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INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

ABDUL AZIZ

1. Introduction and statement of results. If $P(z)$ is a polynomial of degree n, then the inequality

$$(1) \quad \frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta \leqq \underset{|z|=1}{\text{Max}} |P(z)|^2$$

is trivial. It was asked by Callahan [**1**], what improvement results from supposing that $P(z)$ has a zero on $|z| = 1$ and he answered the question by showing that if $P(1) = 0$, then

$$(2) \quad \frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta \leqq \frac{n}{n+1} \underset{|z|=1}{\text{Max}} |P(z)|^2.$$

Donaldson and Rahman [**3**] have shown that if $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$ where β is an arbitrary non-negative number, then

$$(3) \quad \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \leqq \left(\frac{1}{1+\beta^2-2\beta\cos\left(\frac{\pi}{n+1}\right)}\right)\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^2 d\theta$$

whereas if the polynomial $P(z)$ is such that $P(1) = 0$, then [**4**]

$$(4) \quad |P'(1)| \leqq \frac{n}{2} \underset{|z|=1}{\text{Max}} |P(z)|.$$

In this paper we shall estimate

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta$$

in terms of the maximum of $|P(z_k)|$ where z_k, $k = 1, 2, \ldots, n$ are the zeros of $z^n + 1$ and obtain a sharp result. We shall also prove a generalization of (4). In lieu of requiring that the maximum of $|P(z)|$ on the right hand side of (4) be taken on $|z| = 1$, we only assume that it be taken over nth roots of -1.

We prove

THEOREM 1. *If $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$*

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where β is an arbitrary non-negative real number then

$$(5) \quad \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \leqq \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{(1+\beta^n)^2}\max_{1\leqq k\leqq n}|P(z_k)|^2,$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$. The result is best possible and equality in (5) *holds for $P(z) = z^n - \beta^n$.*

Next we present an inequality in the opposite direction.

THEOREM 2. *If $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$ where β is an arbitrary non-negative real number, then*

$$(6) \quad \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \geqq \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{(1+\beta^n)^2}\min_{1\leqq k\leqq n}|P(z_k)|^2,$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$. The result is sharp.

Finally we prove the following generalisation of (4).

THEOREM 3. *If $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$ where β is an arbitrary non-negative real number, then*

$$(7) \quad |P'(\beta)| \leqq \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{1+\beta^n}\max_{1\leqq k\leqq n}|P(z_k)|,$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$. The result is sharp for $\beta = 0$ and $\beta = 1$.

2. For the proofs of these theorems we need the following lemma.

LEMMA. *If $z_1, z_2, \ldots, z_n$ are the zeros of $z^n + 1$, then for an arbitrary non-negative real number β*

$$(8) \quad \frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-\beta|^2} = \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{(1+\beta^n)^2}.$$

Proof of the lemma. If $\beta = 0$, then the assertion is trivial. So we suppose that $\beta \neq 0$. If $P(z)$ is a polynomial of degree n such that $P(\beta) = 0$, then $P(z)/(z - \beta)$ is a polynomial of degree $n - 1$, and therefore, by using Lagrange's interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation we can write

$$\frac{P(z)}{z-\beta} = \sum_{k=1}^{n}\left(\frac{P(z_k)}{z_k-\beta}\right)\left(\frac{z^n+1}{nz_k^{\,n-1}(z-z_k)}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{P(z_k)z_k(z^n+1)}{(z_k-\beta)(z_k-z)},$$

since $z_k^{\,n-1} = -1/z_k$.

Taking in particular $P(z) = z^n - \beta^n$, we obtain

$$(9) \quad z^{n-1} + \beta z^{n-2} + \ldots + \beta^{n-1} = \frac{1+\beta^n}{n}\sum_{k=1}^{n}\frac{z_k(z^n+1)}{(z_k-\beta)(z-z_k)}.$$

Putting $z = 1/\beta$ in (9) and noting that $|z_k| = 1$ for $k = 1, 2, \ldots, n$, we get

$$\frac{1 + \beta^2 + \ldots + \beta^{2(n-1)}}{\beta^{n-1}} = \frac{(1+\beta^n)^2}{n\beta^n} \sum_{k=1}^{n} \frac{z_k\beta}{(z_k - \beta)(1 - \beta z_k)}$$
$$= \frac{(1+\beta^n)^2}{n\beta^{n-1}} \sum_{k=1}^{n} \frac{1}{(z_k - \beta)(\bar{z}_k - \beta)}.$$

Hence

$$1 + \beta^2 + \ldots + \beta^{2(n-1)} = \frac{(1+\beta^n)^2}{n} \sum_{k=1}^{n} \frac{1}{|z_k - \beta|^2},$$

which is equivalent to (8) and the lemma is proved.

3. Proofs.

Proof of Theorem 1. Let $S(\theta) = \sum c_k e^{ik\theta}$ be a trigonometric polynomial. If s and m are two integers of which the first is positive, then it can be easily verified directly (for example see [**2**] and [**5**]) that

$$(10)\quad \sum_{p=0}^{s-1} e^{-2\pi ipm/s} S(t + 2\pi p/s) = s \sum_{k \equiv m (\mathrm{mod}\, s)} c_k e^{ikt}.$$

Since $P(\beta) = 0$, $|P(e^{i\theta})/(e^{i\theta} - \beta)|^2$ is a trigonometric polynomial of degree $n - 1$. We take

$$S(\theta) = |P(e^{i\theta})/(e^{i\theta} - \beta)|^2, \quad s = n, m = 0 \quad \text{and} \quad t = \pi/n.$$

Then (10) reduces to

$$\sum_{p=0}^{n-1} \left| \frac{P(e^{i(1+2p)\pi/n})}{e^{i(1+2p)\pi/n} - \beta} \right|^2 = nc_0 = \frac{n}{2\pi} \int_0^{2\pi} \left| \frac{P(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta.$$

Equivalently

$$(11)\quad \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta = \frac{1}{n} \sum_{k=1}^{n} \left| \frac{P(z_k)}{(z_k - \beta)} \right|^2,$$

where z_k, $k = 1, 2, \ldots, n$ are the zeros of $z^n + 1$. This gives with the help of above lemma

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P(e^{i\theta})}{e^{i\theta} - \beta} \right|^2 d\theta \leqq \left\{ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{|z_k - \beta|^2} \right\} \operatorname*{Max}_{1 \leqq k \leqq n} |P(z_k)|^2$$
$$= \frac{1 + \beta^2 + \ldots + \beta^{2(n-1)}}{(1+\beta^n)^2} \operatorname*{Max}_{1 \leqq k \leqq n} |P(z_k)|^2,$$

which proves the desired result.

Proof of Theorem 2. From (11) we have with the help of the lemma above

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{e^{i\theta}-\beta}\right|^2 d\theta \geqq \left\{\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|z_k-\beta|^2}\right\}\underset{1\leqq k\leqq n}{\text{Min}}|P(z_k)|^2$$

$$=\frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{(1+\beta^n)^2}\underset{1\leqq k\leqq n}{\text{Min}}|P(z_k)|^2,$$

and this completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(\beta) = 0$, $P(z)/(z-\beta)$ is a polynomial of degree $n-1$. Using Lagrange's interpolation formula with $z_1, z_2, \ldots, z_n$ as the basic points of interpolation, we can write

$$\frac{P(z)}{z-\beta} = \sum_{k=1}^{n}\left(\frac{P(z_k)}{z_k-\beta}\right)\left(\frac{z^n+1}{nz_k^{n-1}(z-z_k)}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{P(z_k)z_k(z^n+1)}{(z_k-\beta)(z_k-z)}.$$

Letting $z\to\beta$ we obtain

$$P'(\beta) = \frac{1+\beta^n}{n}\sum_{k=1}^{n}P(z_k)\frac{z_k}{(z_k-\beta)^2}.$$

Hence

$$|P'(\beta)| \leqq \frac{1+\beta^n}{n}\sum_{k=1}^{n}|P(z_k)|\left|\frac{z_k}{(z_k-\beta)^2}\right|$$

$$\leqq \frac{1+\beta^n}{n}\sum_{k=1}^{n}\frac{1}{|z_k-\beta|^2}\underset{1\leqq k\leqq n}{\text{Max}}|P(z_k)|,$$

since $|z_k| = 1$.

Using now the lemma above, it follows that

$$|P'(\beta)| \leqq \frac{1+\beta^2+\ldots+\beta^{2(n-1)}}{1+\beta^n}\underset{1\leqq k\leqq n}{\text{Max}}|P(z_k)|.$$

This proves the desired result.

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University of Kashmir,
Kashmir, India