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INFINITE FAMILIES OF CONGRUENCES FOR OVERPARTITIONS WITH RESTRICTED ODD DIFFERENCES

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Abstract

Let $\bar{t}(n)$ be the number of overpartitions in which (i) the difference between successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. Ramanujan-type congruences for $\bar{t}(n)$ modulo small powers of 2 and 3 have been established. We present two infinite families of congruences modulo 5 and 27 for $\bar{t}(n)$, the first of which generalises a recent result of Chern and Hao ['Congruences for two restricted overpartitions', *Proc. Math. Sci.* **129** (2019), Article 31].

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1. Introduction

A *partition* of a positive integer n is a weakly decreasing sequence of positive integers whose sum is n. The summands are called the *parts* of the partition. Let $p(n)$ denote the number of partitions of n. For example, $p(4) = 5$ and the five partitions of 4 are

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1.$$

The generating function of $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

Ramanujan proved for every nonnegative integer n that

$$p(5n+4) \equiv 0 \pmod 5,$$

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$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Elementary proofs of these three congruences are given in [3]. Ramanujan's congruences have inspired work on similar congruences for other partition functions. We shall investigate congruences for overpartitions with restricted odd differences.

An *overpartition* of n is a partition of n where the final occurrence of parts of each size may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. For example, $\overline{p}(3) = 8$ with the relevant partitions being

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1, 1+1+\overline{1}.$$

Since the overlined parts form a partition into distinct parts and the nonoverlined parts form an ordinary partition,

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.$$

Arithmetic properties of $\overline{p}(n)$ are much studied (see, for example, [5, 8, 10, 11, 13]).

Recently, Bringmann *et al.* [4] considered a new type of overpartitions with restricted odd differences. Let $\overline{t}(n)$ denote the number of overpartitions of n in which (i) the difference between two successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined. For example, $\overline{t}(4) = 8$ because there are eight such overpartitions of 4, namely,

$$4,\ \overline{4},\ 3+\overline{1},\ \overline{3}+\overline{1},\ 2+2,\ 2+\overline{2},\ \overline{2}+1+\overline{1},\ 1+1+1+\overline{1}.$$

By means of certain q-difference equations, Bringmann *et al.* [4] derived the unexpected generating function

$$\sum_{n=0}^{\infty} \overline{t}(n)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}; \tag{1.1}$$

it can also be deduced by a simple combinatorial argument using an elementary analysis of Ferrers diagrams. Subsequently, arithmetic properties of $\overline{t}(n)$ have received attention and congruences modulo small powers of 2 and 3 have been found in [6, 9, 12]. For example, for $n \geq 0$,

$$\begin{aligned}
\overline{t}(6n+4) &\equiv 0 \pmod{2},\\
\overline{t}(6n+6) &\equiv 0 \pmod{2},\\
\overline{t}(24n+4) &\equiv 0 \pmod{4},\\
\overline{t}(32n+4) &\equiv 0 \pmod{4},\\
\overline{t}(96n+4) &\equiv 0 \pmod{8},\\
\overline{t}(96n+36) &\equiv 0 \pmod{8}
\end{aligned}$$

and

$$\bar{t}(4n+2) \equiv 0 \pmod 3,$$
$$\bar{t}(8n+5) \equiv 0 \pmod 9,$$
$$\bar{t}(9n+6) \equiv 0 \pmod 9.$$

From the theory of modular forms, Chern and Hao [6] obtained

$$\bar{t}(45n+30) \equiv 0 \pmod 5$$

and asked for an elementary proof. In this paper, we not only find such a proof but also give a generalisation to an infinite family of congruences.

Theorem 1.1. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\bar{t}(9^{\alpha}(45n+30)) \equiv 0 \pmod 5.$$

Moreover, we present an infinite family of congruences modulo 27 for $\bar{t}(n)$.

Theorem 1.2. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\bar{t}(9^{\alpha}(72n+69)) \equiv 0 \pmod{27}.$$

The rest of the paper is organised as follows. In Section 2, we introduce some preliminary results. In Section 3, we prove Theorems 1.1 and 1.2.

2. Preliminaries

Recall Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$
$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

These two functions can be expressed as infinite products

$$\varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty},$$
$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty},$$

which arise from the Jacobi triple product identity [3, page 11]. Ramanujan [2, page 49] established the following 3-dissections of $\varphi(q)$ and $\psi(q)$.

Lemma 2.1. *We have*

$$\varphi(q) = \varphi(q^9) + 2q(-q^3;q^{18})_\infty(-q^{15};q^{18})_\infty(q^{18};q^{18})_\infty, \tag{2.1}$$
$$\psi(q) = (-q^3;q^9)_\infty(-q^6;q^9)_\infty(q^9;q^9)_\infty + q\psi(q^9). \tag{2.2}$$

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We present two equations satisfied by

$$a(q)=\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Lemma 2.2. *We have*

$$a(q^2)\equiv \frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q;q)_\infty^3(q^6;q^6)_\infty^2}+2q\frac{(q^6;q^6)_\infty^6(q;q)_\infty}{(q^3;q^3)_\infty^3(q^2;q^2)_\infty^2} \pmod 5.$$

Proof. From [7, (22.11.10) and (22.11.6)],

$$a(q^2)=a(q)-6q\frac{\psi(q^3)^3}{\psi(q)},$$
$$a(q)=\frac{\psi(q)^3}{\psi(q^3)}+3q\frac{\psi(q^3)^3}{\psi(q)}.$$

Combining these two identities and using the product expression for $\psi(q)$ gives the desired result. □

Lemma 2.3 [14, Theorem 2.1]. *We have*

$$\frac{1}{(q;q)_\infty(q^2;q^2)_\infty}=\frac{a(q^6)(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^3}+\frac{qa(q^3)(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^4}+\frac{3q^2(q^9;q^9)_\infty^3(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}. \tag{2.3}$$

The following lemmas are useful for our later proofs.

Lemma 2.4. *We have*

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty}=\frac{(q^4;q^4)_\infty(q^6;q^6)_\infty(q^{16};q^{16})_\infty(q^{24};q^{24})_\infty^2}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty(q^{12};q^{12})_\infty(q^{48};q^{48})_\infty}+q\frac{(q^6;q^6)_\infty(q^8;q^8)_\infty^2(q^{48};q^{48})_\infty}{(q^2;q^2)_\infty^2(q^{16};q^{16})_\infty(q^{24};q^{24})_\infty}, \tag{2.4}$$

$$\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}=\frac{(q^4;q^4)_\infty^6(q^6;q^6)_\infty^3}{(q^2;q^2)_\infty^9(q^{12};q^{12})_\infty^2}+3q\frac{(q^4;q^4)_\infty^2(q^6;q^6)_\infty(q^{12};q^{12})_\infty^2}{(q^2;q^2)_\infty^7}. \tag{2.5}$$

Proof. See [7, (30.10.3)] for a proof of (2.4). To obtain (2.5), first replace q by $-q$ in [7, (22.6.2)] and then use $(-q;-q)_\infty=(q^2;q^2)_\infty^3/((q;q)_\infty(q^4;q^4)_\infty)$. □

Lemma 2.5. *Define $u(n)$ by*

$$\sum_{n=0}^{\infty}u(n)q^n=\varphi(q)^2\varphi(q^3)^2.$$

Then

$$\sum_{n=0}^{\infty}u(5n)q^n\equiv\varphi(q)^2\varphi(q^3)^2 \pmod 5.$$

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PROOF. Given an integer n, we can factor it as $n = 2^{\alpha}3^{\beta}N$, where $\alpha \geq 0, \beta \geq 0$ and $(N, 6) = 1$. From [1], for $n \geq 1$,

$$u(n) = \begin{cases} 4\sigma(N) & \text{if } n \equiv 1 \pmod 2, \\ 4(2^{\alpha+1} - 3)\sigma(N) & \text{otherwise,} \end{cases}$$

where $\sigma(N)$ denotes the sum of all of the positive divisors of N. Thus,

$$u(5n) = \begin{cases} 4\sigma(5N) & \text{if } n \equiv 1 \pmod 2, \\ 4(2^{\alpha+1} - 3)\sigma(5N) & \text{otherwise.} \end{cases}$$

Let $N = 5^{\gamma}N_1$, where $(N_1, 5) = 1$. Then

$$\sigma(5N) = \sigma(5^{\gamma+1})\sigma(N_1) = (\sigma(5^{\gamma}) + 5^{\gamma+1})\sigma(N_1) \equiv \sigma(N) \pmod 5,$$

which implies that

$$u(5n) \equiv u(n) \pmod 5.$$

This completes the proof. □

LEMMA 2.6 [8, Theorem 2]. *We have*

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{(q^6;q^6)_\infty^4(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^8(q^{18};q^{18})_\infty^3} + 2q\frac{(q^6;q^6)_\infty^3(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^7} + 4q^2\frac{(q^6;q^6)_\infty^2(q^{18};q^{18})_\infty^3}{(q^3;q^3)_\infty^6}.$$

In addition, we frequently use the following fact without explicitly mentioning it: for any prime p,

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod p.$$

3. Proofs of Theorems 1.1 and 1.2

PROOF OF THEOREM 1.1. Substituting (2.3) into (1.1), extracting those terms on both sides where the power of q is a multiple of 3 and replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} \bar{t}(3n)q^n = (q;q)_\infty \cdot \frac{a(q^2)(q^3;q^3)_\infty^3}{(q;q)_\infty^4(q^2;q^2)_\infty^3}.$$

Applying Lemma 2.2,

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(3n)q^n &\equiv \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty^3(q^2;q^2)_\infty^3}\left(\frac{(q^2;q^2)_\infty^6(q^3;q^3)_\infty}{(q;q)_\infty^3(q^6;q^6)_\infty^2} + 2q\frac{(q^6;q^6)_\infty^6(q;q)_\infty}{(q^3;q^3)_\infty^3(q^2;q^2)_\infty^2}\right)\\
&\equiv \frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^4}{(q;q)_\infty^6(q^6;q^6)_\infty^2} + 2q\frac{(q^6;q^6)_\infty^6}{(q;q)_\infty^2(q^2;q^2)_\infty^5}\\
&\equiv \frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^2}\cdot\frac{(q^3;q^3)_\infty^4}{(q^6;q^6)_\infty^2}\cdot\frac{(q^{10};q^{10})_\infty}{(q^5;q^5)_\infty^2} + 2q\frac{(q;q)_\infty^3(q^6;q^6)_\infty(q^{30};q^{30})_\infty}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty}\\
&\equiv \frac{\varphi(-q)^2\varphi(-q^3)^2}{\varphi(-q^5)} + 2q\frac{(q;q)_\infty^3(q^6;q^6)_\infty(q^{30};q^{30})_\infty}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty} \pmod 5. \qquad (3.1)\end{aligned}$$

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Define $v(n)$ by

$$\sum_{n=0}^{\infty} v(n)q^n = 2q\frac{(q;q)_\infty^3(q^6;q^6)_\infty(q^{30};q^{30})_\infty}{(q^5;q^5)_\infty(q^{10};q^{10})_\infty}.$$

We claim that $v(5n)$ is divisible by 5. By Euler's pentagonal number theorem [3, page 12] and Jacobi's identity [3, page 14],

$$q(q;q)_\infty^3(q^6;q^6)_\infty = q\sum_{r=0}^{\infty}(-1)^r(2r+1)q^{r(r+1)/2}\sum_{s=-\infty}^{\infty}(-1)^s q^{3s(3s+1)}.$$

One can readily verify that $r(r+1)/2 \equiv 0,1,3$ modulo 5 and $3s(3s+1) \equiv 0,1,2$ modulo 5. We therefore see that $1 + r(r+1)/2 + 3s(3s+1) \equiv 0 \pmod 5$ if and only if $r(r+1)/2 \equiv 3 \pmod 5$ and $3s(3s+1) \equiv 1 \pmod 5$. In particular, $r(r+1)/2 \equiv 3 \pmod 5$ if and only if $r \equiv 2 \pmod 5$, which implies that $2r+1$ is a multiple of 5. It follows that $v(5n) \equiv 0 \pmod 5$.

Selecting the terms of the form q^{5n} in (3.1), replacing q^5 by q, applying the above claim and using the definition of $u(n)$,

$$\sum_{n=0}^{\infty}\bar{t}(15n)q^n \equiv \frac{1}{\varphi(-q)}\sum_{n=0}^{\infty}u(5n)(-q)^n \pmod 5.$$

By Lemma 2.5,

$$\sum_{n=0}^{\infty}\bar{t}(15n)q^n \equiv \varphi(-q)\varphi(-q^3)^2 \pmod 5. \tag{3.2}$$

Collecting the terms of the form q^{3n} on both sides of (3.2), applying (2.1) and replacing q^3 by q,

$$\sum_{n=0}^{\infty}\bar{t}(45n)q^n \equiv \varphi(-q)^2\varphi(-q^3) \pmod 5.$$

A similar argument shows that

$$\sum_{n=0}^{\infty}\bar{t}(135n)q^n \equiv \varphi(-q)\varphi(-q^3)^2 \pmod 5.$$

Proceeding by induction on α, it is easy to see that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty}\bar{t}(9^\alpha\cdot 15n)q^n \equiv \varphi(-q)\varphi(-q^3)^2 \pmod 5. \tag{3.3}$$

It follows from (2.1) that there are no powers of q on the right-hand side of (3.3) which are congruent to 2 modulo 3. Therefore,

$$\bar{t}(9^\alpha\cdot 15(3n+2)) \equiv 0 \pmod 5,$$

which completes the proof. □

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Proof of Theorem 1.2. Combining (1.1) and (2.4),

$$\sum_{n=0}^{\infty} \bar{t}(2n+1)q^n = \frac{(q^3;q^3)_\infty}{(q;q)_\infty^3} \cdot \frac{(q^4;q^4)_\infty^2 (q^{24};q^{24})_\infty}{(q^8;q^8)_\infty (q^{12};q^{12})_\infty}.$$

From (2.5),

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(4n+1)q^n &= \frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^3}{(q;q)_\infty^9 (q^6;q^6)_\infty^2} \cdot \frac{(q^2;q^2)_\infty^2 (q^{12};q^{12})_\infty}{(q^4;q^4)_\infty (q^6;q^6)_\infty} \\ &= \left(\frac{(q^3;q^3)_\infty}{(q;q)_\infty^3}\right)^3 \cdot \frac{(q^2;q^2)_\infty^8 (q^{12};q^{12})_\infty}{(q^4;q^4)_\infty (q^6;q^6)_\infty^3}.\end{aligned}$$

Applying (2.5) again,

$$\begin{aligned}\sum_{n=0}^{\infty} \bar{t}(8n+5)q^n &\equiv 9\frac{(q^2;q^2)_\infty^{12} (q^3;q^3)_\infty^6}{(q;q)_\infty^{18} (q^6;q^6)_\infty^4} \frac{(q^2;q^2)_\infty^2 (q^3;q^3)_\infty (q^6;q^6)_\infty^2}{(q;q)_\infty^7} \frac{(q;q)_\infty^8 (q^6;q^6)_\infty}{(q^2;q^2)_\infty (q^3;q^3)_\infty^3} \\ &\equiv 9\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} \cdot \frac{(q^6;q^6)_\infty^3}{(q^3;q^3)_\infty} \pmod{27}.\end{aligned}$$

It follows from Lemma 2.6 that

$$\sum_{n=0}^{\infty} \bar{t}(24n+21)q^n \equiv 9\frac{(q^2;q^2)_\infty^2 (q^6;q^6)_\infty^3}{(q;q)_\infty^6} \cdot \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty} \equiv 9\psi(q)\psi(q^3)^2 \pmod{27}.$$

By (2.2),

$$\sum_{n=0}^{\infty} \bar{t}(72n+45)q^n \equiv 9\psi(q)^2\psi(q^3) \pmod{27}$$

and, applying (2.2) again,

$$\sum_{n=0}^{\infty} \bar{t}(216n+189)q^n \equiv 9\psi(q)\psi(q^3)^2 \pmod{27}.$$

By induction on α, we derive the conclusion that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \bar{t}(9^\alpha(24n+21))q^n \equiv 9\psi(q)\psi(q^3)^2 \pmod{27}. \tag{3.4}$$

Combining (3.4) and (2.2),

$$\bar{t}(9^\alpha(24(3n+2)+21)) \equiv 0 \pmod{27}.$$

This completes the proof. □

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