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Part 4. Random graphs and particle systems

THE NAMING GAME IN LANGUAGE DYNAMICS REVISITED

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Abstract

In this article we study a biased version of the naming game in which players are located on a connected graph and interact through successive conversations in order to select a common name for a given object. Initially, all the players use the same word B except for one bilingual individual who also uses word A . Both words are attributed a fitness, which measures how often players speak depending on the words they use and how often each word is spoken by bilingual individuals. The limiting behavior depends on a single parameter, ϕ , denoting the ratio of the fitness of word A to the fitness of word B . The main objective is to determine whether word A can invade the system and become the new linguistic convention. From the point of view of the mean-field approximation, invasion of word A is successful if and only if $\phi > 3$, a result that we also prove for the process on complete graphs relying on the optimal stopping theorem for supermartingales and random walk estimates. In contrast, for the process on the one-dimensional lattice, word A can invade the system whenever $\phi > 1.053$, indicating that the probability of invasion and the critical value for ϕ strongly depend on the degree of the graph. The system on regular lattices in higher dimensions is also studied by comparing the process with percolation models.

Keywords: Interacting particle system; naming game; language dynamics; semiotic dynamics

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1. Introduction

The naming game was first proposed by Steels [8] to describe the emergence of conventions and shared lexicons in a population of individuals interacting through successive conversations. Several variants of the model have been introduced and studied numerically by statistical physicists; see [2, Section V.B] for a review of these variants. The naming game is popular in the physics literature because it is similar mathematically to traditional models in the field of statistical mechanics.

The model studied in this paper is a biased version of the spatial naming game considered by Baronchelli *et al.* [1]. Their system consists of a population of individuals located on the vertex set of a finite connected graph that is to be regarded as an interaction network. For a given object, each individual is characterized by an internal inventory of words that are synonyms describing the object. All inventories are initially empty and evolve through successive conversations: at each time step, an edge of the network is chosen uniformly at random. This causes the two individuals connected by the edge to converse: one individual is chosen at random to be the ‘speaker’ and the other is then the ‘hearer’. If the speaker has no word to describe the object then she invents one, whereas if she already has some words in her inventory then she

chooses one at random and speaks it to the hearer. As a result of this word being spoken, if the hearer already has the spoken word in her inventory then this word is adopted as the norm by both individuals (i.e. all other words are removed from the inventories of both individuals); otherwise, the hearer adds the spoken word to her inventory. Based on numerical simulations, Baronchelli *et al.* [1] studied the maximum number of words present in the system as well as the time to global consensus, i.e. the time until all inventories consist of the same single word.

In contrast, we use the naming game to study whether a new word can spread into a population that is already using another word as a convention. To do this, we assume that initially all inventories consist of the same single word, B say, except for one individual whose inventory includes another word, A say. Under the symmetric rules of the naming game, the probability that A eventually becomes the new convention tends to 0 as the population size goes to ∞ , so we look at biased versions of the naming game in which each word is attributed a fitness. In our model, the fitness of each word has a dual role: it determines how likely it is for an individual to be selected as a speaker rather than hearer, and how likely it is for each word to be selected and spoken by bilingual individuals, i.e. individuals who possess both words in their internal inventory.

Another significant difference between this paper and previous work on the naming game is that we provide a rigorous analysis of the model on both finite and infinite graphs rather than results based on numerical simulations which are necessarily restricted to finite graphs. Throughout the paper, the network of interactions is denoted by $G = (V, E)$, where V is the vertex set and E the edge set. Also, we describe the dynamics in continuous time, i.e. we assume that conversations occur at rate 1 along each edge; our model is then well defined for both finite and infinite graphs.

1.1. Model description

To describe our biased version of the naming game more formally, let all inventories consist initially of the single word B except for one individual, with inventory A . Then elements of the set $\{A, B, AB\}$ describe all possible inventories in any ensuing conversations. Let ϕ_A and ϕ_B denote the fitnesses of words A and B , respectively. For all $X, Y \in \{A, B, AB\}$, set

$$\phi_{AB} := \frac{1}{2}(\phi_A + \phi_B) \quad \text{and} \quad p_{X,Y} := \phi_X (\phi_X + \phi_Y)^{-1},$$

where ϕ_{AB} is the fitness of a bilingual individual (i.e. whose inventory is AB), and $p_{X,Y}$ is the probability that in a conversation between individuals with fitnesses ϕ_X and ϕ_Y , the former is the speaker. In particular then,

$$p_{X,X} = \frac{1}{2} \quad \text{and} \quad p_{X,Y} + p_{Y,X} = 1.$$

The fitnesses of individuals correspond to the fitnesses of their inventories. Furthermore, when a bilingual individual is the speaker, the conditional probability that word A is spoken is equal to the relative fitness $p_{A,B}$.

Conversations take place between pairs of individuals, each of whom has an inventory that is one of the three types, so that, associating vertices of the network graph with the type of an individual’s inventory, edges are then unordered pairs of vertices, and the result of a conversation is again an unordered pair of vertices; edges (A, A) and (B, B) remain unaltered by a conversation, while no conversation can yield an edge of type (A, B) . An edge must be one of the six possible types (A, A) , (A, AB) , (A, B) , (AB, AB) , (AB, B) , and (B, B) . Assume that each edge becomes active at rate 1, independently for all edges.

Interpret the conversation protocol in terms of possible outcomes. This yields the matrix below of the rates of transitions of any given edge of type (X, Y) to one of type (X', Y') as a

result of a conversation ($q_A = p_{AB,B}p_{A,B}$, $q_B = p_{AB,A}p_{B,A}$):

$$\begin{matrix} & \nearrow & (A, A) & (A, AB) & (A, B) & (AB, AB) & (AB, B) & (B, B) \\ \begin{matrix} (A, A) \\ (A, AB) \\ (A, B) \\ (AB, AB) \\ (AB, B) \\ (B, B) \end{matrix} & & \begin{matrix} 1 \\ 1 - q_B \\ 0 \\ p_{A,B} \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ p_{A,B} \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ q_B \\ 0 \\ 0 \\ q_A \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ p_{B,A} \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ p_{B,A} \\ 1 - q_A \\ 1 \end{matrix} \end{matrix} \quad (1.1)$$

Then, when the fitnesses are equal, we recover the transition probabilities of the unbiased naming game of [1]. While we formulate the dynamics using two parameters so as to have notation that preserves the symmetry between both words, the long-term behavior of the process depends only on the ratio $\phi := \phi_A/\phi_B$, in terms of which

$$p_{A,B} = \frac{\phi}{1 + \phi}, \quad p_{B,A} = \frac{1}{1 + \phi}, \quad q_A = \frac{\phi}{3 + \phi}, \quad q_B = \frac{1}{3\phi + 1}. \quad (1.2)$$

1.2. Mean-field model

Before stating our results for the spatial stochastic model, we consider its nonspatial deterministic mean-field approximation, i.e. the model obtained by assuming that the population is well mixed. This results in the following system of differential equations for the functions u_X that denote the relative frequencies of type- X individuals for $X \in \{A, B, AB\}$:

$$\begin{aligned} \dot{u}_A &= u_A u_{AB} (1 - 2q_B) - u_A u_B p_{B,A} + u_{AB}^2 p_{A,B}, \\ \dot{u}_B &= u_B u_{AB} (1 - 2q_A) - u_A u_B p_{A,B} + u_{AB}^2 p_{B,A}, \\ \dot{u}_{AB} &= u_A u_{AB} (2q_B - 1) + u_B u_{AB} (2q_A - 1) + u_A u_B - u_{AB}^2. \end{aligned}$$

The mean-field model has two trivial equilibria, namely,

$$e_A := (1, 0, 0) \quad \text{and} \quad e_B := (0, 1, 0),$$

corresponding to the configurations in which all individuals are of types A and B , respectively. We say that word A can invade word B in the mean-field model whenever the system starting from any initial state different from e_B converges to the trivial equilibrium e_A .

Irrespective of the ratio $\phi := \phi_A/\phi_B$, the frequency of type- A individuals might decrease because the boundary $u_{AB} = 0$ is repelling. To examine this, consider the difference between the frequencies of individuals using words A and B . This yields

$$\begin{aligned} \dot{u}_A - \dot{u}_B &= u_A u_{AB} (1 - 2q_B) - u_B u_{AB} (1 - 2q_A) + (u_A u_B + u_{AB}^2)(p_{A,B} - p_{B,A}) \\ &= \frac{3\phi - 1}{3\phi + 1} u_A u_{AB} + \frac{\phi - 3}{\phi + 3} u_B u_{AB} + \frac{\phi - 1}{\phi + 1} (u_A u_B + u_{AB}^2), \end{aligned}$$

which is positive for all $\phi > 3$ when $u_A \neq 1$ and $u_B \neq 1$. This implies that, when $\phi > 3$, there is no equilibrium other than the two trivial equilibria and that word A can invade word B . This condition is sharp in the sense that e_B is locally stable when $\phi < 3$. Indeed, the Jacobian matrix of the system of differential equations at the point e_B reduces to

$$\mathcal{J}_{e_B} = \begin{pmatrix} -p_{B,A} & 0 & 0 \\ -p_{A,B} & 0 & 1 - 2q_A \\ 1 & 0 & 2q_A - 1 \end{pmatrix}.$$

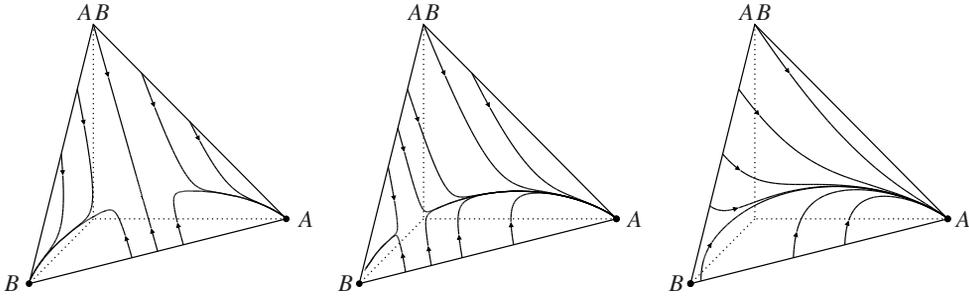


FIGURE 1: Solution curves of the mean-field model with $\phi = 1$, $\phi = 1.5$, and $\phi = 4$, respectively.

The eigenspace associated with the eigenvalue 0 is generated by the vector $(0, 1, 0)$ which is not parallel to the two-simplex containing the solution curves. The other two eigenvalues are $-p_{B,A} = -(\phi + 1)^{-1} < 0$ and $2q_A - 1 = (\phi - 3)(\phi + 3)^{-1}$, both of which are negative when $\phi < 3$. In particular, for all $\phi < 3$, the equilibrium e_B is locally stable; therefore, word A cannot invade.

The obvious symmetry of the model now implies that both trivial equilibria are locally stable when $\frac{1}{3} < \phi < 3$. Numerical simulations of the mean-field model suggest that, in this case, there is an additional nontrivial fixed point which is a saddle point; therefore, the system is bistable: for almost all initial conditions, the system converges to one of the two trivial equilibria (see Figure 1 for the solution curves).

1.3. Spatial stochastic model

We now look at the spatial stochastic naming game defined by (1.1). For the stochastic process, the main objective is to study the probability that word A invades the population and is selected as a new linguistic convention when starting with a single bilingual individual and all other individuals are of type B . Most of our discussion until the end of Section 3 concerns homogeneous graphs; onwards from Section 4 we consider nonhomogeneous graphs for which this probability depends on the location of the initial bilingual individual.

Let $\eta_t(x)$ be the state of the individual at vertex x at time t , and let \mathbb{P}_x denote the law of the process starting with $\eta_0(x) = AB$ and $\eta_0(y) = B$ for all $y \in V, y \neq x$. Define the *probability of invasion* by

$$p_A := \inf_{x \in V} \mathbb{P}_x \left\{ \lim_{t \rightarrow \infty} \eta_t(y) = A \text{ for all } y \in V \right\}.$$

Our results indicate that p_A depends strongly on the topology of the network of interactions, suggesting that on regular graphs it is decreasing with respect to the degree of the network; this property cannot be captured by the mean-field model because that model excludes any spatial structure. In particular, we expect p_A to be minimal on complete graphs, with a critical value for the ratio of the fitnesses equal to 3 as in the mean-field model, and maximal on the one-dimensional lattice, with a critical value for the ratio significantly smaller. Our results strongly support this conjecture.

We start by looking at finite graphs for which our first theorem extends the first result found for the mean-field model.

Theorem 1.1. *Let G be finite, and suppose that $\phi > 3$. Then $p_A \geq 1 - \max\{3\phi^{-1}, 3^{-1}\} > 0$.*

Note that, on finite graphs, p_A is always positive but may tend to 0 as the population size increases to ∞ . In contrast, Theorem 1.1 shows more particularly that p_A is bounded from

below by a constant that depends on ϕ but not on the number of vertices. The idea of the proof is to show first that a certain function of the numbers of type- A and type- B individuals is a supermartingale with respect to the σ -algebra generated by the process and then to apply the optimal stopping theorem. The result can be understood intuitively as follows: when $\phi > 3$, each time there is a conversation between a type- B individual and a bilingual individual, the probability that word A is spoken is given by $p_{AB,B} p_{A,B} = q_A$, with

$$q_A = \frac{\phi_{AB} \phi_A}{(\phi_{AB} + \phi_B)(\phi_A + \phi_B)} = \frac{\phi_A}{\phi_A + 3\phi_B} = \frac{\phi}{\phi + 3} > \frac{1}{2}, \tag{1.3}$$

implying that, in each conversation involving two individuals both of whom know both words, A is always more likely to be spoken and, therefore, A becomes the new linguistic convention with positive probability uniform in the population size.

Our next result indicates that the invadability condition in Theorem 1.1 is sharp for complete graphs in the sense that, when $\phi < 3$, $p_A \rightarrow 0$ as $N \rightarrow \infty$.

Theorem 1.2. *Let G be the complete graph with N vertices. Then, for all $\phi < 3$,*

$$\lim_{N \rightarrow \infty} p_A = 0.$$

In the proof of Theorem 1.1, the dynamics of the numbers of type- A and type- B individuals are expressed as a function of the number of edges of different types. The complete graph is the only graph for which the number of edges of different types can be expressed as a function of the number of individuals of different types. Also, one of the keys to proving the theorem is to use the fact that, on complete graphs, the number of individuals in different states becomes a Markov chain. Taking both theorems together indicates that the dynamics of the naming game on complete graphs are captured well by the mean-field approximation.

Our next result shows more interestingly that this is not true for the process on the infinite one-dimensional lattice, suggesting that the probability of invasion decreases with the degree of the graph.

Theorem 1.3. *In one dimension, $p_A > 0$ whenever $\phi > c$, where*

$$c := \frac{1}{96}(23 + \sqrt{6097}) \approx 1.053 \quad \text{satisfies} \quad 48c^2 - 23c - 29 = 0.$$

The proof of Theorem 1.3 is based on the analysis of the interface between individuals in different states, which is only possible in one dimension. The bound c is not sharp, but our approach to proving the theorem together with the obvious symmetry of the model implies that the critical ratio is between c^{-1} and c , which suggests that the critical ratio is equal to 1, i.e. that the probability of a successful invasion is positive if and only if $\phi > 1$.

Finally, we look at the naming game on regular lattices in higher dimensions. In this case, using a block construction to compare the process properly rescaled in space and time with oriented site percolation (see, e.g. [3]), it can be proved that the probability of invasion is positive for sufficiently large ϕ .

Theorem 1.4. *In any dimension, $p_A > 0$ whenever ϕ is large enough.*

Our approach to proving the theorem can be used to obtain an explicit bound for the critical value of ϕ , but this bound is far from being optimal. We conjecture as in one dimension that the critical ratio is equal to 1; this is supported by numerical simulations of the process. More generally, we conjecture that, on connected graphs in which the degree is uniformly bounded by

a fixed constant d , the critical value is equal to one in the sense that the probability of invasion is bounded from below by a positive constant that depends only on d , in disagreement with the mean-field model.

Before giving details of our proofs, we conclude with some comments on natural generalizations of our biased naming game involving n words, say

words $1, 2, \dots, n$ with respective fitnesses $\phi_1, \phi_2, \dots, \phi_n$.

In this case, the numerical results of Baronchelli *et al.* [1] suggest that, even in the neutral case when all the fitnesses are equal, the process clusters so a word is selected at random to become the linguistic convention. Allowing the fitnesses to be different, standard coupling arguments for particle systems show that the probability of invasion of word 1 is nondecreasing with respect to ϕ_1 and nonincreasing with respect to the other fitnesses, so, everything else being fixed, we again expect the existence of a unique phase transition: word 1 invades with a positive probability uniform in the size of the graph if and only if ϕ_1 is larger than some nondegenerate critical value. In fact, our proof of Theorem 1.4 easily extends to the naming game with n words to show that, for the system starting with a single individual with word 1 in her internal inventory,

$$\mathbb{P}\left\{\lim_{t \rightarrow \infty} \eta_t(x) = 1 \text{ for all } x \in \mathbb{Z}^d\right\} > 0 \quad \text{when } \min_{i \neq 1} \{\phi_1/\phi_i\} \text{ is large.}$$

Note however that the inclusion of additional words may modify the critical value for the ratio of the fitnesses. Assuming for instance that $\phi_2 = \dots = \phi_n$ and observing that the worst-case scenario for word 1 to be spoken is when an individual who knows all n words interacts with an individual who does not know word 1, the heuristic argument (1.3) suggests that word 1 can invade whenever

$$(\phi_1 + (n - 1)\phi_2)(\phi_1 + (2n - 1)\phi_2)^{-1} \phi_1 (\phi_1 + (n - 1)\phi_2)^{-1} > \frac{1}{2},$$

which gives the condition for invadability, $\phi_1/\phi_2 > 2n - 1$. Our proof of Theorem 1.1 easily extends to show that this condition is indeed a sufficient condition for invadability of word 1 on finite graphs. Also, even though our proof of Theorem 1.2 does not extend to this case, we conjecture that the critical value for the ratio ϕ_1/ϕ_2 is strictly increasing with respect to the number of words n .

2. Preliminary results

In this section we describe some basic properties of the naming game that are useful in subsequent sections. A common aspect of all our proofs is to think of the process as being constructed graphically from independent Poisson processes that indicate the time of the interactions, a popular idea in the field of interacting particle systems due to Harris [6]. In the case of the naming game, additional collections of uniform random variables must be introduced to also indicate the outcome of each interaction. More precisely, for every edge $(x, y) \in E$,

- (a) let $\{T_n(x, y) : n \geq 1\}$ be the arrival times of a rate-1 Poisson process;
- (b) let $\{U_n(x, y) : n \geq 1\}$ be independent uniform random variables on $(0, 1)$; and
- (c) collections of random variables attached to different edges are also independent.

TABLE 1: Coupling between the processes (η_t) and (ξ_t) .

	η_t -transition	$U_n(x, y)$ condition	Possible ξ_t -transition
1A	$(A, A) \rightarrow (A, A)$	None	Any
2A	$(A, AB) \rightarrow (A, A)$	$U_n(x, y) < 1 - q_B$	2A, 3A, 3B, 4A, 4B, 5A, 5B, 6B
2B	$(A, AB) \rightarrow (AB, AB)$	$U_n(x, y) > 1 - q_B$	2B, 3B, 4B, 5A, 5B, 6B (excludes 3A, 4A)
3A	$(A, B) \rightarrow (A, AB)$	$U_n(x, y) < p_{A,B}$	3A, 5A, 5B, 6B
3B	$(A, B) \rightarrow (AB, B)$	$U_n(x, y) > p_{A,B}$	3B, 5B, 6B (excludes 5A)
4A	$(AB, AB) \rightarrow (A, A)$	$U_n(x, y) < p_{A,B}$	4A, 5A, 5B, 6B
4B	$(AB, AB) \rightarrow (B, B)$	$U_n(x, y) > p_{A,B}$	4B, 5B, 6B (excludes 5A)
5A	$(AB, B) \rightarrow (AB, AB)$	$U_n(x, y) < q_A$	5A, 6B
5B	$(AB, B) \rightarrow (B, B)$	$U_n(x, y) > q_A$	5B, 6B
6B	$(B, B) \rightarrow (B, B)$	None	Only 6B

The process is then constructed as follows: at time $T_n(x, y)$, the states at x and y are simultaneously updated according to the transitions in the left column of Table 1. Since interactions involving both words can each result in two different outcomes depending on whether word A or word B is spoken, the upper (respectively, lower) bound on the random variable $U_n(x, y)$ in the middle column of the table gives the probability (respectively, the complement of the probability) of the indicated transition, listed earlier at (1.1), where

$$q_A := p_{AB,B} p_{A,B} \quad \text{and} \quad q_B := p_{AB,A} p_{B,A}.$$

Thus, q_A is the probability that word A is spoken in a conversation involving a bilingual individual and a type- B individual. Based on this graphical representation, processes with different parameters or starting from different initial configurations can be coupled to prove important monotonicity results. For instance, our first lemma shows a certain monotonicity of the naming game with respect to its initial configuration; it can be viewed as the analog of attractiveness for spin systems. We use this result in proving Theorem 1.3.

Lemma 2.1. *Let (η_t) and (ξ_t) be two copies of the naming game. Then*

$$\mathbb{P}\{\xi_t(x) = A\} \leq \mathbb{P}\{\eta_t(x) = A\} \quad \text{and} \quad \mathbb{P}\{\xi_t(x) = B\} \geq \mathbb{P}\{\eta_t(x) = B\}$$

for all $(x, t) \in V \times (0, \infty)$ provided this holds for all $(x, t) \in V \times \{0\}$.

Proof. The result follows from a coupling of the two processes constructed simultaneously from the same graphical representation. Assume that, for all $z \in V$,

$$\{\xi_0(z) = A \Rightarrow \eta_0(z) = A\} \quad \text{and} \quad \{\eta_0(z) = B \Rightarrow \xi_0(z) = B\},$$

and that both processes are constructed from the same Poisson processes and the same collections of uniform random variables. The construction given by Harris [6], which relies on arguments from percolation theory, implies that, for any small enough time interval, there exists a partition of the vertex set into almost surely finite connected components such that any two vertices in two different components do not influence each other in the time interval. Since the number of interactions in each component in the time interval is almost surely finite, the result can be proved for each of these finite space-time regions by induction. Assume that, for all $z \in V$,

$$\{\xi_{t-}(z) = A \Rightarrow \eta_{t-}(z) = A\} \quad \text{and} \quad \{\eta_{t-}(z) = B \Rightarrow \xi_{t-}(z) = B\}$$

for some arrival time $t := T_n(x, y)$. To prove that the previous relationship between both

processes is preserved at time t , observe that the interaction between the individuals at x and y can result in ten different transitions depending on the state of both individuals. These transitions are listed in the left-hand column of Table 1 and can be divided into two types:

- (i) the transitions that create an A or remove a B , which are labeled 2A–5A,
- (ii) the transitions that create a B or remove an A , which are labeled 2B–5B.

As previously mentioned, except for transitions 1A and 6B, every other pair of states for the neighbors can result in two possible transitions depending on whether word A or word B is spoken during the conversation. The last column of the table indicates that, for all possible simultaneous updates of both processes, the ordering between both processes is preserved at time t , i.e. for all $z \in V$,

$$\{\xi_t(z) = A \Rightarrow \eta_t(z) = A\} \quad \text{and} \quad \{\eta_t(z) = B \Rightarrow \xi_t(z) = B\}.$$

Referring again to Table 1, to prove, as indicated in the last column, that a transition 2B in the first process indeed excludes the transitions 3A and 4A in the second process, observe that

$$1 - q_B = p_{A,B} + p_{B,A} - p_{AB,A} \quad p_{B,A} \geq p_{A,B},$$

so that

$$\{U_n(x, y) > 1 - q_B\} \implies \{U_n(x, y) > p_{A,B}\}, \tag{2.1}$$

proving the exclusion of type-3A and type-4A transitions in Table 1. Similarly,

$$\{U_n(x, y) > p_{A,B}\} \implies \{U_n(x, y) > p_{AB,B} \quad p_{A,B} = q_A\}, \tag{2.2}$$

showing that the transitions 3B and 4B in the first process exclude transition 5A in the second process. The lemma now follows from the fact that all possible simultaneous updates of both processes given in the last column preserve the desired ordering.

3. The naming game on finite graphs

This section is devoted to the proofs of Theorems 1.1 and 1.2 about the naming game on finite connected graphs. The key to proving Theorem 1.1 is to show that a certain process that depends on the difference between the number of individuals using word A and the number using word B is a supermartingale with respect to the natural filtration of the naming game; this allows us to deduce the theorem directly from the optimal stopping theorem. To prove Theorem 1.2, which specializes Theorem 1.1 to the process on complete graphs, the idea is to observe that, as long as bilingual individuals do not interact with each other, there is no type- A individual in the population and the number of bilingual individuals evolves as a subcritical birth-and-death process that goes extinct quickly. Throughout this section, for all $X, Y \in \{A, B, AB\}$, let

$$X_t := \text{number of type-}X \text{ individuals at time } t,$$

$$e_t(X, Y) := \text{number of edges connecting individuals of types } X \text{ and } Y \text{ at time } t.$$

To motivate our proof of Theorem 1.1 and explain the assumption, observe that the transitions labeled 2A–5A in Table 1, which are the transitions that increase the number of individuals using A or decrease the number of individuals using B , all occur with probability at least one half if and only if $\phi > 3$. As shown in the next lemma, this property can be used to construct a certain supermartingale with respect to the natural filtration of the process: the σ -algebra \mathcal{F}_t generated by the realization of the naming game until time t .

Lemma 3.1. Assume that $\phi \geq 3$, and set $M_t = a^{A_t - B_t}$, where $a := \max\{3/\phi, \frac{1}{3}\}$. Then, for all $s > t$,

$$\mathbb{E}[M_s \mid \mathcal{F}_t] \leq M_t.$$

Proof. Using the transition probabilities in Table 1,

$$\begin{aligned} & \lim_{h \rightarrow 0} h^{-1} \mathbb{E}[M_{t+h} - M_t \mid \mathcal{F}_t] \\ &= \sum_{j=-2}^2 (a^j - 1) M_t \lim_{h \rightarrow 0} h^{-1} \mathbb{P}\{M_{t+h} = M_t + j \mid \mathcal{F}_t\} \\ &= (a - 1) M_t [e_t(A, AB)(1 - q_B) + e_t(A, B)p_{A,B} + e_t(B, AB)q_A] \\ &\quad + (a^{-1} - 1) M_t [e_t(B, AB)(1 - q_A) + e_t(A, B)p_{B,A} + e_t(A, AB)q_B] \\ &\quad + M_t e_t(AB, AB) [(a^2 - 1) p_{A,B} + (a^{-2} - 1) p_{B,A}] \\ &= M_t (e_t(A, AB)[(a - 1)(1 - q_B) + (a^{-1} - 1)q_B] \\ &\quad + e_t(B, AB)[(a - 1)q_A + (a^{-1} - 1)(1 - q_A)] \\ &\quad + e_t(A, B)[(a - 1)p_{A,B} + (a^{-1} - 1)p_{B,A}] \\ &\quad + e_t(AB, AB) [(a^2 - 1)p_{A,B} + (a^{-2} - 1)p_{B,A}]). \end{aligned}$$

In the last expression, M_t is nonnegative and uniformly bounded (because N is finite), and so too are all four quantities $e_t(A, AB), \dots, e_t(AB, AB)$. Each of these quantities has a multiplier of the form

$$(\alpha - 1)(1 - \varpi) + (\alpha^{-1} - 1)\varpi = (\alpha - 1)(1 - \varpi(1 + \alpha^{-1})),$$

where $(\alpha, \varpi) = (a, q_B), (a, 1 - q_A), (a, 1 - p_{A,B}),$ and $(a^2, 1 - p_{A,B}),$ respectively. The multiplier of $(\alpha - 1)$ here is nonnegative if and only if $\varpi^{-1} < 1 + \alpha^{-1}$. In each of these four cases this condition is met if and only if (and here we use the relations at (1.2))

$$a < \frac{1}{3\phi}, \quad a < \frac{3}{\phi}, \quad a < \frac{1}{\phi}, \quad a^2 < \phi,$$

respectively. For all $\phi \geq 3$ and a as stated, all these inequalities hold, and consequently we then have

$$\lim_{h \rightarrow 0} h^{-1} \mathbb{E}[M_{t+h} - M_t \mid \mathcal{F}_t] \leq 0,$$

showing that (M_t) is a supermartingale for $a = \max\{3\phi^{-1}, 3^{-1}\}$.

Applying the optimal stopping theorem to (M_t) gives the following result.

Lemma 3.2. For all $\phi > 3$,

$$p_A := \inf_{x \in V} \mathbb{P}_x\{A_t = N \text{ for some } t\} \geq 1 - \max\{3\phi^{-1}, 3^{-1}\} > 0.$$

Proof. Throughout the proof, we consider the naming game starting with a single bilingual individual at vertex x where vertex x is such that

$$p_A := \mathbb{P}_x\{A_t = N \text{ for some } t\}.$$

Note that the existence of such a vertex follows from the finiteness of the vertex set. In order to apply the optimal stopping theorem, introduce the stopping time

$$T := \inf\{t : A_t - B_t \in \{-N, N\}\},$$

where N denotes the number of vertices. Since the naming game on any finite graph converges almost surely to the configuration in which all individuals are monolingual of the same type, the stopping time T is finite almost surely. Moreover, by the definition of x ,

$$\mathbb{P}_x\{A_T - B_T = N\} = 1 - \mathbb{P}_x\{A_T - B_T = -N\} = p_A.$$

Using the fact that (M_t) is a supermartingale, deduce from the optimal stopping theorem that, whenever $a = \max(3\phi^{-1}, 3^{-1}) < 1$,

$$\mathbb{E}[M_T] = \mathbb{E}[a^{A_T - B_T}] = a^N p_A + a^{-N} (1 - p_A) \leq \mathbb{E}[M_0] = \mathbb{E}[a^{A_0 - B_0}] = a^{-(N-1)}.$$

In particular,

$$p_A \geq \frac{a^{-(N-1)} - a^{-N}}{a^N - a^{-N}} = \frac{1 - a}{1 - a^{2N}} \geq 1 - a = 1 - \max\{3\phi^{-1}, 3^{-1}\} > 0,$$

proving the lemma.

Theorem 1.1 follows directly from Lemma 3.2 because the probability p_A is the same in the statements of both the lemma and the theorem.

We now specialize to the complete graph. In this case, the number of edges of each type can be expressed as a function of the number of individuals of each type, so (A_t, B_t) is now a continuous-time Markov chain. As previously mentioned, to prove that p_A tends to 0 as the number of vertices goes to ∞ , the idea is to observe that, as long as bilingual individuals do not interact with each other, there is no monolingual individual of type A and the number of bilingual individuals evolves according to an inhomogeneous time change of a subcritical birth-and-death process.

To make the argument precise, call the event that two bilingual individuals interact a *collision*. Let

$$\tau_C := \inf\{t : t = T_n(x, y) \text{ for some } x, y \in V \text{ for which } \eta_{t-}(x) = \eta_{t-}(y) = AB\}$$

be the time of the first collision. Observe that word A invades the system only if this stopping time τ_C is finite; this is one of the main keys to our proof. We also introduce the linear birth-and-death process (Z_t) starting with a single individual, and with birth rate q_A and death rate $1 - q_A$, i.e.

$$\lim_{h \rightarrow 0} h^{-1} \mathbb{P}\{Z_{t+h} = j \mid Z_t = i\} = \begin{cases} i q_A & \text{for } j = i + 1, \\ i (1 - q_A) & \text{for } j = i - 1. \end{cases}$$

The next lemma relies on basic properties of (Z_t) to show that the probability that a collision ever happens tends to 0 as the number of individuals goes to ∞ .

Lemma 3.3. *For every $\phi < 3$ and $\varepsilon > 0$, there exists finite N_ε such that*

$$\mathbb{P}\{\tau_C < \infty \mid A_0 = 0 \text{ and } B_0 = N - 1\} < \varepsilon \text{ for all } N > N_\varepsilon.$$

Proof. Start by observing that, before the time τ_C of the first collision, there is no monolingual individual of type A in the population. In particular, using the expression for the transition probabilities in the second column of Table 1 and introducing

$$r_t(i, j) := \lim_{h \rightarrow 0} h^{-1} \mathbb{P}\{A_{t+h} = A_t + i \text{ and } B_{t+h} = B_t + j \mid \mathcal{F}_t\},$$

we obtain, before the first collision,

$$\begin{aligned} r_t(0, -1) &= q_A e_t(B, AB), & r_t(+2, 0) &= p_{A,B} e_t(AB, AB), \\ r_t(0, +1) &= (1 - q_A) e_t(B, AB), & r_t(0, +2) &= p_{B,A} e_t(AB, AB); \end{aligned}$$

while $r_t(i, j) = 0$ for all other (i, j) . The two transition rates on the left indicate that, before the first collision, the number of bilingual individuals evolves according to an inhomogeneous time change of the birth-and-death process (Z_t) . In particular, let

$$J := \text{card}\{t : Z_t \neq Z_{t-}\} \quad \text{and} \quad K := \text{card}\{t < \tau_C : \eta_t \neq \eta_{t-}\}$$

be the total numbers of jumps of the birth-and-death process and of the naming game before the first collision, respectively. Then J is stochastically larger K , that is,

$$\mathbb{P}\{J < n\} \leq \mathbb{P}\{K < n\} \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

In addition, since the fitness ratio $\phi < 3$,

$$q_A = p_{AB,B} p_{A,B} = \frac{\phi_A/2}{\phi_{AB} + \phi_B} = \frac{\phi}{\phi + 3} < \frac{3}{\phi + 3} = 1 - q_A,$$

from which it follows that the birth-and-death process dies out, so

$$\mathbb{P}\{J < \infty \mid Z_0 = 1\} = \mathbb{P}\{Z_t = 0 \text{ for some finite } t \mid Z_0 = 1\} = 1. \tag{3.2}$$

With these preliminary results at hand, the lemma follows by conditioning on the possible values of the number of jumps K before the first collision. First, combining (3.1) and (3.2), we obtain the existence of a large n_ε , fixed from now on, such that

$$\mathbb{P}\{\tau_C < \infty \mid K \geq n_\varepsilon\} \mathbb{P}\{K \geq n_\varepsilon\} \leq \mathbb{P}\{K \geq n_\varepsilon\} \leq \mathbb{P}\{J \geq n_\varepsilon\} < \frac{1}{2}\varepsilon, \tag{3.3}$$

indicating that the number of jumps by time τ_C cannot be too large. Now, in the event that the number of jumps is not too large, the probability that a collision ever occurs is small on large graphs. Indeed, when $K < n_\varepsilon$, the number of bilingual individuals before the first collision cannot exceed n_ε , so, at each jump, the probability of a collision is bounded by $N^{-1}n_\varepsilon$. In particular, the conditional probability given that the number of jumps is small is

$$\mathbb{P}\{\tau_C < \infty \mid K < n_\varepsilon\} \mathbb{P}\{K < n_\varepsilon\} \leq \mathbb{P}\{\tau_C < \infty \mid K < n_\varepsilon\} \leq N^{-1}(n_\varepsilon)^2 < \frac{1}{2}\varepsilon \tag{3.4}$$

for all sufficiently large N . The lemma simply follows by observing that the probability to be estimated is bounded by the sum of the probabilities in (3.3) and (3.4).

Theorem 1.2 directly follows from the next lemma.

Lemma 3.4. Fix $\phi < 3$ and $\varepsilon > 0$. Then, for all sufficiently large N ,

$$\mathbb{P}\{A_t = N \text{ for some } t \mid A_0 = 0 \text{ and } (AB)_0 = 1\} < \varepsilon.$$

Proof. Since there is no type- A individual before the first collision,

$$\begin{aligned} &\mathbb{P}\{A_t = N \text{ for some } t \mid A_0 = 0 \text{ and } (AB)_0 = 1\} \\ &\leq \mathbb{P}\{\eta_t(x) = A \text{ for some } (x, t) \in V \times \mathbb{R}_+ \mid A_0 = 0 \text{ and } (AB)_0 = 1\} \\ &\leq \mathbb{P}\{\tau_C < \infty \mid A_0 = 0 \text{ and } (AB)_0 = 1\} \\ &< \varepsilon \end{aligned}$$

for all sufficiently large N according to Lemma 3.3.

4. The naming game in one dimension

This section is devoted to the proof of Theorem 1.3, in which without loss of generality the network graph consists of the lattice points $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$ on the x -axis, and the only edges joined to vertex x are $(x, x + 1)$ and $(x, x - 1)$. Our ultimate goal is to establish a sufficient condition for it to be possible for a single vertex with inventory A to invade a population in which the common inventory of all others is B , i.e. we give a condition implying that $p_A > 0$. We do this in two stages. First we establish a condition under which, for every vertex x , $\lim_{t \rightarrow \infty} \mathbb{P}\{\eta_t(x) = A\} = 1$ when initially

$$\eta_0(x) = A \quad \text{for all } x \leq 0 \quad \text{and} \quad \eta_0(x) = B \quad \text{for all } x > 0. \tag{4.1}$$

We then use this to show that $p_A > 0$ for the weaker initial configuration

$$\eta_0(0) = A \quad \text{and} \quad \eta_0(x) = B \quad \text{for all } x \in \mathbb{Z} \setminus \{0\}. \tag{4.2}$$

The main difficulty in proving the result for (η_t) starting from (4.1) is that the evolution rules in (1.1) can create infinitely many possible interface configurations, where the *interface* consists of the region from the rightmost type A that has only type A to her left to the leftmost type B that has only type B to her right. In particular, the size of the interface is not bounded.

Numerical simulations, however, suggest that the size of the interface is relatively small most of the time. We therefore investigate a process (ξ_t) which is modified from (η_t) in such a way that its interface is ‘small’ *all* the time; this fact makes it mathematically tractable. We require the process (ξ_t) to satisfy the constraint

$$\xi_t(X_t^\xi + j) = B \quad \text{for all } j \geq 3, \tag{4.3}$$

where $X_t^\xi := \sup\{x \in \mathbb{Z} : \xi_t(y) = A \text{ for all } y \leq x\}$. This constraint is achieved by defining (ξ_t) to be the process that has initial configuration (4.1) and evolves according to the naming game protocol at (1.1) except that, at any instant that the configuration would otherwise violate condition (4.3), the inventory at vertex $X_t^\xi + 3$ is immediately set to B . This new rule and Lemma 2.1 imply that, for all $x \in \mathbb{Z}$ and $t > 0$,

$$\mathbb{P}\{\xi_t(x) = A\} \leq \mathbb{P}\{\eta_t(x) = A\} \quad \text{and} \quad \mathbb{P}\{\xi_t(x) = B\} \geq \mathbb{P}\{\eta_t(x) = B\},$$

from which it follows that $\lim_{t \rightarrow \infty} \mathbb{P}\{\eta_t(x) = A\} = 1$ for all $x \in \mathbb{Z}$ whenever

$$\lim_{t \rightarrow \infty} X_t^\xi = \infty \text{ almost surely.} \tag{4.4}$$

Our goal now is to exhibit conditions under which (4.4) holds.

It is easily checked that at any time $u \geq 0$, the interface configuration $Y(u)$, say, of the process (ξ_t) must be one of the following three types.

- $Y(u) = (0)$: $\xi_u(X_u^\xi + j) = B$ for all $j \geq 1$.
- $Y(u) = (1)$: $\xi_u(X_u^\xi + 1) = AB$ and $\xi_u(X_u^\xi + j) = B$ for all $j \geq 2$.
- $Y(u) = (2)$: $\xi_u(X_u^\xi + 1) = \xi_u(X_u^\xi + 2) = AB$ and $\xi_u(X_u^\xi + j) = B$ for all $j \geq 3$.

Furthermore, for any $s > 0$ at which a transition occurs, starting from $Y(s-) = (0)$ or (1) , the only possible transitions $Y(s-) \mapsto Y(s+)$ are $(0) \mapsto (1)$, $(1) \mapsto (0)$, and $(1) \mapsto (2)$,

TABLE 2: Matrix of transition rates: $(Y(s-)) \mapsto (Y(s+), X_{s+}^\xi - X_{s-}^\xi)$.

\nearrow	(0, 0)	(0, 1)	(0, 2)	(1, -1)	(1, 0)	(1, 1)	(2, -1)	(2, 0)
(0)	0	0	0	$1 - p_{A,B}$	$p_{A,B}$	0	0	0
(1)	$1 - q_A$	$1 - q_B$	0	0	0	0	q_B	q_A
(2)	$1 - p_{A,B}$	0	$p_{A,B}$	0	$1 - q_A$	$1 - q_B$	q_B	q_A

TABLE 3: Nonnegative transition rates of the configuration process $Y(\cdot)$, where $r := q_A + q_B$.

\nearrow	(0)	(1)	(2)
(0)	0	1	0
(1)	$2 - r$	0	r
(2)	1	$2 - r$	r

while, when $Y(s-) = (2)$, so that the vertices $\{X_{s-}^\xi, X_{s-}^\xi + 1, X_{s-}^\xi + 2, X_{s-}^\xi + 3\}$ have the respective inventories $\{A, AB, AB, B\}$, it is one of the three edges determined by these four vertices that becomes active and makes $(\xi_{s+}) \neq (\xi_{s-})$. In Table 2 we present the nonzero rates of all possible transitions from configuration (j) , $j = 0, 1, 2$, to the pair (j, k) , where k denotes the difference $X_{s+}^\xi - X_{s-}^\xi$ at the transition epoch s .

The functional $Y(\cdot)$ of (ξ_t) is in fact a continuous-time Markov chain on the state space $\{(0) (1) (2)\}$; its nonnegative transition rates can be compiled from Table 2 and are shown in Table 3.

Then $Y(\cdot)$, being irreducible on a finite state space, has a stationary distribution (π_0, π_1, π_2) and is ergodic. This distribution can be computed from the stationarity equations in Lemma 4.1 below, where the left-hand side is the (stationary) rate of events $\{Y(s-) = j \neq Y(s+)\}$ (i.e. exit from state j) and the right-hand side the rate of entry into state j . Solving the system of linear equations, two at (4.5) and the relation $\sum_{j=0}^2 \pi_j = 1$, leads to (4.6).

Lemma 4.1. *The limits $\pi_j := \lim_{t \rightarrow \infty} \mathbb{P}\{Y(t) = j\}$ exist and, with $r := q_A + q_B$, satisfy*

$$\pi_0 = (2 - r)\pi_1 + \pi_2, \quad (3 - r)\pi_2 = r \pi_1, \tag{4.5}$$

so that

$$(\pi_0 \ \pi_1 \ \pi_2) = \left(\frac{6 - 4r + r^2}{9 - 4r + r^2} \ \frac{3 - r}{9 - 4r + r^2} \ \frac{r}{9 - 4r + r^2} \right). \tag{4.6}$$

To establish Theorem 1.3, we first prove that (4.4) holds under the conditions of the theorem. The first ingredient in this proof is the stationary distribution at (4.6). For the rest, observe that transitions in the process (ξ_t) can occur only at a subset of the points of a Poisson process at rate 3 (because at any point in time there can never be more than three edges that can become active at the points of the Poisson processes $T_n(\text{edge})$ as in Section 2). Then the process X_t^ξ , which can change only at such points and then by at most 2 as in Table 2, has its rate of change bounded by 6, uniformly. Consequently, $\sup_{t \geq 0} \mathbb{E}[|X_{t+h}^\xi - X_t^\xi|] \leq 6h$ for all $h > 0$. These change points are the jump epochs of the continuous-time irreducible Markov process Y on finite state space, and Y has a well-defined stationary distribution, so it follows that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[X_t^\xi]}{t} = \sum_{j=0}^2 \pi_j \lim_{h \downarrow 0} \frac{\mathbb{E}[X_{t+h}^\xi - X_t^\xi \mid Y(t) = j]}{h} = \sum_{j=0}^2 \pi_j D_j, \tag{4.7}$$

where the mean drift rates $D_j := \lim_{h \downarrow 0} h^{-1} \mathbb{E}[X_{t+h}^\xi - X_t^\xi \mid Y(t) = j]$ for $j = 0, 1, 2$. From Table 2 and (1.2),

$$D_0 = -p_{B,A} = -\frac{1}{\phi + 1}, \tag{4.8a}$$

$$D_1 = (1 - q_B) - q_B = 1 - \frac{2}{3\phi + 1}, \tag{4.8b}$$

$$D_2 = (1 - q_B) - q_B + 2 p_{A,B} = 2 + 2D_0 + D_1. \tag{4.8c}$$

Finally, since Y is ergodic, the Markov chain property

$$X_t^\xi = \sum_{s \in [0,t]} (X_{s+}^\xi - X_{s-}^\xi) \sim (\pi_0 D_0 + \pi_1 D_1 + \pi_2 D_2)t \quad \text{as } t \rightarrow \infty,$$

where the summation is over transition epochs s of Y , shows that (4.4) holds provided the sum at (4.7) is positive, i.e. to prove Theorem 1.3, it suffices to show that

$$\pi_0 D_0 + \pi_1 D_1 + \pi_2 D_2 > 0 \quad \text{for all } \phi > c. \tag{4.9}$$

A standard approach now is to exploit expressions for the π_j and D_j as functions of ϕ as in (4.6) and (4.8). Doing so shows that positivity as at (4.9) holds for ϕ larger than the largest real root of a certain polynomial of degree 6, but it is not obvious how to compute this root. Instead, we observe that, when both fitnesses are close to each other, ϕ is close to 1 and the rate r close to $\frac{1}{2}$. The next two lemmas show that the left-hand side of (4.7) is larger than its counterpart obtained by computing π_j under the assumption that $r = \frac{1}{2}$, which allows us to express c more simply as the largest root of a polynomial of degree 2. (Interestingly, evaluating the polynomial of degree 6 around c indicates that the largest real root of this polynomial differs from c by less than 10^{-6} , showing albeit *a posteriori* the advantage of our approach.)

Lemma 4.2. *For all positive ϕ_A and ϕ_B , $\frac{1}{2} \leq r \leq 1$.*

Proof. For $u > 0$, $2 \leq u + u^{-1} < \infty$, so, for $b \geq 1$, the function

$$h(u; b) := \frac{1}{1 + bu} + \frac{1}{1 + bu^{-1}} = 1 - \frac{b^2 - 1}{1 + b(u + u^{-1}) + b^2}$$

satisfies $2/(1 + b) \leq h(u; b) \leq 1$ for $u > 0$. The lemma now follows immediately on writing, from (1.2),

$$r = \frac{1}{1 + 3/\phi} + \frac{1}{1 + 3\phi} = h\left(\frac{\phi_A}{\phi_B}; 3\right).$$

Lemma 4.3. *For all positive ϕ_A and ϕ_B ,*

$$\text{sgn}(\pi_0 D_0 + \pi_1 D_1 + \pi_2 D_2) \geq \text{sgn}(48\phi^2 - 23\phi - 29). \tag{4.10}$$

Proof. Substitution of the expressions for π_0 into (4.5) and D_2 into (4.8) gives

$$\begin{aligned} \text{sgn}\left(\sum_{j=0}^2 \pi_j D_j\right) &= \text{sgn}([2\pi_1 - (2 - r)\pi_2]D_0 + \pi_1 D_1 + \pi_2[2 + 2D_0 + D_1]) \\ &= \text{sgn}([2D_0 + D_1]\pi_1 + [2 + rD_0 + D_1]\pi_2) \\ &= \text{sgn}([2D_0 + D_1](3 - r) + [2 + rD_0 + D_1]r), \end{aligned}$$

where in the last step we have used (4.6) and deleted the nonnegative factor $1/(9 - 4r + r^2)$. The last argument is expressible, on using (4.8), as

$$2r + [6 - 2r + r^2]D_0 + 3D_1 = 2r - \frac{6 - 2r + r^2}{\phi + 1} + 3 - \frac{6}{3\phi + 1} =: g(\phi, r),$$

where the bivariate function g is defined on $\frac{1}{2} \leq r < 1$ and $\phi \geq 1$. By inspection, g is increasing in ϕ for every r , while $\partial g/\partial r = 2 + 2(1 - r)/(\phi + 1) > 0$, so g is also increasing in r for every ϕ . By Lemma 7, $r \geq \frac{1}{2}$, so, since $\text{sgn}(\cdot)$ is a nondecreasing function,

$$\text{sgn}(g(\phi, r)) \geq \text{sgn}(g(\phi, \frac{1}{2}))$$

and

$$g\left(\phi, \frac{1}{2}\right) = 4 - \frac{21/4}{\phi + 1} - \frac{6}{3\phi + 1} = \frac{48\phi^2 - 23\phi - 29}{4(\phi + 1)(3\phi + 1)}. \tag{4.11}$$

This proves Lemma 4.3. The quadratic expression in the numerator is positive when ϕ exceeds its larger zero which equals c as asserted in Lemma 4.4 below.

Lemma 4.4. *Let c be the larger zero of (4.11). The right-hand side of (4.10) is positive whenever*

$$\phi > c, \quad \text{where } c := \frac{23 + \sqrt{6097}}{96} \approx 1.053.$$

The condition $\phi > c$ implies that $\mathbb{P}\{X_t^\xi \rightarrow \infty \text{ as } t \rightarrow \infty\} = 1$, so $\tau := \sup\{t \geq 0: X_t^\xi \leq 0\}$ is a well-defined random variable that is finite almost surely. It then follows that

$$\mathbb{P}\{X_t^\xi \geq 0 \text{ for all } t\} > 0.$$

We have thus established something stronger than the conclusion of Theorem 1.3, but under the stronger initial condition (4.1). We now retain the condition $\phi > c$, but weaken the initial configuration on (η_t) to (4.2).

When word A invades (η_t) starting from (4.2), such invasion is monitored via the progress of the boundary processes

$$X_t^+ := \sup_{x \geq 0} \{\eta_t(0) = \eta_t(1) = \dots = \eta_t(x - 1) = \eta_t(x) = A\},$$

$$X_t^- := \sup_{x \geq 0} \{\eta_t(0) = \eta_t(-1) = \dots = \eta_t(-x + 1) = \eta_t(-x) = A\},$$

for which $X_0^+ = X_0^- = 0$. Now couple these processes with the boundary process (X_t^η) defined, analogously to the process (X_t^ξ) introduced around (4.3), for the process starting from (4.1). This coupling implies that, under the assumptions of the theorem, the probability that the word A invades the population starting from (4.2) is larger than

$$\begin{aligned} &\mathbb{P}\{X_t^+ \geq 0 \text{ and } X_t^- \geq 0 \text{ for all } t > 0\} \\ &\geq \mathbb{P}\{X_t^+ \geq 3 \text{ and } X_t^- \geq 3 \text{ for all } t > 1 \mid X_1^+ = X_1^- = 3\} \mathbb{P}\{X_1^+ = X_1^- = 3\} \\ &\geq [\mathbb{P}\{X_t \geq 0 \text{ for all } t > 0\}]^2 \mathbb{P}\{X_1^+ = X_1^- = 3\} \\ &> 0. \end{aligned}$$

Since there is positive probability that, for the process starting with a single bilingual individual at the origin, the origin is of type A at time 1, this completes the proof of Theorem 1.3.

5. The naming game in higher dimensions

This section is devoted to proving Theorem 1.4, which relies on a block construction. We use ideas from oriented site percolation (see, e.g. Durrett’s work [3, 5]). For economy of notation, we prove the result only in the case $d = 2$, but our approach easily extends to higher dimensions. The idea of the block construction is to couple a certain collection of good events related to the process properly rescaled in space and time with the set of open sites of oriented site percolation on the oriented graph \mathcal{H}_1 with vertex set

$$H := \{(z, n) = ((z_1, z_2), n) \in \mathbb{Z}^2 \times \mathbb{Z}_+ : z_1 + z_2 + n \text{ is even}\}$$

and in which there is an oriented edge

$$(z, n) \rightarrow (z', n') \quad \text{if and only if} \quad z' = z \pm e_i \quad \text{for some } i = 1, 2 \text{ and } n' = n + 1,$$

where e_i is the i th unit vector in two dimensions. (See the left-hand side of Figure 3 below for a picture of this oriented graph in $d = 1$.) To rescale the process and define the collection of good events later in the proof of Lemma 5.1, let $T := \sqrt{\phi}$ and, for all $(z, n) \in H$, introduce the collection of space–time blocks

$$B(z, n) := \{(x, t) = ((x_1, x_2), t) \in \mathbb{Z}^2 \times [0, \infty) \text{ such that} \\ x_j \in \{z_j, z_j + 1\} \text{ for } j = 1, 2 \text{ and } nT \leq t < (n + 1)T\}.$$

This implicitly defines two coverings of space into 2×2 squares and one partition of time into intervals of length T which together define a covering of the space–time universe. The key to proving invasion of word A is to show that the set of sites

$$(z, n) \in H \quad \text{such that} \quad \eta_t(x) = A \quad \text{for all } (x, t) \in B(z, n),$$

which we call A -sites for short, dominates stochastically the set of wet sites in an oriented site percolation process whose parameter can be made arbitrarily close to 1 by choosing the parameter ϕ sufficiently large. More precisely, introduce

$$\mathbb{X}_n := \{z \in \mathbb{Z}^2 : (z, n) \in H \text{ and is an } A\text{-site}\},$$

and let \mathbb{W}_n^ε be the set of wet sites at level n in a 2-dependent oriented site percolation process in which sites are open with probability $1 - \varepsilon$.

Lemma 5.1. *For all $\varepsilon > 0$, there exists large $\phi > 0$ and a coupling of the naming game and oriented site percolation such that*

$$\mathbb{W}_n^\varepsilon \subset \mathbb{X}_n \quad \text{for all } n \quad \text{whenever} \quad \mathbb{X}_0 = \mathbb{W}_0^\varepsilon.$$

Proof. Say that the interaction along edge (x, y) at time $T_n(x, y)$ is

$$\text{a good interaction if } U_n(x, y) < q_A = \phi(\phi + 3)^{-1}$$

and a bad interaction otherwise. Referring to Figure 2, let $G(z, n)$ be the event that

- (i) between time nT and time $(n + 1)T$, there are at least two good interactions along each of the eight edges labeled 1 on the left-hand side; and
- (ii) between time nT and time $(n + 1)T$, there is no bad interaction along any of the sixteen edges labeled 2 on the left-hand side.

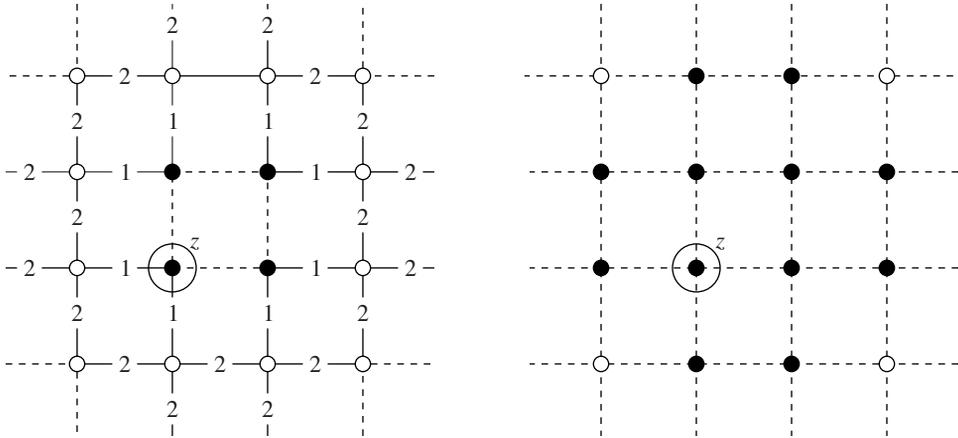


FIGURE 2: Picture related to the block construction.

From (2.1)–(2.2) and the probabilities in Table 1, it follows that an interaction involving at least one individual using word A can only result in one of the transitions 1A–5A in the table. In particular, whenever site (z, n) is an A -site and our good event $G(z, n)$ occurs, all twelve vertices marked with a filled circle on the right-hand side of the figure are monolingual of type A at time $(n + 1)T$. In other words, with $A(z, n)$ denoting the event that (z, n) is an A -site,

$$A(z, n) \cap G(z, n) \subset A(z \pm e_i, n + 1) \quad \text{for } i = 1, 2. \tag{5.1}$$

Now, let X and Y be the number of good and bad interactions that occur along one given edge in a given time interval of length T . Since interactions occur along each edge of the lattice at rate 1 and are independently good with probability q_A ,

$$X = \text{Poi}(Tq_A) \quad \text{and} \quad Y = \text{Poi}(T(1 - q_A)).$$

In particular, for all $\varepsilon > 0$, the probability of the good event equals

$$\begin{aligned} \mathbb{P}\{G(z, n)\} &\geq 1 - 8\mathbb{P}\{X \leq 1\} - 16\mathbb{P}\{Y \neq 0\} \\ &= 1 - 8(1 + Tq_A)e^{-Tq_A} - 16(1 - e^{-T(1-q_A)}) \\ &\geq 1 - 8(1 + Tq_A)e^{-Tq_A} - \frac{48T}{\phi + 3} \\ &= 1 - 8\left[1 + \frac{\phi^{3/2}}{\phi + 3}\right]e^{-\phi^{3/2}/(\phi+3)} - 48\frac{\phi^{1/2}}{\phi + 3} \\ &\geq 1 - \varepsilon \end{aligned} \tag{5.2}$$

for all large enough ϕ . Finally, observe that the good event $G(z, n)$ is measurable with respect to the graphical representation in the space–time region

$$(z, nT) + \{-2, -1, 0, 1, 2, 3\}^2 \times [0, T) \subset \mathbb{Z}^2 \times [0, \infty).$$

This, inclusion (5.1), and lower bound (5.2) are exactly the comparison assumptions of Theorem 4.3 of [5], from which the lemma directly follows.

Introduce the function $f: \{0, 1\}^H \mapsto \{0, 1\}$ defined by

$$f(\{\mathbb{W}_n^\varepsilon: n \geq 0\}) := \mathbf{1}\{\text{card}\{n: z \in \mathbb{W}_n^\varepsilon\} = \infty \text{ for all } z \in \mathbb{Z}^2\}.$$

Taking positive $\varepsilon < 1 -$ (the critical value of 2-dependent oriented site percolation), and using the coupling given in the previous lemma for this value of ε as well as the monotonicity of the function f , we obtain, for all $(x, t) \in \mathbb{Z}^2 \times \mathbb{R}_+$,

$$\begin{aligned} \mathbb{P}\{\eta_s(x) = A \text{ for some } s > t\} &\geq \mathbb{P}\{\text{card}\{n : z \in \mathbb{X}_n\} = \infty \text{ for all } z \in \mathbb{Z}^2\} \\ &= \mathbb{E}[f(\{\mathbb{X}_n : n \geq 0\})] \\ &\geq \mathbb{E}[f(\{\mathbb{W}_n^\varepsilon : n \geq 0\})] \\ &= \mathbb{P}\{\text{card}\{n : z \in \mathbb{W}_n^\varepsilon\} = \infty \text{ for all } z \in \mathbb{Z}^2\} \\ &> 0. \end{aligned}$$

This proves that the probability of survival of word A in the naming game starting with a single bilingual individual is positive, but it does not show that extinction of word B has positive probability as required by the theorem. In fact, a weak form of survival can be proved in the general case when $\phi > 3$ by simply using techniques similar to those in the proof of Lemma 3.1 to show that the number of individuals using word A is a submartingale. However, extinction of word B with positive probability cannot be deduced from this approach.

Instead, to complete the proof of the theorem, we use the coupling of the rescaled naming game with oriented site percolation given by Lemma 5.1 combined with an idea of the author [7] that extends a result of Durrett [4] from discrete-time to continuous-time processes. This result states that sites which are not wet do not percolate for oriented site percolation models in which sites are open with probability close to 1.

Lemma 5.2. *For all large enough ϕ , $p_A > 0$.*

Proof. Throughout the proof, think of the naming game as being coupled with oriented site percolation as in the statement of Lemma 5.1. To begin with, we follow [7] by introducing the new oriented graph \mathcal{H}_2 with the same vertex set as \mathcal{H}_1 but in which there is an oriented edge $(z, n) \rightarrow (z', n')$ if and only if either $(z' = z \pm e_i$ for some $i = 1, 2$ and $n' = n + 1)$ or $(z' = z \pm 2e_i$ for some $i = 1, 2$ and $n' = n)$; see the right-hand side of Figure 3 for a picture in $d = 1$. Say that a site is dry if it is not wet for oriented site percolation on the graph \mathcal{H}_1 . Also, for $j = 1, 2$, write

$$(w, 0) \rightarrow_j (z, n)$$

and say that there is a dry path in \mathcal{H}_j connecting both sites if there exist

$$(z_0, 0) = (w, 0), (z_1, n_1), \dots, (z_{k-1}, n_{k-1}), (z_k, n_k) = (z, n) \in H$$

such that the following two conditions hold:

- (i) for all $i = 0, 1, \dots, k - 1$, $(z_i, n_i) \rightarrow (z_{i+1}, n_{i+1})$ is an oriented edge in \mathcal{H}_j ; and
- (ii) the site (z_i, n_i) is dry for all $i = 0, 1, \dots, k$.

Note that a dry path in \mathcal{H}_1 is also a dry path in \mathcal{H}_2 , but the converse is false because the latter has more oriented edges than the former.

The key to the proof is the following result: if sites are closed with small enough probability $\varepsilon > 0$ and $|C_0| = \infty$ is the event that percolation occurs, then

$$\lim_{n \rightarrow \infty} \mathbb{P}\{(w, 0) \rightarrow_2 (z, n) \text{ for some } w \in \mathbb{Z}^2 \mid |C_0| = \infty\} = 0. \tag{5.3}$$

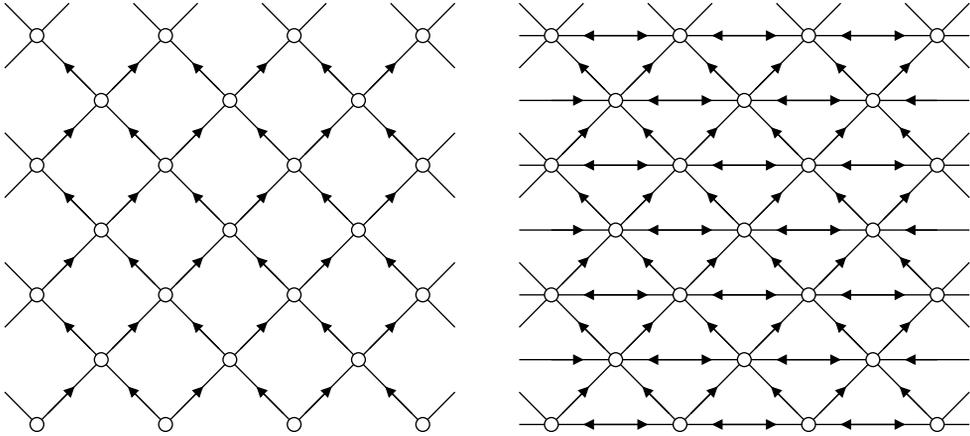


FIGURE 3: Picture of the oriented graphs \mathcal{H}_1 and \mathcal{H}_2 in dimension $d = 1$.

In other words, if the density of open sites is close enough to 1 then dry sites do not percolate even with the additional edges in \mathcal{H}_2 . The proof for dry paths in the graph \mathcal{H}_1 follows directly from Lemmas 4–11 of [4], but, as pointed out in [7], the proof easily extends to give the analog for dry paths in the oriented graph \mathcal{H}_2 .

The last step in the proof is to show the connection between dry paths and A -sites. Assume that

$$\eta_t(x) \neq A \quad \text{for some } (x, t) \in B(z, n), \text{ where } (z, n) \in H. \tag{5.4}$$

Since word B cannot appear spontaneously, this implies the existence of

$$x_0, x_1, \dots, x_m = x \in \mathbb{Z}^2 \quad \text{and} \quad s_0 = 0 < s_1 < \dots < s_{m+1} = t$$

such that the following two conditions hold:

- (i) for all $j = 0, 1, \dots, m$, $\eta_s(x_j) \neq A$ for $s \in [s_j, s_{j+1}]$; and
- (ii) for all $j = 0, 1, \dots, m - 1$, vertices x_j and x_{j+1} are connected by an edge.

With the coupling in Lemma 5.1, this further implies that

$$(w, 0) \rightarrow_2 (z, n) \quad \text{for some } w \in \mathbb{Z}^2. \tag{5.5}$$

Note however that this does not imply the existence of a dry path in \mathcal{H}_1 , which is the reason why we introduced a new graph with additional edges. Now the event at (5.4) is a subset of (5.5), so the probability of the former is dominated by that of the latter for which the limit is shown at (5.3). These facts imply that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{P}\{\eta_t(x) \neq A \mid |C_0| = \infty\} \\ & \leq \lim_{n \rightarrow \infty} \mathbb{P}\{(w, 0) \rightarrow_2 (z, n) \text{ for some } w \in \mathbb{Z}^2 \mid |C_0| = \infty\} \\ & = 0 \end{aligned}$$

for all sufficiently large ϕ according to Lemma 5.1. In particular, the probability that word A invades the lattice for the naming game starting with all four sites in $\{0, 1\}^2$ occupied by

individuals of type A is larger than the percolation probability, which itself is close to 1 when ϕ is large. Since, for the process starting with a single bilingual individual at the origin, there is positive probability that at time 1 all sites in $\{0, 1\}^2$ are of type A , the lemma and Theorem 1.4 follow.

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