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Very Ample Linear Systems on Blowings-Up at General Points of Projective Spaces

Marc Coppens

Abstract. Let $\mathbf{P}^n$ be the n-dimensional projective space over some algebraically closed field k of characteristic 0. For an integer $t \geq 3$ consider the invertible sheaf $O(t)$ on $\mathbf{P}^n$ (Serre twist of the structure sheaf). Let $N = \binom{t+n}{n}$, the dimension of the space of global sections of $O(t)$, and let k be an integer satisfying $0 \leq k \leq N - (2n + 2)$. Let $P_1, \dots, P_k$ be general points on $\mathbf{P}^n$ and let $\pi\colon X \to \mathbf{P}^n$ be the blowing-up of $\mathbf{P}^n$ at those points. Let $E_i = \pi^{-1}(P_i)$ with $1 \leq i \leq k$ be the exceptional divisor. Then $M = \pi^*\big(O(t)\big) \otimes O_X(-E_1 - \cdots - E_k)$ is a very ample invertible sheaf on X.

In their paper [5], J. d'Almeida and A. Hirschowitz prove the following theorem:

Theorem of d'Almeida and Hirschowitz *Let t, k be integers satisfying $t \geq 2$, $0 \leq k \leq \binom{t+2}{2} - 6$ and let $P_1, \dots, P_k$ be general points on $\mathbf{P}^2$. Let $\pi\colon X \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ at $P_1, \dots, P_k$ and let $E_i = \pi^{-1}(P_i)$ be the exceptional divisors. Then $M = \pi^*\Big(O_{\mathbf{P}^2}(t)\Big) \otimes O_X(-E_1 - \cdots - E_k)$ is very ample on X.*

The bound on k is natural. Indeed $\dim\Big(\Gamma\Big(\mathbf{P}^2, O_{\mathbf{P}^2}(t)\Big)\Big) = \binom{t+2}{2}$ hence for the invertible sheaf M as in the previous theorem (but only assuming $k \leq \binom{t+2}{2}$) one finds $\dim\big(\Gamma(X, M)\big) = \binom{t+2}{2} - k$. Since X is a surface one expects M is not very ample if $\dim\big(\Gamma(X, M)\big) \leq 5$ (most surfaces cannot be embedded in $\mathbf{P}^4$).

Let Y be a smooth n-dimensional projective variety, and let L be a very ample invertible sheaf on Y with $\dim\big(\Gamma(Y; L)\big) = N + 1$. Inspired by the theorem of d'Almeida and Hirschowitz we define:

Very Ampleness Property for Blowings-Up of (Y, L) at k General Points Let $P_1, \dots, P_k$ be general points on Y, let $\pi\colon X \to Y$ be the blowing-up of Y at $P_1, \dots, P_k$ and let $E_i = \pi^{-1}(P_i)$. Then $M = \pi^*(L) \otimes O_X(-E_1 - \cdots - E_k)$ is very ample on X.

Optimal Very Ampleness Property for Blowings-Up of (Y, L) at General Points The very ampleness property for blowings-up of (Y, L) at k general points holds that for all integers $k \leq N - (2n + 1)$.

The natural general problem becomes: find sufficient conditions on (Y, L) such that the optimal very ampleness property for blowings-up of (Y, L) at general points holds.

From now on we assume the ground field has characteristic zero.

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Let Y be an arbitrary smooth projective variety Y and let L' be a very ample invertible sheaf on Y. Let $L \cong L'^{\otimes a}$ for some $a \geq 3\dim(Y) + 1$. In [4] it is proved that the optimal very ampleness property holds for blowings-up of (Y, L) at general points. In that paper we also consider the case of $K3$ surfaces Y. In the case of $K3$-surfaces we obtain non-trivial examples not satisfying the optimal very ampleness property for blowings-up at general points. Recently the very ampleness property for blowings-up at general points is also studied for other special surfaces: for rational surfaces in [2] and [6], for ruled surfaces in [1] and [9] and for abelian surfaces in [8].

In this paper we study the very ampleness property for blowings-up at general points for the case $Y = \mathbf{P}^n$ $(n \geq 3)$ extending the theorem of d'Almeida and Hirschowitz to all projective spaces. The very ample invertible sheafs are $O_{\mathbf{P}^n}(t)$ with $t \geq 1$. Of course for $t = 1$ there is nothing to consider while the case $t = 2$ is very bad because of the following argument. Assume $n \geq 3$ and let P_1, P_2 be two different points on $\mathbf{P}^n$. Let L be the line in $\mathbf{P}^n$ joining P_1 and P_2; let $P \in L$ with $P \notin \{P_1, P_2\}$. Let Q be a quadric in $\mathbf{P}^n$ containing P_1 and P_2. In case $P \in Q$ then $L \subset Q$, hence using quadrics in $\mathbf{P}^n$ containing P_1 and P_2 one cannot separate 2 general points on L. Let $\pi\colon X \to \mathbf{P}^n$ be the blowing-up of $\mathbf{P}^n$ at P_1 and P_2; let $E_i = \pi^{-1}(P_i)$ for $i = 1, 2$. Then this implies $M = \pi^*\big(O_{\mathbf{P}^n}(2)\big) \otimes O_X(-E_1 - E_2)$ is not very ample on X.

So now assume $n \geq 2$, $t \geq 3$ and $L = O_{\mathbf{P}^n}(t)$ on $Y = \mathbf{P}^n$. In [3] we proved the very ampleness property for blowings-up at k general points in case $k \leq \binom{n+t}{t} - (n-1)(n+1) - 4$. Now we prove the property using the optimal bound on k.

***Theorem* (Optimal Very Ampleness Property for Blowings-Up of Projective Spaces at General Points)** *Let n, t, k be integers with $n \geq 2$, $t \geq 3$ and $0 \leq k \leq \binom{n+t}{t} - (2n+2)$. Let $P_1, \dots, P_k$ be general points on $\mathbf{P}^n$; let $\pi\colon X \to \mathbf{P}^n$ be the blowing-up of $\mathbf{P}^n$ at $P_1, \dots, P_k$ and let $E_i = \pi^{-1}(P_i)$ be the exceptional divisors. Then the invertible sheaf $M = \pi^*\big(O_{\mathbf{P}^n}(t)\big) \otimes O_X(-E_1 - \dots - E_k)$ is very ample on $\mathbf{P}^n$.*

The proof of the theorem follows the steps of the proof of Theorem 1 in [4]. We refer to that proof for some of the details; hence this paper is dependent on [4].

1 Proof of the Theorem

Consider $\nu\colon \mathbf{P}^n \to \mathbf{P}^N$ with $N = \binom{n+t}{t} - 1$ the t-th Veronese embedding of $\mathbf{P}^n$ and let Y be the image, so $Y \subset \mathbf{P}^N$. We also consider $P_1, \dots, P_k$ as general points on Y. We write $\mathbf{P}$ to denote the set $\{P_1, \dots, P_k\}$ (both on $\mathbf{P}^n$ and on Y); we consider $\mathbf{P}$ as a reduced closed subscheme. We write P to denote the linear span of $\mathbf{P} \subset \mathbf{P}^n$. (By definition, if Z is a closed subscheme of some projective space $\mathbf{P}^a$, then the linear span $\langle Z \rangle$ is the intersection of all hyperplanes in $\mathbf{P}^a$ containing Z as a subscheme.) From (1.3.4) in [4] we know we have to prove the following statement: For all curvilinear subschemes $Z \subset Y$ of length $k+2$ containing $\mathbf{P}$ one has $\dim(\langle Z \rangle) = \dim(P) + 2 = k + 1$. This is equivalent to the following statement on $\mathbf{P}^n$: For all curvilinear subschemes $Z \subset \mathbf{P}^n$ of length $k+2$ containing $\mathbf{P}$, one has $\dim\Big(\Gamma\big(\mathbf{P}^n; I_Z(t)\big)\Big) = \binom{t+n}{t} - k - 2$. (Here $I_Z(t) = I_Z \otimes O_{\mathbf{P}^n}(t)$ and I_Z is the sheaf of ideals of Z.) Let $\mathbf{P}_t$ be the complete linear system of hypersurfaces of degree t on $\mathbf{P}^n$ (*i.e.*, the complete linear system associated to the invertible sheaf $O_{\mathbf{P}^n}(t)$.) For such curvilinear subschemes Z

we need to prove that Z imposes $k+2$ independent conditions on $\mathbf{P}_t$. We write $\mathbf{P}_t(Z)$ to denote the linear subsystem of hypersurfaces containing Z and we need to prove $\dim\big(\mathbf{P}_t(Z)\big) = \binom{t+n}{t} - k - 3$.

2

We are going to use induction on k. Since $\mathbf{P}_t$ is very ample on $\mathbf{P}^n$ there is nothing to prove in case $k=0$. So assume $k>0$. Assume there exists for $\mathbf{P}$ general as above, a curvilinear subscheme Z of $\mathbf{P}^n$ containing $\mathbf{P}$ such that Z does not impose independent conditions on $\mathbf{P}_t$. (Of course we also consider Z as a curvilinear subscheme of Y.) Let $T' \subset \mathrm{Hilb}^{k+2}(\mathbf{P}^n) \times (\mathbf{P}^n)^k$ be the closure of the set of points $(Z; P_1, \ldots, P_k)$ with $P_i \neq P_j$ for $i \neq j$ and such that Z is a curvilinear subscheme of length $k+2$ containing $P_1, \ldots, P_k$ such that $\dim\big(\mathbf{P}_t(Z)\big) > \binom{t+n}{t} - (k+3)$. Let T be an irreducible component T' dominating $(\mathbf{P}^n)^k$ (such a component exists by assumption), so $\dim(T) \geq nk$. Let $G(N-n+1; N)$ be the Grassmannian of $(N-n+1)$-planes in $\mathbf{P}^N$ (remember $Y \subset \mathbf{P}^N$) and consider $I \subset T \times G(N-n+1; N)$ with $\big((Z; P_1, \ldots, P_k); \Lambda\big) \in I$ if and only if $\Lambda \supset \langle Z \rangle$ in $\mathbf{P}^N$ (so here we consider $Z \subset \mathbf{P}^N$).

Since $\dim(\langle Z \rangle) \leq k$, $\langle Z \rangle \supset P$, $\dim(P) = k-1$ and $P \cap Y = \mathbf{P}$ (the last two facts are true because $\mathbf{P}$ is a general set of k points on Y), we find $\dim(\langle Z \rangle) = k$. Therefore the fibers of the projection $I \to T$ have dimension $(N-n+1-k)(n-1)$, and hence $\dim(I) \geq nk + (N-n+1-k)(n-1)$. We consider the projection $\tau\colon I \to G(N-n+1; N)$. For $\Lambda \in \tau(I)$ we consider the scheme theoretic intersection $\Lambda \cap Y$; we denote by $Z(\Lambda) \subset \mathbf{P}^n$ the associated closed subscheme on $\mathbf{P}^n$ (of course $Z(\Lambda) \cong \Lambda \cap Y$).

3

Claim For $\Lambda \in \tau(I)$ general $\Lambda \cap Y$ is not a smooth curve.

Assume for some $\Lambda \in \tau(I)$ the scheme $\Lambda \cap Y$ is a smooth curve (since $\dim(\Lambda) = N-n+1$ we know $\dim(\Lambda \cap Y) \geq 1$; hence this assumption is equivalent to $\Lambda \cap Y$ being a smooth curve for a general $\Lambda \in \tau(I)$). Let g be the linear system on $\Lambda \cap Y$ induced by $\Gamma\big(Y, O_Y(1)\big)$. It is the same as the linear system on $Z(\Lambda)$ induced by $\mathbf{P}_t$. Since $Z(\Lambda)$ is a complete intersection curve (scheme theoretical intersection of $n-1$ hypersurfaces of degree t) it is a complete linear system on $Z(\Lambda)$. Let $V^{k+1}_{k+2}(g)$ be the space of effective divisors of degree $k+2$ on $Z(\Lambda)$ imposing only $k+1$ conditions on g, then elements of $\tau^{-1}(\Lambda)$ give rise to elements Z belonging to a subvariety V of $V^{k+1}_{k+2}(g)$ with $\dim(V) \geq k-n+1$ (see [4, the proof of (1.2.1)]). Assume, for the next arguments, that $k \geq n+1$; at the end the conclusion will be true for $k < n+1$ too (and we do not use the induction hypothesis yet). Using a result from the theory of linear systems on smooth curves, it is explained in [4] that Z contains a closed subscheme S (hence an effective divisor on $Z(\Lambda)$) of length $m+2 \leq 3n+2$ such that $S \in V^{m+1}_{m+2}(g)$. Since Z is obtained from a general element of $\tau^{-1}(\Lambda)$ and Λ is a general element of $\tau(I)$, we find Z comes from a general element of T. Hence Z contains a set of k general points of $\mathbf{P}^n$, and it follows that S contains a set of m general points

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of $\mathbf{P}^n$. So, we obtain the following situation. There is an integer $m \le 3n$ such that for m general points $P_1, \dots, P_m$ of $\mathbf{P}^n$ there exists a curvilinear subscheme $S \subset \mathbf{P}^n$ of length $m+2$ with $S \supset \{P_1, \dots, P_m\}$ and $\dim\big(\mathbf{P}_t(S)\big) \le \binom{n+t}{t} - m - 4$ (*i.e.*, S does not impose $m+2$ independent conditions on $\mathbf{P}_t$). This situation also holds if $k < n+1$! The claim will be proved by deducing a contradiction to this statement. This contradiction will be obtained in Section 5 using a lemma proved in Section 4.

4

Lemma *Let a, t be integers at least 1, and let $W \subset \mathbf{P}^a$ be a curvilinear closed subscheme of length $x \le a+3$ with $\langle W \rangle = \mathbf{P}^a$. Let $\mathbf{P}_t$ be the linear system of hypersurfaces of degree t in $\mathbf{P}^a$. If $t \ge 3$, then W imposes independent conditions on $\mathbf{P}_t$.*

Proof It is possible to make a chain of subschemes

$$\varnothing = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_X = W$$

such that W_i has length i and either $\langle W_i \rangle \cap W = W_i$ or $W_{i-1} \subset \langle W_i \rangle \cap W$. It is enough to prove that for each $0 \le i < x$ there exists $F \in \mathbf{P}_t$ with $F \cap W = W_i$.

Assume $\langle W_{i+1} \rangle = \langle W_i \rangle$. Take $H \in \mathbf{P}_1$ general with the condition $H \supset \langle W_i \rangle$. Then, since $\langle W_i \rangle \cap W = W_i$, one has $H \cap W = W_i$. For $Q \in \mathbf{P}_{t-1}$ general one has $Q \cap W = \varnothing$. Take $F = H + Q \in \mathbf{P}_t$; then $F \cap W = W_i$.

Next assume $\langle W_{i+1} \rangle = \langle W_i \rangle$ (hence $i \ge 1$) but $\langle W_i \rangle \ne \langle W_{i-1} \rangle$. Let $P \in W_i$ be defined by $O_{W_i,\mathbf{P}} \ne O_{W_{i-1},\mathbf{P}}$ and take $H_1 \in \mathbf{P}_1$ general with the condition $H_1 \supset \langle W_{i-1} \rangle$. Since $\langle W_{i-1} \rangle \cap W = W_{i-1}$ we find $H_1 \cap W = W_{i-1}$. Take $H_2 \in \mathbf{P}_1$ general with the condition $P \in H_2$. Then $H_2 \cap \mathbf{P} = \{P\}$ and, since W is curvilinear, it follows $(H_1 + H_2) \cap W = W_i$. Take $Q \in \mathbf{P}_{t-2}$ general; hence $Q \cap W = \varnothing$. For $F = H_1 + H_2 + Q \in \mathbf{P}_t$ we obtain $F \cap W = W_i$. Finally assume $\langle W_{i+1} \rangle = \langle W_i \rangle = \langle W_{i-1} \rangle$ (hence $i - 1 \ge 1$). Since $\langle W \rangle = \mathbf{P}^a$ and W has length at most $a+3$, it follows that $\langle W_{i-2} \rangle \ne \langle W_{i-1} \rangle$. Let $P \in W_i$ be as before and let $P' \in W_{i-1}$ with $O_{W_{i-1},P'} \ne O_{W_{i-2},P'}$. Take $H_1 (H_2; H_3) \in \mathbf{P}_1$ general with the condition $H_1 \supset \langle W_{i-2} \rangle$ (resp. $P' \in H_2$, $P \in H_3$). Since $H_1 \cap W = W_{i-2}$ and W is curvilinear we find $(H_1 + H_2 + H_3) \cap W = W_i$. Take $Q \in \mathbf{P}_{t-3}$ general; hence $Q \cap W = \varnothing$. Let $F = H_1 + H_2 + H_3 + Q \in \mathbf{P}_t$; then $F \cap W = W_i$.

5

Now we prove that the situation at the end of Section 3 can not occur.

In case $m \le n+1$, since $P_1, \dots, P_m$ are general points of $\mathbf{P}^n$, $\dim(\langle P_1, \dots, P_m \rangle) = m-1$ and so $\dim(\langle S \rangle) = a \ge m-1$ while S has length $m+2 \le a+3$. So we apply the lemma taking $W = S$ and $\mathbf{P}^a = \langle S \rangle$.

So assume $m > n+1$. We write $[S]$ to denote the cycle associated to the subscheme S (formal $\mathbf{Z}$-linear combination of the points of S with coefficients at P equal to the multiplicity of S at P). Let $S' \subset S$ be a closed subscheme of S of length n such that $S' \supset \{P_1, \dots, P_{n-2}\}$ and $[S] - [S'] = P_{n-1} + \cdots + P_m$. Since $\dim(\langle P_1, \dots, P_{n-2} \rangle) =$

$n-3$, we find $n-3 \leq \dim(\langle S' \rangle) \leq n-1$ and so $\langle S' \rangle \cap \{P_1, \ldots, P_m\}$ contains at most n points. Let $S_0 = S \cap \langle S' \rangle$. Since $P_1, \ldots, P_m$ are general points of $\mathbf{P}^n$, the scheme S_0 has length at most $n+2$. If S_0 has length $n+2$, then $S_0 \cap \{P_1, \ldots, P_m\}$ contains n points; hence $\dim(\langle S_0 \rangle) = n-1$. If S_0 has length $n+1$, then $S_0 \cap \{P_1, \ldots, P_m\}$ contains at least $n-1$ points; hence $\dim(\langle S_0 \rangle) \geq n-2$. So in all cases we can apply the lemma to $W = S_0$ and $\mathbf{P}^a = \langle S_0 \rangle$. Hence S_0 imposes independent conditions on $\mathbf{P}'_t$ (here $\mathbf{P}'_t$ is the linear system in $\langle S_0 \rangle$). Using suited cones in $\mathbf{P}^n$ on elements of $\mathbf{P}'_t$ we find $\varnothing = W_0 \subset W_1 \subset \cdots \subset W = S_0$ as in the proof of the lemma and $F_i \in \mathbf{P}_t$ (here $\mathbf{P}_t$ is the linear system in $\mathbf{P}^n$) with $F_i \cap S = W_i$.

Let $H \in \mathbf{P}_1$ be a general hyperplane in $\mathbf{P}^n$ with the assumption $H \supset \langle S_0 \rangle$. Since $\langle S_0 \rangle \cap S = S_0$, we find $H \cap S = S_0$. Since $[S] - [S_0] \leq P_{n-1} + \cdots + P_m$ it is enough to prove that the reduced closed subscheme associated to $P_{n-1} + \cdots + P_m$ imposes independent conditions on $\mathbf{P}_{t-1}$ (linear system of $\mathbf{P}^n$). Since $P_{n-1}, \ldots, P_m$ are $m-n+2$ general points of $\mathbf{P}^n$, it is enough to prove $\dim(\mathbf{P}_{t-1}) \geq m-n+1$. But $t-1 \geq 2$; hence $\dim(\mathbf{P}_{t-1}) \geq \dim(\mathbf{P}_2) = \binom{n+2}{2} - 1$. Since $m \leq 3n$ and $n \geq 2$, we find $\binom{n+2}{2} - 1 \geq m-n+1$. This gives a contradiction to the statement at the end of Section 3, finishing the proof of the claim.

6

Take $(Z; P_1, \ldots, P_k) \in I$ general, and assume $\langle Z \rangle \cap Y$ is a 0-dimensional subscheme of Y. In the case $\langle Z \rangle \cap Y$ is a curvilinear subscheme, then, using Bertini's Theorem as in [4, (1.2.3.1)], we find a contradiciton to the claim in Section 3. So $\langle Z \rangle \cap Y$ is not curvilinear but then, as explained in [4, (1.2.3.2)], we find a contradiction to the induction hypothesis on k. So we conclude $\dim(\langle Z \rangle \cap Y) \geq 1$.

7

Since P is a hyperplane in $\langle Z \rangle \subset \mathbf{P}^N$ and $P \cap Y = \mathbf{P}$ is finite, we find $\dim(\langle Z \rangle \cap Y) = 1$. Let Γ be a 1-dimensional irreducible component of $\langle Z \rangle \cap Y$. Let $\Gamma \cap P$ (a hyperplane section of Γ) be $\{P_1, \ldots, P_b\}$. Then $\dim(\langle \Gamma \cap P \rangle) = b-1$, and hence $\dim(\langle \Gamma \rangle) = b$ and $\deg(\Gamma) = b$, so Γ is a rational normal curve on Y. On $\mathbf{P}^n$ we find a smooth rational curve of degree b/t which is also denoted by Γ. So we obtain the following situation. There are integers d, t with $d \geq 1$, $t \geq 3$ such that for $P_1, \ldots, P_{dt}$ general points in $\mathbf{P}^n$ there exists a smooth rational curve $\Gamma \subset \mathbf{P}^n$ of degree d containing $P_1, \ldots, P_{dt}$. We are going to prove that this is impossible, finishing the proof of the theorem.

8

Let $H_{d,n}$ be the Hilbert scheme of smooth rational curves of degree d in $\mathbf{P}^n$. We are going to use the well-known fact that $\dim(H_{d,n}) = nd + d + n - 3$. This can be proved in an elementary way using dimension arguments for the space of linear systems on $\mathbf{P}^1$. It also follows from the following considerations.

Let Γ be a smooth rational curve of degree d in $\mathbf{P}^n$, and let N_Γ be the normal

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bundle. From Corollary (11.3) in [7] it follows that $h^1(N_\Gamma) = 0$. Then from Corollaries (8.5) and (8.6) in [7] it follows that the dimension of $H_{d,n}$ at Γ is equal to $h^0(N_\Gamma)$. It follows from the computations at the beginning of Chapter 11 in [7] that $h^0(N_\Gamma) = nd + d + n - 3$, proving the statement.

Consider the incidence variety $I^b_{d,n} \subset (\mathbf{P}^n)^b \times H_{d,n}$ defined by $(P_1, \ldots, P_b; \Gamma) \in I^b_{d,n}$ if and only if $P_i \in \Gamma$ for $1 \leq i \leq b$, and let $\pi_1 \colon I^b_{d,n} \to (\mathbf{P}^n)^b$ and $\pi_2 \colon I^b_{d,n} \to H_{d,n}$ be the projection morphisms. The fibers of π_2 have dimension b, hence $\dim(I^b_{d,n}) = nd+d+n-3+b$. We found that π_1 is dominating, and hence $nd+d+n-3+b \geq nb$, *i.e.*, $(n-1)b \leq (n+1)d+n-3$. Remember $b = td$; hence $(n-1)td \leq (n+1)d+n-3$. Since $t \geq 3$, we obtain $3(n-1)d \leq (n+1)d+n-3$. In the case $n = 2$, this inequality becomes $3d \leq 3d-1$, a contradiction. If $n \geq 3$, then $2(n-1) \geq n+1$, and hence we obtain $(n-1)d \leq n-3$, *i.e.*, $n(d-1) \leq d-3$ and so $3(d-1) \leq d-3$, a contradiction.

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Katholieke Hogeschool Kempen
Departement Industrieel Ingenieur en Biotechniek
Campus H. I. Kempen
Kleinhoefstraat 4
B 2440 Geel
Belgium
email: marc.coppens@khk.be