in \mathbb{N} \cup \{0\}\} \cup \{(a - \mu)^{-m} : \mu \in \sigma(k),\ m \in \mathbb{N}\}.$$

Then S is a subset of $K(a)$ and, since k is infinite, we have $\sharp S = \sharp k$. We now establish (a)–(d) below, which will show that S is an orthonormal base of $\overline{K(a)}$.

(a) *S is an orthonormal system.*
(b) *$\overline{[S]}$ is a subring of $\overline{K(a)}$.*
(c) *For each $\beta \in K$, $(a - \beta)^{-1} \in \overline{[S]}$.*
(d) $K(a) \subset \overline{[S]}$.

Proof of (a) Clearly each member of S has length 1. Take a linear combination

$$\Phi := \sum_{r=0}^{s} \xi_r\, a^r + \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}\, (a - \mu_i)^{-j}$$

(where $s \in \mathbb{N} \cup \{0\}$, $m, n \in \mathbb{N}$, $\xi_r, \lambda_{ij} \in K$, $\mu_i \in \sigma(k)$). For orthonormality of S it suffices to show, assuming $|\Phi| < 1$, that all $|\xi_r|$ and $|\lambda_{ij}|$ are < 1. Via (downward) induction on n we only need to prove that all $|\xi_r|$ and all $|\lambda_{in}|$ are < 1 for $r \in \{0, \dots, s\}$, $i \in \{1, \dots, m\}$.

To obtain polynomials, we multiply Φ by $L(a) := (a - \mu_1)^n \cdots (a - \mu_m)^n$, which does not change the absolute value, as $|L(a)| = 1$. The assumption $|\Phi| < 1$ turns into

$$|V_1(a) + V_2(a)| < 1, \tag{2.1}$$

where $V_1, V_2 \in K[X]$. In fact, for $x \in L$,

$$V_1(x) := \Big(\sum_{r=0}^{s} \xi_r x^r\Big) L(x), \quad V_2(x) := \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} L_{ij}(x),$$

where $L_{ij}(x) := (x - \mu_i)^{n-j} \prod_{l \neq i} (x - \mu_l)^n$.

Let $r \in \{0, \dots, s\}$. If $\xi_r \neq 0$, the degree of V_1 is $\geq mn$, whereas V_2 has degree $< mn$. Thus, ξ_r is a coefficient of the polynomial $V_1 + V_2$, so that by Lemma 1.2 we have $|\xi_r| \leq |V_1(a) + V_2(a)| < 1$. So all $|\xi_r|$ are < 1 and $|V_1(a)| < 1$, hence (2.1) reduces to $|V_2(a)| < 1$. Choose $q \in \{1, \dots, m\}$. Then since

$$|L_{ij}(\mu_q)| = \begin{cases} 1 & \text{if } q = i,\ j = n, \\ 0 & \text{otherwise,} \end{cases}$$

we find by Lemma 1.2 that $1 > |V_2(a)| \geq |V_2(\mu_q)| = |\lambda_{qn}|$, so that $|\lambda_{1n}|, \dots, |\lambda_{mn}|$ are all less than 1.

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Proof of (b) It suffices to show that $S \,.\, S \subset [S]$. For $\mu \in \sigma(k)$ and $m \in \mathbb{N}$ the identity

$$a(a-\mu)^{-m} = (a-\mu)^{1-m} + \mu\,(a-\mu)^{-m}$$

shows that $aS \subset [S]$. Then $a^2 S = aaS \subset a[S] = [aS] \subset [S]$, and so on, proving that $a^r S \subset [S]$ for each $r \in \mathbb{N} \cup \{0\}$. It remains to be shown that $(a-\mu_1)^{-m_1}(a-\mu_2)^{-m_2} \in [S]$ for $m_1, m_2 \geq 1$, $\mu_1, \mu_2 \in \sigma(k)$. If $\mu_1 = \mu_2$, this is clear, so suppose $\mu_1 \neq \mu_2$. We use induction with respect to $n := m_1 + m_2$. If $n = 2$ (so $m_1 = m_2 = 1$), the formula

$$(a-\mu_1)^{-1}(a-\mu_2)^{-1} = \frac{1}{\mu_1 - \mu_2}\big((a-\mu_1)^{-1} - (a-\mu_2)^{-1}\big) \tag{2.2}$$

does the job. For the step $n-1 \to n$, observe that we have, using (2.2),

$$(a-\mu_1)^{-m_1}(a-\mu_2)^{-m_2} = \frac{1}{\mu_1-\mu_2}\big((a-\mu_1)^{-1} - (a-\mu_2)^{-1}\big)(a-\mu_1)^{-m_1+1}(a-\mu_2)^{-m_2+1},$$

which is a linear combination of the elements $(a-\mu_1)^{-m_1}(a-\mu_2)^{-m_2+1}$ and $(a-\mu_1)^{-m_1+1}(a-\mu_2)^{-m_2}$, and these are in $[S]$ by the induction hypothesis.

Proof of (c) If $|\beta| > 1$, we have $(a-\beta)^{-1} = -\beta^{-1}\sum_{n=0}^{\infty}(\beta^{-1}\,a)^n \in \overline{[S]}$, so let $|\beta| \leq 1$. Then there is a $\mu \in \sigma(k)$ with $|\beta - \mu| < 1$ and

$$\begin{aligned}(a-\beta)^{-1} - (a-\mu)^{-1} &= (\beta-\mu)\,(a-\mu)^{-2}\Big(1 - \frac{\beta-\mu}{a-\mu}\Big)^{-1} \\ &= (\beta-\mu)\,(a-\mu)^{-2}\sum_{n=0}^{\infty}(\beta-\mu)^n(a-\mu)^{-n} \in \overline{[S]}.\end{aligned}$$

Proof of (d) Every element of $K(a)$ can be written as $P(a)Q(a)^{-1}$ for some polynomials $P, Q \in K[X]$, $Q(a) \neq 0$. By algebraic closedness we can decompose Q into linear factors whose inverses are in $\overline{[S]}$ by (c). Then since $\overline{[S]}$ is a ring by (b), $P(a)Q(a)^{-1} \in \overline{[S]}$. ■

3 Some Consequences and Final Remarks

As an immediate consequence of Theorem 2.1 we derive the next result.

Corollary 3.1 (i) *For each $t \in (0,1)$ $\overline{K(a)}$ has a t-orthogonal base.*

(ii) *$\overline{K(a)}$ has an orthogonal base if and only if* $\operatorname{dist}(a, K)$ *is attained.*

(iii) *$\overline{K(a)}$ is not of countable type if and only if* $\operatorname{dist}(a, K)$ *is attained and in $|K|$ and k is uncountable.*

Remark 3.2 To see that cases (ii) and (iii) of Corollary 3.1 really occur, choose K such that k is uncountable. Let L be the completion of $(K(X), |\cdot|)$, and choose $a := X$.

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Another consequence of Theorem 2.1 concerns reflexivity of $\overline{K(a)}$. Recall that a set is *small* if it has a non-measurable cardinality [2, p. 31].

Corollary 3.3 *$\overline{K(a)}$ is reflexive except when either* (i) *K is spherically complete, or* (ii) *K is not spherically complete,* $\operatorname{dist}(a, K)$ *is attained and in $|K|$, and k is not small.*

Proof Suppose K is spherically complete (case (i)). Since K is algebraically closed, $\overline{K(a)}$ is an infinite-dimensional Banach space, so it is not reflexive [2, Theorem 4.16].

Now suppose that K is not spherically complete (case (ii)). If the remaining assumptions of (ii) hold, then by Theorem 2.1(iii), $\overline{K(a)}$ is isometrically isomorphic to $c_0(I)$ with $\sharp I = \sharp k$. As $\sharp I$ is not small, it follows from [2, Exercise 4.M] that $\overline{K(a)}$ is not reflexive.

On the other hand, if K is not spherically complete and some of the remaining assumptions of (ii) fail, then we have that either

- we are in case (i) or (ii) of Theorem 2.1 (so $\overline{K(a)}$ is of countable type, hence reflexive by [2, Corollary 4.18]), or
- we are in case (iii) of Theorem 2.1 with k small (so, $\overline{K(a)}$ is isometrically isomorphic to $c_0(I)$ with $\sharp I = \sharp k$ and, as $\sharp I$ is small, $\overline{K(a)}$ is reflexive, by [2, Theorem 4.21.(iii)]).

■

Remark 3.4 To see that case (ii) of Corollary 3.3 really occurs, choose K non-spherically complete and such that k is small. Then proceed as in Remark 3.2.

Also, as an application of Theorem 2.1 we obtain the following interesting fact.

Corollary 3.5 *Suppose that K is not spherically complete. Then on c_0 there exists an equivalent norm $\|\cdot\|$ with $\|c_0\| = |K|$ such that*

(i) *there is a multiplication on c_0 making $(c_0, \|\cdot\|)$ into a valued field;*
(ii) *no two non-zero vectors are $\|\cdot\|$-orthogonal.*

Proof Let $L := K^{\vee}$ be the spherical completion of K and let $a \in L \setminus K$. Then $\operatorname{dist}(a, K)$ is not attained, since L is an immediate valued field extension of K. So we are in case (i) of Theorem 2.1, showing that $\overline{K(a)}$ is of countable type (and infinite-dimensional, as K is algebraically closed), hence linearly homeomorphic to c_0 ([2, Theorem 3.16.(ii)]).

If $\lambda, \mu \in \overline{K(a)}$, $\lambda \perp \mu$ with respect to the valuation $|\cdot|$ on $\overline{K(a)}$ and $\lambda, \mu \neq 0$, then $\lambda\mu^{-1} \neq 0$, $\lambda\mu^{-1} \perp K$, which is not possible since, as $\overline{K(a)}$ is an immediate valued field extension of K, 0 is the only element of $\overline{K(a)}$ that is orthogonal to K ([2, pp. 57, 162]).

Finally, let $\|\cdot\|$ be the norm on c_0 inherited from $|\cdot|$ through the bijective linear homeomorphism $\overline{K(a)} \to c_0$. From the facts previously showed in this proof we obtain that $(c_0, \|\cdot\|)$ satisfies the required conditions. ■

Problem Now let us drop the condition of algebraic closedness of K. Again, let L be a valued field extension of K. Let a be in $L \setminus K$, a not algebraic over K. What conclusions about $\overline{K(a)}$ of Theorem 2.1 and its corollaries remain valid in this more general context?

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https://doi.org/10.4153/CMB-2011-133-2 Published online by Cambridge University Press