FINITE RECIPROCITIES

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Introduction. A. P. Guinand [1] has shown that for appropriate infinite sequences a_n, α_n, (a_n, α_n real and positive for all n) the relation

$$(1) \qquad g(x) = \sum_{n=1}^{\infty} \frac{a_n}{x} f\left(\frac{\alpha_n}{x}\right)$$

implies that

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{x} g\left(\frac{\alpha_n}{n}\right) .$$

This suggests the question: Do there exist transformations similar to (1) in which we have a finite sum only? The following example

$$(2) \qquad \begin{aligned} g(x) &= \frac{1}{x} f\left(\frac{1}{x}\right) \\ f(x) &= \frac{1}{x} g\left(\frac{1}{x}\right) \end{aligned} \qquad x \neq 0$$

shows that a reciprocal transformation with one term exists - but do there exist such reciprocities with two or more terms? It turns out that no such transformation exists in which the α_n are all real and positive, but if we permit the α_n to have negative or complex values, then such transformations can be found.

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In section 1 reciprocities with two terms are investigated. In section 2 reciprocities with three terms are studied. In section 3 a special case of n-term reciprocities is investigated.

1. Two-term reciprocities. For which values of a, b, α, β, $(\alpha \neq \beta)$ does the relation

$$(1.1) \qquad g(x) = \frac{a}{x} f\left(\frac{\alpha}{x}\right) + \frac{b}{x} f\left(\frac{\beta}{x}\right)$$

imply

$$(1.2) \qquad f(x) = \frac{a}{x} g\left(\frac{\alpha}{x}\right) + \frac{b}{x} g\left(\frac{\beta}{x}\right) \ ?$$

By substituting (1.1) in (1.2) we get

$$(1.3) \qquad f(x) = \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta}\right) f(x) + \frac{ab}{\alpha} f\left(\frac{\beta x}{\alpha}\right) + \frac{ab}{\beta} f\left(\frac{x}{\beta}\right) .$$

If this equation is to hold for all functions $f(x)$ we must have $\frac{\beta}{\alpha} = \frac{\alpha}{\beta}$, or $\beta^2 = \alpha^2$. The solution $\beta = \alpha$ will not be considered since we assume $\beta \neq \alpha$. Therefore $\alpha = -\beta$, and we must have $\frac{a^2}{\alpha} + \frac{b^2}{\beta} = 1$, or

$$(1.4) \qquad a^2 - b^2 = \alpha .$$

Condition (1.4) is equivalent to

$$a = \frac{\alpha + \lambda^2}{2\lambda} , \qquad b = \frac{\alpha - \lambda^2}{2\lambda} .$$

Therefore we have:

THEOREM 1.1. If $x \neq 0$ and $f(x)$ is defined for all x, and $g(x)$ is defined by

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$$g(x) = \frac{\alpha + \lambda^2}{2\lambda x} f(\frac{\alpha}{x}) + \frac{\alpha - \lambda^2}{2\lambda x} f(-\frac{\alpha}{x}) ,$$

then

$$f(x) = \frac{\alpha + \lambda^2}{2\lambda x} g(\frac{\alpha}{x}) + \frac{\alpha - \lambda^2}{2\lambda x} g(-\frac{\alpha}{x}) .$$

2. <u>Three-term reciprocities.</u> For which values of a, b, c, α, β, γ (α, β, γ all unequal) does

$$g(x) = \frac{a}{x} f(\frac{\alpha}{x}) + \frac{b}{x} f(\frac{\beta}{x}) + \frac{c}{x} f(\frac{\gamma}{x})$$

imply

$$f(x) = \frac{a}{x} g(\frac{\alpha}{x}) + \frac{b}{x} g(\frac{\beta}{x}) + \frac{c}{x} g(\frac{\gamma}{x}) \ ?$$

Using similar methods one arrives at the condition

$$(2.1) \quad f(x) = \left(\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma}\right) f(x) + \frac{ab}{\alpha} f(\frac{\beta x}{\alpha}) + \frac{ab}{\beta} f(\frac{\alpha x}{\beta})$$

$$+ \frac{bc}{\gamma} f(\frac{\beta x}{\gamma}) + \frac{ca}{\alpha} f(\frac{\gamma x}{\alpha}) + \frac{cb}{\beta} f(\frac{\gamma x}{\beta}) + \frac{ac}{\gamma} f(\frac{\alpha x}{\gamma}) .$$

For this equation to hold for all functions $f(x)$ we must have

$$(2.2) \qquad \frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} = 1 ,$$

and the last six terms on the right of (2.1) must cancel out. We consider two cases.

1) If they cancel out in two groups of three, then we could have

$$\frac{\beta}{\alpha} = \frac{\alpha}{\gamma} = \frac{\gamma}{\beta} = w, \quad \frac{\gamma}{\alpha} = \frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{1}{w},$$

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(2.3) $$\frac{ab}{\beta} + \frac{bc}{\gamma} + \frac{ca}{\alpha} = 0 ,$$

(2.4) $$\frac{ab}{\alpha} + \frac{bc}{\beta} + \frac{ac}{\gamma} = 0 .$$

Since $\frac{\beta}{\alpha} = \frac{\alpha}{\gamma} = \frac{\gamma}{\beta} = w$, we obtain

$$w^3 = 1 .$$

Thus we obtain three values of w, namely $1, \omega, \omega^2$, where $\omega = \exp(\frac{2\pi i}{3})$. Hence a solution is given by

$$\beta = \alpha\omega , \qquad \gamma = \alpha\omega^2 .$$

With these values of β and γ it is possible to show that (2.3) implies (2.4). To obtain a simple example take $\alpha = 1$. Thus (2.2) gives

$$a^2 + b^2\omega^2 + c^2\omega^{-2} = 1$$

that is,

(2.5) $$a^2\omega + b^2 + c^2\omega^2 = \omega .$$

Equation (2.5) together with (2.3) give the conditions

(2.6) $$ac\omega^2 + ab\omega + bc = 0, \quad a + b\omega + c\omega^2 = 1 .$$

Putting $a = A$, $b\omega = B$, $c\omega^2 = C$ we obtain, from (2.6),

$$AC + BA + BC = 0, \quad A + B + C = 1 .$$

Hence A, B, C are the roots of $z^3 - z^2 + k = 0$, k arbitrary. Therefore we have,

THEOREM 2.1. If $x \neq 0$, $\omega = \exp(\frac{2\pi i}{3})$, and $g(x)$ is defined by

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$$g(x) = \frac{z_1}{x} f(\frac{1}{x}) + \frac{z_2\omega^2}{x} f(\frac{\omega}{x}) + \frac{z_3\omega}{x} f(\frac{\omega^2}{x}) ,$$

then

$$f(x) = \frac{z_1}{x} g(\frac{1}{x}) + \frac{z_2\omega^2}{x} g(\frac{\omega}{x}) + \frac{z_3\omega}{x} g(\frac{\omega^2}{x}) ,$$

where z_1, z_2, z_3, are roots of

$$z^3 - z^2 + k = 0$$

and k is arbitrary.

2) If they cancel out in groups of two we could have the following relations among α, β, γ:

(2.7) $$\frac{\alpha}{\beta} = \frac{\beta}{\alpha}, \quad \frac{\alpha}{\gamma} = \frac{\gamma}{\beta}, \quad \frac{\beta}{\gamma} = \frac{\gamma}{\alpha} .$$

On examining all possible cases it turns out that the only solution is given by

$$\frac{\alpha}{\beta} = \frac{\beta}{\alpha} = t .$$

This implies $t^2 = 1$. We consider $t = -1$ since $t = 1$ gives $\alpha = \beta$. Hence $\frac{\beta}{\gamma} = \frac{\gamma}{\alpha}$, that is $\beta\alpha = \gamma^2$. This gives $\gamma = \pm i\alpha$ since $\beta = -\alpha$. Hence a possible solution is $\alpha = 1$, $\beta = -1$, $\gamma = i$.

Let us find the values of a, b, c to satisfy

(2.8) $$g(x) = \frac{a}{x} f(\frac{1}{x}) + \frac{b}{x} f(\frac{-1}{x}) + \frac{c}{x} f(\frac{i}{x}) ,$$

that is the values of a, b, c, when $\alpha = +1$, $\beta = -1$, $\gamma = i$. Relation (2.8) would imply

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$$f(x) = (a^2-b^2-ic^2)\,f(x) + (ba-ab)\,f(x) + (-iac-bc)\,f(-xi) + (ac-cbi)\,f(xi).$$

For the above to hold we must have

$$a^2 - b^2 - ic^2 = 1, \quad c(ai+b) = 0, \quad c(a-ib) = 0 .$$

Since we want $c \neq 0$, it follows that $a = bi$, and consequently $c^2 = (1-2a^2)i$, that is $c = \sqrt{(1-a^2)i}$ with $2a^2 \neq 1$. Hence we have

THEOREM 2.2. If $x \neq 0$, a is any number such that $2a^2 \neq 1$, and $g(x)$ is defined by

$$g(x) = \frac{a}{x} f(\frac{1}{x}) - \frac{ia}{x} f(\frac{-1}{x}) + \frac{\sqrt{(1-2a^2)i}}{x} f(\frac{i}{x}) ,$$

then

$$f(x) = \frac{a}{x} g(\frac{1}{x}) - \frac{ia}{x} g(\frac{-1}{x}) + \frac{\sqrt{(1-2a^2)i}}{x} g(\frac{i}{x}) .$$

For more than three terms it becomes too complicated to investigate all possible cases. However, if we consider the following problem, we can show that there exist n-term reciprocities.

3. <u>N-term reciprocities.</u> Problem. What conditions must the coefficients a_n satisfy if

$$g(x) = \sum_{n=0}^{k-1} \frac{a_n}{x} \exp(\pi ni/k)\, f(\frac{1}{x} \exp(2\pi in/k))$$

is to imply

$$f(x) = \sum_{n=0}^{k-1} \frac{a_n}{x} \exp(\pi ni/k)\, g(\frac{1}{x} \exp(2\pi in/k)) \; ?$$

Suppose that the reciprocity (3.1) is true, then

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$$f(x) = \sum_{m=0}^{k-1} \frac{a_m}{x} \exp(\pi i m/k) \sum_{n=0}^{k-1} a_n\, x \exp\left(\frac{\pi i}{k}(n-2m)\right) f\left(x \exp \frac{2\pi i}{k}(n-m)\right)$$

$$= \sum_{m=0}^{k-1} \sum_{n=0}^{k-1} a_m a_n \exp((n-m)\pi i/k)\, f(x \exp(2\pi i(n-m)/k)) \, .$$

If (3.2) holds, we must have

$$\sum_{n=0}^{k-1} a_n^2 = 1 \, ,$$

(3.3)

$$\sum_{m=0}^{k-1} \sum_{n=0}^{k-1} a_m a_n \exp(-\pi i(m-n)/k) = 0 \, ,$$

where the latter summation is considered for all values of m, n such that $n - m \equiv h (\mathrm{mod}\ k)$, where $h = \pm 1, \pm 2, \ldots \pm(k-1)$. We shall consider only positive values of h, since $n - m \equiv h(\mathrm{mod}\ k)$ implies $m - n \equiv -h(\mathrm{mod}\ k)$ and this is equivalent to the exchange of n and m in (3.3).

Put $n = m + h$ if the domain of m is $0 \le m \le k - h - 1$,
$n = m + h - k$ if the domain of m is $k - h \le m \le k - 1$,
then the sum (3.3) splits into two parts, giving

$$\sum_{m=0}^{k-h-1} a_m a_{m+h} \exp(\pi i h/k) + \sum_{m=k-h}^{k-1} a_m a_{m+h-k} \exp(-\pi i(k-h)/k) = 0 \, ,$$

that is

$$\text{(3.4)} \qquad \sum_{m=0}^{k-h-1} a_m a_{m+h} = \sum_{m=k-h}^{k-1} a_m a_{m+h-k} \, .$$

Next, put $n = m + h - k$ in the second sum of (3.4) i.e. $m = n+k-h$, therefore n varies from $n = 0$ to $n = h - 1$. Put $q = k - h$. With these substitutions, (3.4) becomes

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$$\sum_{m=0}^{q-1} a_m a_{m+h} = \sum_{n=0}^{h-1} a_n a_{n+q}, \quad h + q = k, \ 1 \leq h \leq k - 1, \ 1 \leq q \leq k - 1 .$$

Therefore we have the following conditions:

$$(3.5) \qquad \sum_{n=0}^{q-1} a_n a_{n+h} = \sum_{n=0}^{h-1} a_n a_{n+q} ,$$

with $h + q = k$, $1 \leq h \leq k - 1$, $1 \leq q \leq k - 1$,

$$(3.6) \qquad \sum_{n=0}^{k-1} a_n^2 = 1 .$$

The equations considered in (3.5) are not independent. There are two cases.

(i) If $k = 2p$, the set of equations (3.5) contains an identity which corresponds to the case $h = q = \frac{k}{2}$. The rest are identical in pairs; that is we have at most $p - 1$ independent equations in the set (3.5). Therefore there are at most p independent equations in the system of equations determined by (3.5) and (3.6).

(ii) If $k = 2q + 1$ the set of equations (3.5) contains q distinct equations. Hence the system of equations determined by (3.5) and (3.6) contains $q + 1$ equations.

Therefore in either case we get $[\frac{1}{2}k + \frac{1}{2}]$ equations for determining the quantities $a_0, a_1, \ldots, a_{k-1}$; that is, at least $[\frac{1}{2}k]$ of these quantities are independent, and reciprocities with k terms certainly exist for any $k \geq 1$.

REFERENCE

1. A. P. Guinand, Finite Summation Formulae, The Quarterly Journal of Mathematics, Oxford Series Vol. 10 (38-44).

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