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THE TOP LEFT DERIVED FUNCTORS OF THE GENERALISED I-ADIC COMPLETION

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Abstract

We study the top left derived functors of the generalised I-adic completion and obtain equivalent properties concerning the vanishing or nonvanishing of the modules $L_i\Lambda_I(M, N)$. We also obtain some results for the sets $\mathrm{Coass}(L_i\Lambda_I(M; N))$ and $\mathrm{Cosupp}_R(H_i^I(M; N))$.

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1. Introduction

Let R be a noetherian commutative ring and I an ideal of R. In [10] the generalised I-adic completion $\Lambda_I(M, N)$ of the R-modules M, N is defined by

$$\Lambda_I(M, N) = \varprojlim_t (M/I^t M \otimes_R N).$$

When $M = R$, we have $\Lambda_I(R, N) \cong \Lambda_I(N)$, the I-adic completion of N. For each R-module M, there is a covariant functor $\Lambda_I(M, -)$ from the category R-modules to itself. Let $L_i\Lambda_I(M, -)$ be the ith left derived functor of $\Lambda_I(M, -)$. The ith generalised local homology module $H_i^I(M, N)$ of M, N with respect to I is defined by (see [12])

$$H_i^I(M, N) = \varprojlim_t \mathrm{Tor}_i^R(M/I^t M, N).$$

This definition of generalised local homology modules is in some sense dual to the definition of generalised local cohomology modules of Herzog [5] and in fact a generalisation of the usual local homology $H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M)$. In [10] we also studied some basic properties of the left derived functor $L_i\Lambda_I(M, -)$ of $\Lambda_I(M, -)$

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and showed that if M is a finitely generated R-module and N a linearly compact R-module, then $L_i\Lambda_I(M,N) \cong H_i^I(M,N)$ for all $i \geq 0$. However, the nonvanishing of the modules $L_i\Lambda_I(M,N)$ is a rather difficult problem. In Section 2, Theorem 2.1 gives us equivalent statements for vanishing and nonvanishing of the modules $L_i\Lambda_I(M,N)$. In Theorem 2.5 we study the set of co-associated primes of the modules $L_i\Lambda_I(M,N)$ and show that if M is a finitely generated R-module and N an R-module with $l_+^I(N) = d$ and $r = \mathrm{pd}(M) < \infty$, then there is submodule X of $\mathrm{Tor}_r^R(M; L_{d-1}\Lambda_I(N))$ such that

$$\mathrm{Coass}(L_{d+r-1}\Lambda_I(M;N)) \subseteq \mathrm{Coass}(\mathrm{Tor}_{r-1}^R(M; L_d\Lambda_I(N))) \cup \mathrm{Coass}(X).$$

It should be mentioned that when M is a finitely generated R-module, the left derived functors $L_i\Lambda_I(M,-)$ and the generalised local homology functors $H_i^I(M,-)$ are coincident on the category of linearly compact R-modules. In the final section we study the co-localisation of generalised local homology modules $H_i^I(M,M)$ when N is a semi-discrete linearly compact R-module and prove that if $r = \mathrm{pd}(M) < \infty$ and $\mathrm{Ndim}\, N = d$, then $\mathrm{Cosupp}_R(H_{d+r}^I(M;N)) \subseteq \{\mathfrak{m}\}$ (Theorem 3.3). Note that $\mathrm{Ndim}\, M$ is the noetherian dimension defined by Roberts [14] (see also [6]). It should be mentioned that the class of linearly compact modules is large, containing important classes of modules. Even its subclass of semi-discrete linearly compact modules contains artinian modules, as well as finitely generated modules over a complete ring. Further information on linearly compact modules can be found in [7] or [2].

2. The top left derived functors of the generalised I-adic completion

For two R-modules M and N, we put

$$\mathrm{tor}_+(M,N) = \sup\{i \mid \mathrm{Tor}_i^R(M,N) \neq 0\},$$
$$l_+^I(M,N) = \sup\{i \mid L_i\Lambda_I(M,N) \neq 0\}$$

and

$$l_+^I(N) = \sup\{i \mid L_i\Lambda_I(N) \neq 0\}.$$

Theorem 2.1. *Let M be a finitely generated R-module and N an R-module such that $r = \mathrm{pd}(M) < \infty$ and $d = l_+^I(N) < \infty$. Then the following statements are equivalent:*

(i) $r = \mathrm{tor}_+(M, L_d\Lambda_I(N))$;
(ii) $L_{r+d}\Lambda_I(M,N) \neq 0$;
(iii) $l_+^I(M,N) = r + d$.

To prove Theorem 2.1 we need the following lemmas.

Lemma 2.2 [11, Lemma 2.5]. *Let M be a finitely generated R-module and F a free R-module. Then*

$$M \otimes_R \Lambda_I(F) \cong \Lambda_I(M,F).$$

Lemma 2.3. *Let M be a finitely generated R-module and N an R-module. If $\mathrm{Tor}_p^R(M; L_q\Lambda_I(N)) = 0$ for all $p > r$ or $q > d$, then*

$$\mathrm{Tor}_r^R(M; L_d\Lambda_I(N)) \cong L_{r+d}\Lambda_I(M,N).$$

PROOF. Let us consider functors $F = M\otimes_R$- and $G = \Lambda_I$. The functor F is obviously right exact. On the other hand, it follows from [1, Theorem 1.4.7] that a projective module P implies $\Lambda_I(P)$ is flat and then is F-acyclic. Hence, combining [15, Theorem 11.39] with Lemma 2.2 yields a Grothendieck spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^R_p(M, L_q\Lambda_I(N)) \underset{p}{\to} L_{p+q}\Lambda_I(M, N).$$

Thus there is a filtration Φ of $L_{p+q}\Lambda_I(M, N)$ with

$$0 = \Phi^{-1}H_{r+d} \subseteq \cdots \subseteq \Phi^{r+d}H_{r+d} = L_{r+d}\Lambda_I(M, N)$$

and

$$E^\infty_{p,r+d-p} \cong \Phi^p H_{r+d}/\Phi^{p-1}H_{r+d}, \quad 0 \le p \le r + d.$$

As $\operatorname{Tor}^R_p(M; L_q\Lambda_I(N)) = 0$ for all $p > r$ or $q > d$, $E^2_{p,r+d-p} = 0$ for all $p \neq r$.

We have

$$\Phi^{r-1}H_{r+d} = \Phi^{r-2}H_{r+d} = \cdots = \Phi^{-1}H_{r+d} = 0$$

and

$$\Phi^r H_{r+d} = \Phi^{r+1}H_{r+d} = \cdots = \Phi^{r+d}H_{r+d} = L_{r+d}\Lambda_I(M, N).$$

It follows that $\Phi^r H_{r+d} \cong E^\infty_{r,d}$, which means that $L_{r+d}\Lambda_I(M, N) \cong E^\infty_{r,d}$. To finish the proof we consider homomorphisms of the spectral sequence

$$E^k_{r+k,d-k+1} \longrightarrow E^k_{r,d} \longrightarrow E^k_{r-k,d+k-1}.$$

The hypothesis gives $E^k_{r+k,d-k+1} = E^k_{r-k,d+k-1} = 0$ for all $k \ge 2$. Therefore

$$\operatorname{Tor}^R_r(M, L_d\Lambda_I(N)) = E^2_{r,d} = E^3_{r,d} = \cdots = E^\infty_{r,d} \cong L_{r+d}\Lambda_I(M, N).$$

The proof is complete. □

COROLLARY 2.4. *Let M be a finitely generated R-module and N an R-module such that $r = \operatorname{pd}(M) < \infty$ and $d = l^I_+(N) < \infty$. Then*

$$\operatorname{Tor}^R_r(M; L_d\Lambda_I(N)) \cong L_{d+r}\Lambda_I(M; N).$$

We are now in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. (i) ⇒ (ii) By Corollary 2.4, we have the isomorphism

$$\operatorname{Tor}^R_r(M; L_d\Lambda_I(N)) \cong L_{r+d}\Lambda_I(M; N).$$

As $r = \operatorname{tor}_+(M, L_d\Lambda_I(N)$, we get $L_{r+d}\Lambda_I(M, N) \neq 0$.

(ii) ⇒ (iii) Assume that $L_{r+d}\Lambda_I(M, N) \neq 0$. For all $j > d + r$, we have

$$\operatorname{Tor}^R_r(M; L_{j-r}\Lambda_I(N)) \cong L_j\Lambda_I(M; N)$$

by Lemma 2.3. It follows that $L_j\Lambda_I(M; N) = 0$ and then that $l^I_+(M, N) = r + d$.

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(iii) ⇒ (i) We have $\operatorname{Tor}_r^R(M; L_d\Lambda_I(N)) \cong L_{d+r}\Lambda_I(M; N) \neq 0$. For all $i > r$,

$$\operatorname{Tor}_i^R(M; L_d\Lambda_I(N)) \cong L_{i+d}\Lambda_I(M; N) = 0,$$

as $l_+^I(M, N) = r + d$. Therefore $r = \operatorname{tor}_+(M, L_d\Lambda_I(N))$. □

A prime ideal $\mathfrak{p}$ is said to be *co-associated* to a nonzero R-module M if there is an artinian homomorphic image T of M with $\mathfrak{p} = \operatorname{Ann}_R T$. The set of co-associated primes of M is denoted by $\operatorname{Coass}_R(M)$. It should be noted that if M is a semi-discrete linearly compact R-module, then the set $\operatorname{Coass}_R(M)$ is finite [18, Property 1(L4)].

If $0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$ is an exact sequence of R-modules, then $\operatorname{Coass}_R(K) \subseteq \operatorname{Coass}_R(M) \subseteq \operatorname{Coass}_R(N) \cup \operatorname{Coass}_R(K)$ [17, Theorem 1.10].

THEOREM 2.5. *Let M be a finitely generated R-module and N an R-module. If $l_+^I(N) = d$ and $r = \operatorname{pd}(M) < \infty$, then there is a submodule X of $\operatorname{Tor}_r^R(M; L_{d-1}\Lambda_I(N))$ such that*

$$\operatorname{Coass}(L_{d+r-1}\Lambda_I(M; N)) \subseteq \operatorname{Coass}(\operatorname{Tor}_{r-1}^R(M; L_d\Lambda_I(N))) \cup \operatorname{Coass}(X).$$

PROOF. We have the Grothendieck spectral sequence

$$E_{p,q}^2 = \operatorname{Tor}_p^R(M; L_q\Lambda_I(N)) \underset{p}{\to} L_{p+q}\Lambda_I(M; N).$$

Then there is a filtration Φ of $L_{p+q}\Lambda_I(M; N)$ with

$$0 = \Phi^{-1}H_{d+r-1} \subseteq \cdots \subseteq \Phi^{d+r-1}H_{d+r-1} = L_{d+r-1}\Lambda_I(M; N)$$

and

$$E_{p,d+r-1-p}^\infty = \Phi^p H_{d+r-1}/\Phi^{p-1}H_{d+r-1}, \quad 0 \le p \le d + r - 1.$$

Note that $L_q\Lambda_I(N) = 0$ for all $q > d$, so $E_{p,d+r-1-p}^2 = 0$ for all $p \neq r, r-1$. Thus

$$\Phi^r H_{d+r-1} = \Phi^{r+1}H_{d+r-1} = \cdots = \Phi^{d+r-1}H_{d+r-1} = L_{d+r-1}\Lambda_I(M; N)$$

and

$$\Phi^{r-2}H_{d+r-1} = \Phi^{r-3}H_{d+r-1} = \cdots = \Phi^{-1}H_{d+r-1} = 0.$$

It follows that

$$\Phi^{r-1}H_{d+r-1} \cong E_{r-1,d}^\infty \quad \text{and} \quad E_{r,d-1}^\infty \cong L_{d+r-1}\Lambda_I(M; N)/\Phi^{r-1}H_{d+r-1}.$$

We now consider homomorphisms of the spectral sequence

$$E_{r-1+k,d-k+1}^k \longrightarrow E_{r-1,d}^k \longrightarrow E_{r-1-k,d+k-1}^k,$$
$$E_{r+k,d-k}^k \longrightarrow E_{r,d-1}^k \longrightarrow E_{r-k,d+k-2}^k.$$

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As $E^k_{r-1+k,d-k+1} = E^k_{r-1-k,d+k-1} = E^k_{r+k,d-k} = 0$ for all $k \geq 2$ and $E^k_{r-k,d+k-2} = 0$ for all $k \geq 3$,

$$\mathrm{Tor}^R_{r-1}(M; L_d\Lambda_I(N)) = E^2_{r-1,d} = E^3_{r-1,d} = \cdots = E^\infty_{r-1,d}$$

and $E^3_{r,d-1} = E^4_{r,d-1} = \cdots = E^\infty_{r,d-1}$, a submodule of $E^2_{r,d-1}$. Setting $X = E^\infty_{r,d-1}$, we have a short exact sequence

$$0 \longrightarrow \mathrm{Tor}^R_{r-1}(M; L_d\Lambda_I(N)) \longrightarrow L_{d+r-1}\Lambda_I(M; N) \longrightarrow X \longrightarrow 0$$

which finishes the proof. □

3. Co-support of local homology modules

In this section we study the the generalised local homology functors $H^I_i(M, -)$ when M is a finitely generated R-module. Note that the left derived functors $L_i\Lambda_I(M, -)$ and the generalised local homology functors $H^I_i(M, -)$ are coincident on the category of linearly compact R-modules.

LEMMA 3.1 [2, Proposition 3.5] and [10, Theorem 3.6]. *Let M be a finitely generated R-module and N a linearly compact R-module. Then:*

(i) $L_i\Lambda_I(N) \cong H^I_i(N)$ *for all* $i \geq 0$;
(ii) $L_i\Lambda_I(M, N) \cong H^I_i(M, N)$ *for all* $i \geq 0$.

Let S be a multiplicative subset of R. Following [9], the co-localisation of an R-module M with respect to S is the module ${}_SM = \mathrm{Hom}(R_S, M)$. If M is a linearly compact R-module, then ${}_SM$ is also a linearly compact R-module by [2, Lemma 2.5]. If M is an artinian R-module, then ${}_SM$ is not necessarily artinian (see [9, Section 4]) but is a linearly compact R-module. Let $\mathfrak{p}$ be a prime of R and $S = R - \{\mathfrak{p}\}$; then instead of ${}_SM$ we write ${}_\mathfrak{p}M$.

For an R-module M, Melkersson and Schenzel [9] defined the *co-support* of M to be the set

$$\mathrm{Cos}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid {}_\mathfrak{p}M \neq 0\}.$$

In [17, Definition 2.1] Yassemi defined the co-support $\mathrm{Cosupp}_R(M)$ of an R-module M to be the set of primes $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\mathrm{Ann}(L) \subseteq \mathfrak{p}$. Note that a module is cocyclic if it is a submodule of $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m} \in R$. We have $\mathrm{Cos}_R(M) \subseteq \mathrm{Cosupp}_R(M)$, but the equation is in general not true (see [17, Theorem 2.6]). If M is a linearly compact R-module, we proved that $\mathrm{Coass}_R(M) \subseteq \mathrm{Cos}_R(M) = \mathrm{Cosupp}_R(M)$ and every minimal element of $\mathrm{Cosupp}_R(M)$ belongs to $\mathrm{Coass}_R(M)$ [13, Theorem 3.8] and [4, Theorem 4.2].

The following lemma is used to prove Theorem 3.3.

LEMMA 3.2. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. If N is a linearly compact R-module, then for all $i, j \geq 0$:*

(i) $\mathrm{Tor}^R_i(M; N) \cong \mathrm{Tor}^{\hat{R}}_i(\hat{M}; N)$ *and, especially,* $\mathrm{Tor}^{\hat{R}}_i(\hat{M}; H^I_j(N)) \cong \mathrm{Tor}^R_i(M; H^I_j(N))$;
(ii) $H^{I\hat{R}}_i(\hat{M}; N) \cong H^I_i(M; N)$.

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Proof. (i) It follows from [2, Lemma 7.1] that N has a natural linearly compact module structure over $\widehat{R}$. As $\widehat{R}$ is a flat R-module, we have, by [15, Theorem 11.53],

$$\operatorname{Tor}_i^R(M; N) \cong \operatorname{Tor}_i^R(M; \widehat{R} \otimes_{\hat{R}} N)$$
$$\cong \operatorname{Tor}_i^{\hat{R}}(M \otimes_R \widehat{R}; N) \cong \operatorname{Tor}_i^{\hat{R}}(\hat{M}; N).$$

The second statement is an immediate consequence, since $H_j^I(N)$ is linearly compact by [2, Proposition 3.3].

(ii) For all $i \geq 0$, $t > 0$, by (i),

$$\operatorname{Tor}_i^R(M/I^t M; N) \cong \operatorname{Tor}_i^{\hat{R}}(\hat{M}/(I\hat{R})^t \hat{M}; N).$$

Passing to inverse limits, we have the isomorphism as required. □

We now recall the concept of *noetherian dimension* of an R-module M, denoted by $\operatorname{Ndim} M$. Note that the notion of noetherian dimension was introduced first by Roberts [14] under the term 'Krull dimension'. Kirby [6] later changed Roberts's terminology and referred to *noetherian dimension* to avoid confusion with the well-known Krull dimension of finitely generated modules. Let M be an R-module. When $M = 0$ we put $\operatorname{Ndim} M = -1$. Then by induction, for any ordinal α, we put $\operatorname{Ndim} M = \alpha$ when (i) $\operatorname{Ndim} M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M, there exists a positive integer m_0 such that $\operatorname{Ndim}(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$. Thus M is nonzero and finitely generated if and only if $\operatorname{Ndim} M = 0$. If $0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0$ is a short exact sequence of R-modules, then $\operatorname{Ndim} M = \max\{\operatorname{Ndim} M'', \operatorname{Ndim} M'\}$. For each subset B of R, let $V(B)$ denote the set of all primes of R which contain B.

Theorem 3.3. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and N a semi-discrete linearly compact R-module.*

(i) *If* $\dim R = d$, *then*

 (a) $\operatorname{Cosupp}_R(H_d^I(N)) \subseteq \{\mathfrak{m}\}$*;*

 (b) $\operatorname{Cosupp}_R(H_{d-1}^I(N))$ *is finite.*

(ii) *If M is a finitely generated R-module with $r = \operatorname{pd}(M) < \infty$, and* $\operatorname{Ndim} N = d$, *then* $\operatorname{Cosupp}_R(H_{d+r}^I(M; N)) \subseteq \{\mathfrak{m}\}$.

Proof. (i) (a) Let us first give a proof in the special case where N is artinian. From [17, Proposition 2.3], $\operatorname{Cosupp}_R(N) = V(\operatorname{Ann}_R(N))$. It should be noted that, by [16, Lemma 1.11], N has a natural artinian module structure over $\widehat{R}$ and the going-down theorem holds for the canonical $R \longrightarrow \widehat{R}$. Therefore we may assume that $(R, \mathfrak{m})$ is complete by Lemma 3.2. We mention that $d = \dim R \geq \operatorname{Ndim} N$. If $d > \operatorname{Ndim} N$, then $H_d^I(N) = 0$ because of [2, Theorem 4.8]. We need only give a proof when $d = \operatorname{Ndim} N$. Note that $H_d^I(N)$ is finitely generated by [2, Theorem 5.3]. From [17, Theorem 2.10] we get $\operatorname{Cosupp}(H_d^I(N)) \subseteq \{\mathfrak{m}\}$.

We now turn to the case where M is semi-discrete linearly compact. By [2, Corollary 4.5], there is an isomorphism $H_i^I(N) \cong H_i^I(\Gamma_{\mathfrak{m}}(N))$ for all $i \geq 1$ and $\Gamma_{\mathfrak{m}}(N)$

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is artinian. So the lemma is true for $d \geq 1$. When $d = 0$, there is, from [18, Theorem], a short exact sequence $0 \longrightarrow B \longrightarrow N \longrightarrow A \longrightarrow 0$ where A is artinian and B is finitely generated. It induces an exact sequence $H_0^I(B) \longrightarrow H_0^I(N) \longrightarrow H_0^I(A) \longrightarrow 0$. According to the above proof, we have $\mathrm{Cosupp}(H_0^I(A)) \subseteq \{\mathfrak{m}\}$. On the other hand, combining [2, Corollary 3.11] with [17, Theorem 2.10] gives $\mathrm{Cosupp}(H_0^I(B)) = \mathrm{Cosupp}(B) = \{\mathfrak{m}\}$. This finishes the proof of (a).

(b) We first deal with the special case where N is artinian. By an analysis similar to that in the proof of (i), we may assume that $(R, \mathfrak{m})$ is complete. Let $D(N)$ denote the Matlis dual of N. We have $H_{d-1}^I(N) \cong D(H_I^{d-1}(D(N)))$ by [3, Proposition 3.3]. Applying [17, Corollary 2.9] yields

$$\begin{aligned} \mathrm{Cosupp}_R(H_{d-1}^I(N)) &= \mathrm{Cosupp}_R(D(H_I^{d-1}(D(N)))) \\ &= \mathrm{Supp}_R(H_I^{d-1}(D(N))). \end{aligned}$$

The last set is finite by [8, Theorem 2.4] and so $\mathrm{Cosupp}_R(H_{d-1}^I(N))$ is finite.

We can now proceed analogously to the proof of (a) for the case where N is semi-discrete linearly compact and the proof of (b) is complete.

(ii) From [2, Theorem 4.8], $H_i^I(N) = 0$ for all $i > d$. Thus, combining Lemmas 2.3 and 3.1 yields $H_{d+r}^I(M; N) \cong \mathrm{Tor}_r^R(M; H_d^I(N))$. As M is finitely generated, there is a free resolution of M with finitely many free modules. Then $\mathrm{Tor}_r^R(M; H_d^I(N))$ is isomorphic to a subquotient of a finite direct sum of copies of $H_d^I(N)$. Therefore $\mathrm{Cosupp}_R(\mathrm{Tor}_r^R(M; H_d^I(N))) \subseteq \mathrm{Cosupp}_R(H_d^I(N))$ and the conclusion follows from (i). □

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