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A UNIFORMLY ASYMPTOTICALLY REGULAR MAPPING WITHOUT FIXED POINTS

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ABSTRACT. We construct a uniformly asymptotically regular, Lipschitzian mapping acting on a weakly compact convex subset of l_2 which has no fixed points.

Let K be a weakly compact convex subset of a Banach space X. A mapping $f:K \to K$ is said to be *asymptotically regular* if $\lim_{n\to\infty} \|f^{n+1}(x) - f^n(x)\| = 0$ for all $x \in K$. f is said to be *uniformly asymptotically regular* if for any $\delta > 0$ there exists an N such that for all $x \in K$ and for all $n \geqq N$, $\|f^{n+1}(x) - f^n(x)\| < \delta$. It is known [3] that if f is nonexpansive i.e. $\|f(x) - f(y)\| \leqq \|x - y\|$, then $F_\lambda = \lambda I + (1 - \lambda)f$ is uniformly asymptotically regular for all $0 < \lambda < 1$. It is also known [1] that if X is uniformly convex and f is nonexpansive, then f has a fixed point. Recently, D. Tingley [8] has constructed an asymptotically regular mapping acting on a weakly compact subset of l_2, which has no fixed points. The question arises as to whether f has a fixed point when f is uniformly asymptotically regular and X is uniformly convex. The purpose of this paper is to show the answer is negative.

For more results of a nonexpansive or the more general nonexpansive mapping, we suggest the reader consult [1-8].

Let $\{e_i\}$ be an orthonormal basis of l_2 and let

$$K = \{\sum a_i e_i : \sum a_1^2 \leqq 1 \text{ and } a_1 \geqq a_2 \geqq a_3 \geqq \ldots \geqq 0\}.$$

For each $x = \sum a_i e_i \in K$, let $g(x) = \max(a_1, 1 - \|x\|)e_1 + \sum_{i=2}^{\infty} a_{i-1} e_i$ and define the function f by

$$f(x) = \frac{g(x)}{\|g(x)\|}.$$

It is clear that f has no fixed points. Now, we claim that f is Lipschitzian and uniformly asymptotically regular.

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LEMMA 1. *f is Lipschitzian.*

PROOF. If $x \in K$, then $2 \geqq \|g(x)\| \geqq \frac{1}{2}$. So

$$\begin{aligned}\|f(x) - f(y)\| &= \left\|\frac{g(x)}{\|g(x)\|} - \frac{g(y)}{\|g(y)\|}\right\| \\ &\leqq \left\|\frac{g(x) - g(y)}{\|g(x)\|}\right\| \\ &\quad + \|g(y)\| \left|\frac{1}{\|g(x)\|} - \frac{1}{\|g(y)\|}\right| \\ &\leqq 2 \cdot 2\|x - y\| + 2 \cdot 4 \cdot 2\|x - y\| \\ &= 20\|x - y\|.\end{aligned}$$

LEMMA 2. f is *uniformly asymptotically regular*.

PROOF. We need the following facts.

FACT 1. If $x \in K$, then $\|f(x)\| = 1$.

FACT 2. $f^{n+1}(x) = \sum_{i=1}^{\infty} a_i e_i$, then $a_1 = a_2 = \ldots = a_n \leqq 1/\sqrt{n}$.

FACT 3. If $x = \sum_{i=1}^{\infty} a_i e_i \in K$ and $g(x) = \sum_{i=1}^{\infty} b_i e_i$, then $a_n \leqq b_n$ for all $n \in \mathbf{N}$.

Since l_2 is uniformly convex, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x\| \leqq 1$, $\|y\| \leqq 1$, and $\|x - y\| > \delta$ then $\|x + y\|/2 < 1 - \epsilon$.
Hence, if $1/\sqrt{n} \leqq \epsilon$, then

$$1 + \epsilon \geqq 1 + \frac{1}{\sqrt{n}} \geqq \|g(f^{n+1}(x))\| \geqq \|g(f^{n+1}(x)) + f^{n+1}(x)\|/2 \geqq 1.$$

So $\|g(f^{n+1}(x)) - f^{n+1}(x)\| < \delta(1 + 1/\sqrt{n})$, and

$$\begin{aligned}&\|f^{n+2}(x) - f^{n+1}(x)\| \\ &\leqq \|f^{n+2}(x) - g(f^{n+1}(x))\| + \|g(f^{n+1}(x)) - f^{n+1}(x)\| \\ &\leqq \|g(f^{n+1}(x))\| - 1 + \delta\left(1 + \frac{1}{\sqrt{n}}\right) \\ &\leqq \frac{1}{\sqrt{n}} + \delta\left(1 + \frac{1}{\sqrt{n}}\right).\end{aligned}$$

(Note: $g(f^{n+1}(x)) = \|g(f^{n+1}(x))\| f^{n+2}(x)$.) □

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