Chapter 1

James Clerk Maxwell and Structural Mechanics

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1.1 Introduction

James Clerk Maxwell was a remarkable man. In his brief forty-eight years of life, he made profound discoveries in many fields. His writings are full of fundamental ideas that cannot be completely distilled into textbooks or scholarly summaries. It is only by reading and studying Maxwell's original writings that today's mathematicians, scientists, and engineers can hope to utilize and extend the full potential of his insights.

The importance of Maxwell's writings was clear to his contemporaries. His friend, classmate, and collaborator, Peter Guthrie Tait, wrote shortly after Maxwell's death that "The spirit of Clerk-Maxwell still lives with us in his imperishable writings..." (Tait, 1880). Maxwell's writings were assembled and published in a two-volume set, *The Scientific Papers of James Clerk Maxwell*, by W. D. Niven in 1890. The study of Maxwell's writings has continued until the present day as evidenced by the three-volume set, *The Scientific Letters and Papers of James Clerk Maxwell* by Peter Harman, published between 1990 and 2002.

Maxwell's writings are rich in ideas but are packaged in difficult and multilayered text that, at times, seems impenetrable. Many of his contributions are complex and defy concise summaries. As such, readers who are trying to access Maxwell are in a dilemma. They should read the original text to get the complete depth of Maxwell's ideas, but the text is a barrier to access.

The prose is somewhat disjointed and difficult to follow, and the figures are minimal; furthermore, terminology has changed. But the profound problem for today's reader is that Maxwell's writings were based on the reasonable assumption that he and his readers shared a common body of knowledge. While this was likely true for the leading mathematicians, scientists, and engineers who were his contemporaries, it is generally not true for a reader today.

Statements like the following are now unintelligible to most readers:

These points and planes are reciprocally polar in the ordinary sense with respect to the paraboloid of revolution

$$2cz = x^2 + y^2.$$

We have thus arrived at a construction for reciprocal diagrams by considering each as a plane projection of a plane-sided polyhedron, these polyhedra being reciprocal to one another, in the geometrical sense, with respect to a certain paraboloid of revolution.

(Maxwell, 1870a)

To most readers today, the casualness of his phrase "reciprocally polar in the ordinary sense" clearly demonstrates the gap of common knowledge between technical audiences in the 19th and 21st centuries.

This book is intended to serve as a bridge between the centuries, with a focus on key articles by Maxwell about structural mechanics. Although the scope is limited to structural mechanics, many of the topics addressed in this book could be of use to those who wish to better understand the development of Maxwell's work in other fields such as optics, thermodynamics, or electromagnetism.

This introductory chapter provides a short summary of Maxwell's life, his contributions to structural mechanics, and recommendations as to how to approach his writings and this book.

1.2 James Clerk Maxwell (1831–1879)

James Clerk Maxwell was the premier "natural philosopher" of the 19th century, meaning someone we would today primarily call a physicist but with an overlay of metaphysics. A professorship of Natural Philosophy was a common position in British universities in the 19th century. Maxwell is often placed with Newton and Einstein as one of the top physicists of all time. He is best known for his unification of electricity and magnetism into the single theory of electromagnetism, an advancement that ushered in countless advances in science and engineering. Maxwell's intellect cast a wide net. In addition to electricity and magnetism, his interests included geometry, optics,

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color perception, mechanical devices and instruments, dynamics, fluid mechanics, thermodynamics, and, the subject of this book, structural mechanics.

Much has been written about the life of Maxwell. His childhood friend, Lewis Campbell, and Cambridge colleague, William Garnett, wrote an extensive biography soon after Maxwell's death (Campbell and Garnett, 1882), summarized concisely by Niven (1890). Maxwell's educational development and intellectual contributions are also described in the more recent book by Harman (1998).

Maxwell was born in Edinburgh in 1831 and educated at Edinburgh Academy, the University of Edinburgh, and the University of Cambridge. After his studies, he taught at the University of Cambridge, Marischal College Aberdeen, King's College London, and then, again, at the University of Cambridge. He died in Cambridge in 1879 and is buried in Parton, Scotland, near his family estate, Glenlair (Campbell and Garnett, 1882).

During the mid-19th century, Britain was an epicenter of scientific and industrial innovation. As an active member of the country's leading scientific and mathematical societies, Maxwell was integrally involved in the rapid advancements taking place across the natural sciences. At the same time, railways were being extended across the country and new structures were needed to carry heavy dynamic loads over longer spans. Advances in structural engineering attracted attention from leading scientists such as Maxwell and George Biddell Airy (1801–1892), the Astronomer Royal. Maxwell interacted regularly with several contemporaries in civil engineering, including the Regius Professor of Civil Engineering and Mechanics at the University of Glasgow, William John Macquorn Rankine (1820–1872). Rankine was simultaneously making major contributions in both engineering and physics and the boundaries between engineering and science were very porous in this period of rapid theoretical development.

Maxwell's own work in structural mechanics dated back to his time as an undergraduate student at the University of Edinburgh. He studied catenaries in Professor J.D. Forbes' Natural Philosophy class (Harman, 1998) and, when he was only eighteen years old, he published *On the Equilibrium of Elastic Solids* in the Transactions of the Royal Society of Edinburgh (Maxwell, 1850). His interest in structural mechanics would extend throughout his career. For the Royal Society of London, he reviewed George Biddell Airy's seminal paper *On the Strains in the Interior of Beams* (Airy, 1863) and, as a professor of Natural Philosophy, Maxwell first introduced his own structural mechanics theories in his courses on applied mechanics (Maxwell, 1870b).

Additionally, Maxwell had personal connections to civil and structural engineering. His cousin and dearest friend, William Dyce Cay (1838–1925), was a practicing civil engineer. Maxwell also wrote an article for a publication called *The Engineer*, which was intended to be read by practitioners (Maxwell, 1867). Thus, Maxwell took note of recent developments in the field of civil and structural engineering and the engineering requirements of 19th–century industry in Britain.

1.3 Maxwell's Work in Structural Mechanics

The beginning of the theoretical study of structural mechanics can be traced to the works of Galileo (1564–1642) in the 17th century, as well as continental Europeans like Charles-Augustin Coulomb (1736–1806) and Leonhard Euler (1707–1783) in the 18th century (Heyman, 1998). But despite these advances there was still much work left to be done in the mid-19th century when Maxwell began to publish his discoveries. Although trusses had been used in construction for centuries, a theoretical basis of truss design dates to the 1830s (Kurrer, 2008) and the first metal truss bridge was built in Britain in 1845 (Timoshenko, 1953). It was only in the 1840s and 1850s that concepts of conservation of energy, including heat, were being expounded. It was in this context that Maxwell made fundamental and profound contributions to the theory of structures.

Maxwell's contributions to structural theory outlined in this book are primarily concerned with equilibrium of two-dimensional and three-dimensional frameworks, as previous authors have noted (Kurrer, 2008; Heyman, 1998). Many mechanics researchers since Maxwell's time have focused on elasticity and deformation of solids, rather than on geometry and equilibrium, which are critical for structural designers. This over-reliance on elasticity theory, termed "Navier's straitjacket" by Heyman (1999), has precluded research and university teaching in the fertile ground that Maxwell uncovered. By returning to the source of Maxwell's focus on geometry and equilibrium, and by placing his discoveries in context, this book aims to reignite and extend his work.

In structural mechanics, Maxwell's discoveries can be divided into four categories: (1) the work that is still commonly known; (2) the work that was formerly known, then generally forgotten; (3) the work that is just now emerging; and (4) the work that has yet to be applied.

Several of Maxwell's discoveries are still in common use. These include the following:

- the Maxwell count for determining the level of redundancy of truss structures;
- the Maxwell reciprocal work theorem. This has many applications and explains why stiffness matrices are symmetric;
- the unit load method for analysis of trusses;
- the flexibility method for the analysis of indeterminate trusses;
- the Maxwell stress function (a member of a family of stress functions that includes Airy, Beltrami, Morera, Prandtl, Beltrami–Schaefer, and Gurtin).

Surprisingly, some of Maxwell's discoveries were in use by engineers in the late 19th and early 20th centuries, but were generally forgotten or fell out of use over time (Kurrer, 2008). These include two-dimensional graphic statics, Maxwell's load path theorem, and the Airy stress function for trusses. More recently, researchers and designers, including many of the contributors to this book, have been using digital and graphical computation to adapt these ideas to contemporary structural design problems.

With today's computational and graphic tools, researchers and designers are finding applications for several of Maxwell's discoveries that failed to find general use in earlier times due to technological limitations. These discoveries include concepts that relate to form-finding and analysis of three-dimensional structures. Maxwell's writings are packed with ideas and it is highly likely that there are other applications of his work that have yet to be found. It is hoped that this book can help researchers and designers better understand Maxwell's insights and discover new applications.

1.4 How to Read Maxwell

Maxwell's structural mechanics writings do not lend themselves to a linear examination, nor would such an approach be desirable. However, the following is one possible roadmap for approaching his work: The reader might start with *On Reciprocal Figures and Diagrams of Forces* (Maxwell, 1864a) and concurrently read Chapter 6, *A Brief Introduction to Mid-19th Century Projective Geometry and Topology*. If the reader finds that sections of Maxwell (1864a) become inaccessible or confusing, they should refer to other chapters in Part II of this book that reference the pertinent paragraphs of Maxwell (1864a). After Maxwell (1864a), it is recommended that the reader then move on to *On Reciprocal Figures, Frames, and Diagrams of Forces* (Maxwell, 1870a). This seminal article was read on February 7, 1870, at the Royal Society of Edinburgh and published that same year, expanding upon a series of ideas that Maxwell had previously published, primarily in the 1860s, which are also included in this book. Once again, the chapters in Part II of this book have numerous references to paragraphs in Maxwell (1870a). Finally, with the understanding gained from these two articles, the reader is invited to read Maxwell's other articles.

1.5 Guide to this Book

This book is organized in three parts.

Part I

Maxwell's contributions to structural mechanics have a broad reach that goes well beyond engineering applications. The same documents that are the source for the engineering of a bridge or skyscraper also apply to computer vision and scene analysis in computer graphic animations. Maxwell's ideas work at scales that range from large civil structures to the behavior of foams and the interaction of granular particles down to the interactions of molecules. Maxwell's observations on networks are used by rigidity theorists for autonomous robots in formation and by engineers in the design of a gridshell structure.

The four chapters that follow this introductory chapter discuss the 21st–century relevance of Maxwell's contributions to structural mechanics in both applications and research across a wide range of disciplines. Chapter 2 focuses on topics closely related to engineering applications and the next three on topics beyond engineering.

Chapter 3 summarizes the state of the art in rigidity theory. Rigidity theory takes the counts and other relationships of layouts put forth by Maxwell and others and creates a theoretical world with far–reaching applications in 2D, 3D, and higher dimensions. It explores the multiple ways rigidity can be defined and how that rigidity can be achieved. It also explores the converse, identifying the movements that can be realized by various layouts. The results have many practical applications that are currently being explored in metamaterials, computer graphics, robotics, machinery, and, coming full circle to where Maxwell began, structural engineering. In fact, rigidity theory has applications in any problem where specialized knowledge is needed to control the relative positions of "nodes" of objects such as drones flying in a desired formation.

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A major challenge of the 21st century is the creation of new materials and the utilization of existing materials in previously unexplored ways. Chapter 4 looks at the physics of mechanical metamaterials and how Maxwell's work in counts, rigidity, mechanisms, dualities, and elastic energy is used in the design of materials ranging from the nanoscale to the physics of the jamming of granular media. Its relevance extends to biological science including the design of proteins.

Geometry was central to the work of Maxwell. So, it is fitting that his work would be central to the creation of a geometry, namely isotropic geometry. The innovations of isotropic geometry are explored in Chapter 5. Isotropic geometry is a "top-view" geometry that utilizes the Maxwell innovation of the paraboloid of revolution as the equivalent of the Gaussian unit sphere in Euclidean geometry. Isotropic geometry also utilizes Maxwell's enhancements to our understanding of the Airy stress function. Current applications include the architectural geometry of complex shapes and gridshells and 2D representation of 3D objects in computer graphics.

At the end of each chapter is a suggested roadmap to relevant essays in Part II and Maxwell's original texts in Part III.

Projective geometry and topology are central to much of Maxwell's writings on structural mechanics; therefore, Chapter 6 provides a summary of these topics as they would have been understood in the mid–19th century. One of the major barriers for a 21st–century reader is a lack of knowledge about projective geometry. The reader is urged to read this chapter prior to or in conjunction with reading Maxwell's original texts.

Part II

Part II is comprised of Chapters 7 to 16. Many of Maxwell's ideas are developed through multiple articles, using language and nomenclature that are foreign to the 21st–century reader. Furthermore, many of Maxwell's ideas were later refined by others. Each of the chapters of Part II takes a topic studied by Maxwell and provides an explanation and examination of the topic for today's reader.

Part III

Part III consists of transcriptions of Maxwell's original articles with each paragraph numbered and annotations added to facilitate better comprehension. On occasion, errors in the original publication have been identified and corrected with the corrections noted. The readers of this book are urged to spend time exploring and understanding Maxwell's original texts.

1.6 Summary

The writings of Maxwell still have much to offer to today's reader. By reading his original writings, mathematicians, scientists, and engineers have direct access to fundamental knowledge that could lead them to new discoveries and designs that might otherwise be unattainable.

The intent of this book is to assist the reader in this journey to better understand Maxwell's contributions, and to inspire and inform new innovations and insights, just as Maxwell did in his own lifetime.