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CONSTRUCTION OF SUNNY NONEXPANSIVE RETRACTIONS IN BANACH SPACES

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Let $\mathcal{J}$ be a commutative family of nonexpansive self-mappings of a closed convex subset C of a uniformly smooth Banach space X such that the set of common fixed points is nonempty. It is shown that if a certain regularity condition is satisfied, then *the sunny nonexpansive retraction from* C *to* F *can be constructed in an iterative way.*

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space X and let D be a nonempty subset of C. A *retraction* from C to D is a mapping $Q : C \to D$ such that $Qx = x$ for $x \in D$. A retraction Q from C to D is *nonexpansive* if Q is nonexpansive (that is, $\|Qx - Qy\| \leqslant \|x - y\|$ for $x, y \in C$). A retraction Q from C to D is *sunny* if Q satisfies the property:

$$Q\big(Qx + t(x - Qx)\big) = Qx \quad \text{for } x \in C \text{ and } t > 0 \text{ whenever } Qx + t(x - Qx) \in C.$$

A retraction Q from C to D is sunny nonexpansive if Q is both sunny and nonexpansive. It is known ([2]) that in a smooth Banach space X, a retraction Q from C to D is a sunny nonexpansive retraction from C to D if and only if the following inequality holds:

$$\big\langle x - Qx, J(y - Qx)\big\rangle \leqslant 0, \quad x \in C,\ y \in D, \tag{1.1}$$

where $J : X \to X^*$ is the duality map defined by

$$Jx := \big\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\big\}, \quad x \in X.$$

Hence a sunny nonexpansive retraction must be unique (if it exists).

If C is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_C from H to C is the sunny nonexpansive retraction. This however is not true for Banach spaces since nonexpansivity of projections P_C characterises Hilbert spaces.

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On the other hand, one sees by (1.1) that a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in a Hilbert space. So an interesting problem is this: In what kind of Banach spaces, does a sunny nonexpansive retraction exist? If it does exist, how can one find it? It is known ([**2**]) if C is a closed convex subset of a uniformly smooth Banach space X and there is a nonexpansive retraction from X to C, then there exists a sunny nonexpansive retraction from X to C. But Bruck's proof is not constructive.

The purpose of the present paper is to construct sunny nonexpansive retractions in a uniformly smooth Banach space in an iterative way. More precisely, we show that if F is the nonempty common fixed point set of a commutative family of nonexpansive self-mappings of a closed convex subset C of a uniformly smooth Banach space X satisfying certain regularity condition, then we are able to construct a sequence that converges to the sunny nonexpansive retraction Q of C to F. This extends a result of Reich [**7**] where the case of a single nonexpansive mapping is dealt with.

2. Two Lemmas

Lemma 2.1. *Let (s_n) be a sequence of nonnegative numbers satisfying the condition:*

$$s_{n+1} \leqslant (1-\alpha_n)(s_n+\beta_n)+\alpha_n\gamma_n, \quad n \geqslant 0, \tag{2.1}$$

where (α_n), (β_n), (γ_n) are sequences of real numbers satisfying

(i) $(\alpha_n) \subset [0,1]$, $\lim\limits_n \alpha_n = 0$, *and* $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,* $\prod\limits_{n=0}^{\infty}(1-\alpha_n) := \lim\limits_{n\to\infty} \prod\limits_{k=0}^{n}(1-\alpha_k) = 0$;

(ii) $\limsup\limits_{n\to\infty} \beta_n \leqslant 0$;

(iii) $\limsup\limits_{n\to\infty} \gamma_n \leqslant 0$.

Then $\lim\limits_{n\to\infty} s_n = 0$.

Proof: For any $\varepsilon > 0$, let $N \geqslant 1$ be an integer big enough so that

$$\beta_n < \varepsilon/2, \quad \gamma_n < \varepsilon, \quad \alpha_n < 1/2, \quad n \geqslant N. \tag{2.2}$$

It follows from (2.1) and (2.2) that, for $n > N$,

$$\begin{aligned} s_{n+1} &\leqslant (1-\alpha_n)(s_n+\beta_n)+\varepsilon\alpha_n \\ &\leqslant (1-\alpha_n)(1-\alpha_{n-1})(s_{n-1}+\beta_{n-1})+\varepsilon\Big(1-(1-\alpha_n)(1-\alpha_{n-1})+\frac{1}{2}(1-\alpha_n)\Big) \\ &\leqslant (1-\alpha_n)(1-\alpha_{n-1})(s_{n-1}+\beta_{n-1})+\varepsilon\Big(1-\Big(\frac{1}{2}-\alpha_n\Big)\Big(\frac{1}{2}-\alpha_{n-1}\Big)\Big). \end{aligned}$$

Hence by induction we obtain

$$s_{n+1} \leqslant \prod_{N}^{n}(1-\alpha_j)(s_N+\beta_N)+\varepsilon\Big[1-\prod_{j=N}^{n}(1-\widetilde{\alpha}_j)\Big], \quad n>N,$$

where $\widetilde{\alpha}_j := 1/2+\alpha_j < 1$, $j \geqslant N$. By condition (i) we deduce, after taking the limsup as $n \to \infty$ in the last inequality, that $\limsup_{n\to\infty} s_{n+1} \leqslant \varepsilon$. □

Lemma 2.2. *(The Subdifferential Inequality, see* [**3**].*) In a Banach space X there holds the inequality:*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j\rangle, \quad x,y \in X,\ j \in J(x+y),$$

where J is the duality map of X.

3. Iterative Processes

Let G be an unbounded subset of $\mathbf{R}^+$ such that $s+t \in G$ whenever $s,t \in G$. (Often $G = \mathcal{N}$, the set of nonnegative integers or $\mathbf{R}^+$.) Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, and $\mathcal{J} = \{T_s : s \in G\}$ a commutative family of nonexpansive self-mappings of C. Denote by F the set of common fixed point of $\mathcal{J}$, that is, $F = \{x \in C : T_s x = x,\ s \in G\}$. Throughout this section we always assume that F is nonempty. Our purpose is to construct the sunny nonexpansive retraction Q from C to F. We shall introduce two iterative processes to construct Q. The first one is implicit, while the second one is explicit. But before introducing the iterative processes, we recall that the uniform smoothness of X is equivalent to the following statement:

$$\lim_{\lambda\to 0^+}\frac{\|x+\lambda y\|^2-\|x\|^2}{\lambda} = 2\langle y, Jx\rangle \quad \text{uniformly for bounded } x,y \in X.$$

Let $u \in C$ be given arbitrarily and let $(\alpha_s)_{s\in G}$ be a net in the interval $(0,1)$ such that $\lim_{s\to\infty}\alpha_s = 0$. By Banach's contraction principle, for each $s \in G$, we have a unique point $z_s \in C$ satisfying the equation

$$z_s = \alpha_s u + (1-\alpha_s)T_s z_s. \tag{3.1}$$

Theorem 3.1. *Let X be a uniformly smooth Banach space. Assume $\mathcal{J}$ is uniformly asymptotically regular on bounded subsets of C; that is, for each bounded subset $\widetilde{C}$ of C and each $r \in G$,*

$$\lim_{\substack{s\in G\\ s\to\infty}}\ \sup_{x\in\widetilde{C}}\|T_r T_s x - T_s x\| = 0. \tag{3.2}$$

Then the net (z_s) defined in (3.1) converges in norm and the limit defines the sunny nonexpansive retraction Q from C to F.

PROOF: First we observe that (z_s) is bounded. Indeed, for $p \in F$, we have

$$\|z_s - p\| \leqslant \alpha_s \|u - p\| + (1 - \alpha_s)\|T_s z_s - p\| \leqslant \alpha_s \|u - p\| + (1 - \alpha_s)\|z_s - p\|.$$

This implies that $\|z_s - p\| \leqslant \|u - p\|$ and (z_s) is bounded. Thus $\|z_s - T_s z_s\| = \alpha_s \|u - T_s z_s\| \to 0$ $(s \to \infty)$. Let (s_n) be a subsequence of G such that $\lim_n s_n = \infty$. Next we define a function f on C by

$$f(x) := \mathrm{LIM}_n \|z_n - x\|^2, \quad x \in C,$$

where LIM is a Banach limit and $z_n := z_{s_n}$. We have for each $r \in G$,

$$\begin{aligned} f(T_r x) &= \mathrm{LIM}_n \|z_n - T_r x\|^2 \\ &= \mathrm{LIM}_n \|T_r T_{s_n} z_n - T_r x\|^2 \\ &\leqslant \mathrm{LIM}_n \|T_{s_n} z_n - x\|^2 \end{aligned}$$

using (3.2). Therefore,

$$f(T_r x) \leqslant f(x), \quad r \in G,\ x \in C. \tag{3.3}$$

Let

$$K := \{x \in C : f(x) = \min_C f\}.$$

Then it can be seen that K is a closed bounded convex nonempty subset of C. By (3.3) we see that K is invariant under each T_r; that is, $T_r(K) \subset K$, $r \in G$. By Lim [5] the family $\mathcal{J} = \{T_s : s \in G\}$ has a common fixed point, that is, $K \cap F \neq \emptyset$. Let $q \in K \cap F$. Since q is a minimiser of f over C and since X is uniformly smooth, it follows that for each $x \in C$,

$$\begin{aligned} 0 &\leqslant \lim_{\lambda \to 0^+} \frac{1}{\lambda}\Big[f\big(q + \lambda(x - q)\big) - f(q)\Big] \\ &= \mathrm{LIM}_n \Big(\lim_{\lambda \to 0^+} \frac{1}{\lambda}\Big[\big\|(z_n - q) + \lambda(q - x)\big\|^2 - \|z_n - q\|^2\Big]\Big) \\ &= \mathrm{LIM}_n \big\langle q - x, J(z_n - q)\big\rangle. \end{aligned}$$

Thus,

$$\mathrm{LIM}_n \big\langle x - q, J(z_n - q)\big\rangle \leqslant 0, \quad x \in C.$$

In particular,

$$\mathrm{LIM}_n \big\langle u - q, J(z_n - q)\big\rangle \leqslant 0, \quad x \in C. \tag{3.4}$$

On the other hand, by equation (3.1) we have for any $p \in F$,

$$z_n - p = (1 - \alpha_n)(T_n z_n - p) + \alpha_n(u - p),$$

where $\alpha_n = \alpha_{s_n}$ and $T_n = T_{s_n}$. It follows that for $p \in F$,

$$\begin{aligned}\|z_n - p\|^2 &= (1-\alpha_n)\langle T_n z_n - p, J(z_n - p)\rangle + \alpha_n \langle u - p, J(z_n - p)\rangle \\ &\leqslant (1-\alpha_n)\|z_n - p\|^2 + \alpha_n \langle u - p, J(z_n - p)\rangle.\end{aligned}$$

Hence

$$\|z_n - p\|^2 \leqslant \langle u - p, J(z_n - p)\rangle. \tag{3.5}$$

Combining (3.4) and (3.5) with p replaced with q, we get

$$\mathrm{LIM}_n \|z_n - q\|^2 \leqslant 0.$$

So we have a subsequence (z_{n_j}) of (z_n) such that $z_{n_j} \xrightarrow{s} q$. Assume there exists another subsequence (z_{m_k}) of (z_s) such that $z_{m_k} \xrightarrow{s} \widetilde{q}$. Then (3.5) implies

$$\|q - \widetilde{q}\|^2 \leqslant \langle u - \widetilde{q}, J(q - \widetilde{q})\rangle. \tag{3.6}$$

Similarly we have

$$\|\widetilde{q} - q\|^2 \leqslant \langle u - q, J(\widetilde{q} - q)\rangle. \tag{3.7}$$

Adding up (3.6) and (3.7) obtains $q = \widetilde{q}$. Therefore (z_s) converges in norm to a point in F.

Now define $Q : C \to F$ by

$$Qu := s - \lim_{s \to \infty} z_s.$$

Then Q is a retraction from C to F. Moreover, by (3.5) we get for $p \in F$,

$$\|Qu - p\|^2 \leqslant \langle u - p, J(Qu - p)\rangle \quad \Rightarrow \quad \langle u - Qu, J(p - Qu)\rangle \leqslant 0, \quad p \in F.$$

Therefore Q is a sunny nonexpansive retraction from C to F. ∎

Next we introduce an explicit iterative process to construct the sunny nonexpansive retraction Q from C to F.

Let $u \in C$ be arbitrary. Take a sequence (r_n) in G and a sequence (α_n) in the interval $[0,1]$. Starting with an arbitrary initial $x_0 \in C$ we define a sequence (x_n) recursively by the formula:

$$x_{n+1} := \alpha_n u + (1-\alpha_n) T_{r_n} x_n, \quad n \geqslant 0. \tag{3.8}$$

The following is a convergence result for the process (3.8).

Theorem 3.2. *Let X be a uniformly smooth Banach space. Assume*

(i) $\alpha_n \to 0$, $\alpha_n/\alpha_{n+1} \to 1$, *and* $\sum_n \alpha_n = \infty$;

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(ii) $r_n \to \infty$;

(iii) $\mathcal{J}$ *is semigroup (that is,* $T_r T_s = T_{r+s}$ *for* $r, s \in G$*) and satisfies the uniformly asymptotically regularity condition:*

$$\lim_{\substack{r \in G \\ r \to \infty}} \sup_{x \in \widetilde{C}} \|T_s T_r x - T_r x\| = 0 \quad \text{uniformly in } s \in G, \tag{3.9}$$

where $\widetilde{C}$ *is any bounded subset of* C*. Then the sequence* (x_n) *generated by* (3.8) *converges in norm to* Qu*, where* Q *is the sunny nonexpansive retraction from* C *to* F *established in Theorem 3.1.*

PROOF: 1. First prove the sequence (x_n) is bounded. As a matter of fact, for $p \in F$, we have

$$\|x_{n+1} - p\| \leqslant \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\|.$$

This together with an induction implies that

$$\|x_n - p\| \leqslant \max\{\|u - p\|, \|x_0 - p\|\}, \quad \text{for all } n \geqslant 0.$$

Thus (x_n) is bounded and it follows that

$$\|x_{n+1} - T_{r_n} x_n\| = \alpha_n \|u - T_{r_n} x_n\| \to 0. \tag{3.10}$$

2. Now prove $\|x_{n+1} - x_n\| \to 0$. Indeed we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|(\alpha_n - \alpha_{n-1})(u - T_{r_{n-1}} x_{n-1}) \\
&\qquad + (1 - \alpha_n)(T_{r_n} x_n - T_{r_n} x_{n-1}) + (1 - \alpha_n)(T_{r_n} x_{n-1} - T_{r_{n-1}} x_{n-1})\big\| \\
&\leqslant |\alpha_n - \alpha_{n-1}| \|u - T_{r_{n-1}} x_{n-1}\| + (1 - \alpha_n)\|x_n - x_{n-1}\| \\
&\qquad + (1 - \alpha_n)\|T_{r_n} x_{n-1} - T_{r_{n-1}} x_{n-1}\| \\
&\leqslant (1 - \alpha_n)\big(\|x_n - x_{n-1}\| + \beta_n\big) + \alpha_n \gamma_n,
\end{aligned}$$

where $\beta_n := \|T_{r_n} x_{n-1} - T_{r_{n-1}} x_{n-1}\|$ and $\gamma_n := (\alpha_n^{-1})|\alpha_n - \alpha_{n-1}| \|u - T_{r_{n-1}} x_{n-1}\|$. Since (x_n) is bounded, by condition (i), we have $\gamma_n \to 0$. It is easily seen that condition (iii) implies $\beta_n \to 0$. Indeed, if $r_n > r_{n-1}$, since $\mathcal{J}$ is a semigroup, we have $\beta_n = \|T_{r_n - r_{n-1}} T_{r_{n-1}} x_{n-1} - T_{r_{n-1}} x_{n-1}\| \to 0$ by step 1 and (3.9). Interchanging r_n and r_{n-1} if $r_n < r_{n-1}$ finishes the proof of $\beta_n \to 0$. Hence by Lemma 2.1 we get $\|x_{n+1} - x_n\| \to 0$.

3. Next we show for each fixed $s \in G$, $\|T_s x_n - x_n\| \to 0$. Indeed (3.10) and step 2 imply that $\|x_n - T_{r_n} x_n\| \to 0$. Let $\widetilde{C}$ be any bounded subset of C which contains the sequence (x_n). It follows that

$$\begin{aligned}
\|T_s x_n - x_n\| &\leqslant \|T_s x_n - T_s T_{r_n} x_n\| + \|T_s T_{r_n} x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - x_n\| \\
&\leqslant 2\|x_n - T_{r_n} x_n\| + \sup_{x \in \widetilde{C}} \|T_s T_{r_n} x - T_{r_n} x\|.
\end{aligned}$$

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So condition (iii) implies that $\|T_s x_n - x_n\| \to 0$.

4. Now we prove $\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \leqslant 0$, where $q = Q(u)$. Recall that z_s satisfies (3.1) and we have shown that $z_s \xrightarrow{s} q$. Writing

$$z_s - x_n = \alpha_s(u - x_n) + (1 - \alpha_s)(T_{r_s} z_s - x_n)$$

and applying Lemma 2.2 we get

$$\begin{aligned}\|z_s - x_n\|^2 &\leqslant (1 - \alpha_s)^2 \|T_{r_s} z_s - x_n\|^2 + 2\alpha_s \langle u - x_n, J(z_s - x_n)\rangle \\ &\leqslant (1 + \alpha_s^2)\|z_s - x_n\|^2 + M\|x_n - T_{r_s} x_n\| + 2\alpha_s\langle u - z_s, J(z_s - x_n)\rangle,\end{aligned}$$

where M is a constant such that $\|x_n - T_{r_s} x_n\| + 2\|z_s - x_n\| \leqslant M$ for all $n, s \in G$. It follows from the last inequality that

$$\langle u - z_s, J(x_n - z_s)\rangle \leqslant \frac{\alpha_s}{2}\|z_s - x_n\|^2 + \frac{M}{2\alpha_s}\|T_{r_s} x_n - x_n\|.$$

By step 3 we find that

$$\limsup_{n\to\infty} \langle u - z_s, J(x_n - z_s)\rangle \leqslant O(\alpha_s) \to 0 \quad \text{as } s \to \infty. \tag{3.11}$$

Since X is uniformly smooth, the duality map J is norm-to-norm uniformly continuous on bounded subsets of X, letting $s \to \infty$ in (3.11) we obtain $\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \leqslant 0$.

5. Finally we show $x_n \to q$ in norm. Apply Lemma 2.2 to get

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \big\|(1 - \alpha_n)(T_{r_n} x_n - q) + \alpha_n(u - q)\big\|^2 \\ &\leqslant (1 - \alpha_n)^2\|T_{r_n} x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle \\ &\leqslant (1 - \alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle.\end{aligned}$$

Now using condition (i), step 4 and Lemma 2.1 ($\beta_n \equiv 0$) we conclude that $\|x_n - q\| \to 0$. ∎

REMARKS. 1. The case where the family $\mathcal{J}$ consists of a single nonexpansive mapping was studied by Reich [7]. For the framework of a Hilbert spaces see also Halpern [4], Lions [6] and Wittmann [8].

2. For the case where $\mathcal{J}$ is a finite family of nonexpansive mappings and X is a Hilbert space, similar results involving with a periodic iteration process were proved in Lions [6] and Bauschke [1].

REFERENCES

[1] H.H. Bauschke, 'The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces', *J. Math. Anal. Appl.* **202** (1996), 150–159.

[2] R.E. Bruck, 'Nonexpansive projections on subsets of Banach spaces', *Pacific J. Math.* **47** (1973), 341–355.

[3] I. Cioranescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Mathematics and its Applications **62** (Kluwer Academic Press, Dordrecht, 1990).

[4] B. Halpern, 'Fixed points of nonexpanding maps', *Bull. Amer. Math. Soc.* **73** (1967), 957–961.

[5] T.C. Lim, 'A fixed point theorem for families on nonexpansive mappings', *Pacific J. Math.* **53** (1974), 487–493.

[6] P.L. Lions, 'Approximation de points fixes de contractions', *C.R. Acad. Sci. Paris, Sér. A-B* **284** (1977), 1357–1359.

[7] S. Reich, 'Strong convergence theorems for resolvents of accretive operators in Banach spaces', *J. Math. Anal. Appl.* **75** (1980), 287–292.

[8] R. Wittmann, 'Approximation of fixed points of nonexpansive mappings', *Arch. Math.* **58** (1992), 486–491.

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