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V-MODULES WITH KRULL DIMENSION

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Boyle and Goodearl proved that if R is a left V-ring then R has left Krull dimension if and only if R is left Neotherian. In this paper we extend this result to arbitrary V-modules.

Introduction and definitions. All rings considered are associative, have an identity and all modules are unitary left R-modules. We write $J(M)$, $Z(M)$ and $\mathrm{Soc}(M)$ for the Jacobson radical, the singular submodule and the socle of M, respectively. Let M and U be R-modules. Following Azumaya, we say that U is M*-injective* if for each submodule K of M every R-homomorphism from K into U can be extended to an R-homomorphism from M into U. Following Tominaga [**7**] and Hirano [**5**] a module M is called a V*-module* if every proper submodule of M is an intersection of maximal submodules (equivalently if every simple module is M-injective). Such a module M has also been called "co-semisimple" by Fuller in [**3**]. It was shown in [**3**] that the class of V-modules is closed under submodules, homomorphic images and direct sums. The reader is assumed to be familiar with the notion of Krull dimension as in [**4**]. We will make frequent use of the fact that every module with Krull dimension is finite dimensional [**4**, Proposition 1.4]. If $0 \to N \to M \to M/N \to 0$ is an exact sequence of modules then M has krull dimension if and only if both N and M/N have Krull dimension [**4**, Lemma 1.1(i)]. Finally a module M is *cofinitely generated* if M has a finitely generated essential socle.

THEOREM 1. *Let M be a V-module. Then M has Krull dimension if and only if M is Noetherian.*

PROOF: By [**4**, Proposition 1.3] every Noetherian module has Krull dimension. Before we begin to show the converse, we need the following lemma which is motivated by the work of Kurshan in [**6**]. ∎

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Now, since K is a proper essential submodule of M and a maximal submodule of N, by (i) there exists a maximal submodule L of M, such that $K \subseteq L$ and $N \not\subseteq L$. If $-: M \to M/\mathrm{Soc}(M)$ is the canonical quotient map, then $\overline{M}/\overline{K} = \overline{N}/\overline{K} \oplus \overline{L}/\overline{K}$. And if $\tilde{f}: \overline{N}/\overline{K} \to S$ is the map induced by f in the obvious way, then clearly $\tilde{f}$ can be extended to an R-homomorphism $\tilde{g}: \overline{M}/\overline{K} \to S$. And if we define $g: \overline{M} \to S$ by $g(\overline{m}) = \tilde{g}(\overline{m} + \overline{K})$ for every $m \in M$, then clearly $g: \overline{M} \to S$ is an R-homomorphism which extends f.

We can now proceed with the proof of Theorem 3. Since M is a GV-module, it follows from Lemma 4 that $M/\mathrm{Soc}(M)$ is a V-module. Inasmuch as M has Krull dimension and hence $M/\mathrm{Soc}(M)$ has Krull dimension, we infer from Theorem 1 that $M/\mathrm{Soc}(M)$ is a Noetherian module. And since M is finite dimensional and hence $\mathrm{Soc}(M)$ is finitely generated, it follows that M is Noetherian. ∎

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