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Induced Coactions of Discrete Groups on C^*-Algebras

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Abstract. Using the close relationship between coactions of discrete groups and Fell bundles, we introduce a procedure for inducing a C^*-coaction $\delta\colon D \to D \otimes C^*(G/N)$ of a quotient group G/N of a discrete group G to a C^*-coaction $\operatorname{Ind}\delta\colon \operatorname{Ind} D \to \operatorname{Ind} D \otimes C^*(G)$ of G. We show that induced coactions behave in many respects similarly to induced actions. In particular, as an analogue of the well known imprimitivity theorem for induced actions we prove that the crossed products $\operatorname{Ind} D \times_{\operatorname{Ind}\delta} G$ and $D \times_\delta G/N$ are always Morita equivalent. We also obtain nonabelian analogues of a theorem of Olesen and Pedersen which show that there is a duality between induced coactions and twisted actions in the sense of Green. We further investigate amenability of Fell bundles corresponding to induced coactions.

1 Introduction

One of the most important constructions in ergodic theory and dynamical systems is the construction of an induced action (or induced flow): if H is a closed subgroup of the group G and Y is an H-space, then the induced G-space $G \times_H Y$ is defined as the quotient of $G \times Y$ with respect to the equivalence relation $(s, hy) \sim (sh, y)$ for $h \in H$, and G-action given by translation on the first factor. The induced G-action behaves in almost all respects similarly to the original H-action on Y, and the theory is particularly useful when it is possible to identify a given G-space as one which is induced from a more manageable H-space.

The analogue of the induced G-space in the theory of C^*-dynamical systems is the induced C^*-algebra $\operatorname{Ind}_H^G A$ together with the induced action $\operatorname{Ind}\alpha$, where we start with an action $\alpha\colon H \to \operatorname{Aut}(A)$. Needless to say, if $A = C_0(Y)$, then $\operatorname{Ind}_H^G A = C_0(G \times_H Y)$. As for G-spaces, the importance of this construction comes from the fact that induced actions enjoy in most respects the same properties as the original ones. The most important manifestation of this statement is certainly Green's imprimitivity theorem (see [11, Theorem 17]), which implies that the crossed product $\operatorname{Ind}_H^G A \times_{\operatorname{Ind}\alpha} G$ of the induced system is always Morita equivalent to the crossed product $A \times_\alpha H$ of the original system. To see the importance of this result, note that Morita equivalent algebras have naturally homeomorphic representation spaces and the same K-theory.

In this paper we are concerned with the question whether a similar theory of induced algebras can be obtained in the theory of coactions of locally compact groups. Recall that the theory of coactions of a group G (or rather of the group C^*-algebra $C^*(G)$ equipped with a natural comultiplication) is in a natural way dual to the theory of actions of G: if

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$\alpha\colon G \to \mathrm{Aut}(A)$ is an action of G on the C^*-algebra A, then there is a canonical coaction $\hat\alpha$ of G on $A \times_\alpha G$ such that the double crossed product $A \times_\alpha G \times_{\hat\alpha} G$ is stably isomorphic to A (see [13] and [26]). This generalizes the Takesaki-Takai duality theorem for actions of abelian groups, where $\hat\alpha$ is an action of the dual group $\hat G$ of G. Conversely, starting with any coaction δ of G on A, there exists a dual action $\hat\delta\colon G \to \mathrm{Aut}(A \times_\delta G)$, and Katayama obtained a similar duality theorem [14], which works for all normal coactions (see the preliminary section for the notation). Of course, in order to develop the full power of this duality theory, it is most desirable to have an as complete as possible dual mirror of the usual constructions for actions. In particular it would certainly be interesting to have a working notion of induced coactions and induced C^*-algebras by coactions. At least if G is discrete we will see here that there is indeed such a theory, and that it enjoys many properties which are known in the theory of induced actions.

Our results are based heavily on observations due to the second author, which connect the theory of coactions of discrete groups to the theory of Fell bundles (or C^*-algebraic bundles) over G [23]. Recall that a Fell bundle $(\mathcal{A}, G)$ over G is a family of Banach spaces A_s for $s \in G$, together with a multiplication $A_s \times A_t \to A_{st}$ and an involution $A_s \to A_{s^{-1}}$, which satisfies some further conditions. The set $\Gamma_c(\mathcal{A})$ of sections of finite support forms a *-algebra, and a cross sectional algebra A of $(\mathcal{A}, G)$ is a completion of $\Gamma_c(\mathcal{A})$ with respect to a given C^*-norm.

If $\delta\colon A \to A \otimes C^*(G)$ is a coaction of a discrete group G on A, then the spectral subspaces $\{A_s : s \in G\}$ (*i.e.*, $a_s \in A_s \Leftrightarrow \delta(a_s) = a_s \otimes s$) of δ form a Fell bundle $(\mathcal{A}, G)$ over G and A is a *topologically graded* cross sectional algebra for $\mathcal{A}$ (see Section 2 for more details). Similarly to other situations in the theory of C^*-algebras, there may exist more than one C^*-norm on $\Gamma_c(\mathcal{A})$, but if we insist that the corresponding completions are topologically graded, then there always exists a maximal and a minimal one. We denote the respective cross sectional algebras by $C^*(\mathcal{A})$ (for the maximal norm) and $C^*_r(\mathcal{A})$ (for the minimal one; see [7] for a detailed treatment of this). Note that both algebras, $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$, carry natural coactions $\delta_{\mathcal{A}}$ and $\delta^n_{\mathcal{A}}$ which are both determined by the property that they map $a_s \in A_s$ to the element $a_s \otimes s \in A \otimes C^*(G)$. Thus any other coaction δ lies "between" $\delta_{\mathcal{A}}$ and $\delta^n_{\mathcal{A}}$, if $\mathcal{A}$ is the bundle associated to δ. Note that just recently, Fell bundles over discrete groups were studied extensively by several people [23], [25], [7], [1], [6], partly due to the discovery that many important C^*-algebras appear as cross sectional algebras of Fell bundles.

Due to the above-described connection between coactions and Fell bundles for discrete groups, we are able to use Fell bundles, rather than the coactions themselves, in order to define induced coactions. Starting with a Fell bundle $(\mathcal{D}, G/N)$ over a quotient G/N by a normal subgroup N of a discrete group G, we define the induced coaction simply as the dual coaction of the maximal cross sectional algebra $\operatorname{Ind} D := C^*(q^*\mathcal{D})$ of the pull-back bundle $(q^*\mathcal{D}, G)$, where $q\colon G \to G/N$ denotes the quotient map. Note that there is a certain arbitrariness in our definition, since we could also have taken the dual coaction (if it exists) of any other topologically graded cross sectional algebra of $q^*\mathcal{D}$ (*e.g.*, $C^*_r(q^*\mathcal{D})$) instead of the maximal one. Thus there is no canonical choice unless $q^*\mathcal{D}$ is amenable in the sense of Exel [7], which roughly means that all topologically graded cross sectional algebras are the same.

We will show that induced coactions behave in almost all respects similarly to induced actions; for example, if G is abelian, the induced coactions of G correspond exactly to the induced actions of $\hat{G}$ under the usual identification between coactions of G and actions of $\hat{G}$ (see Section 2). In Section 3 we show that crossed products by coactions of discrete groups can be realized as cross sectional algebras of certain Fell bundles over the transformation groupoid $G \times G$. In fact if $(\mathcal{A}, G)$ is the Fell bundle associated to the coaction $\delta\colon A \to A \otimes C^*(G)$, then $A \times_\delta G$ is the *enveloping C^*-algebra* of $\Gamma_c(\mathcal{A} \times G)$, where $\mathcal{A} \times G$ is the product bundle over the groupoid $G \times G$. Using this result we show in Section 4 that there is an analogue, for induced coactions, of Green's imprimitivity theorem: if $\delta\colon D \to D \otimes C^*(G/N)$ is a coaction of G/N, then there is a natural Morita equivalence between the crossed products $D \times_\delta G/N$ and $\operatorname{Ind} D \times_{\operatorname{Ind}\delta} G$. Notice that both crossed products only depend on the underlying Fell bundles $\mathcal{D}$ and $q^*\mathcal{D}$, and not on the particular choices of the cross sectional algebras D and $\operatorname{Ind} D$.

In Section 5 we show that there is an analogue of Olesen and Pedersen's classical result about twisted group actions (see [20], [24]): using a very useful general characterization of induced coactions, which is the analogue of the characterization of induced actions given by the first author in [3], we will see that a dual coaction $\hat{\beta}$ of a crossed product $B \times_\beta G$ is induced from a quotient G/N if and only if the action β is twisted over N in the sense of Green [11]. This leads to a negative result concerning the possibility of a "Mackey machine" for coactions: there is an important feature of induced actions of compact groups which fails for induced coactions of discrete groups. Namely, if β is an action of a *compact* group G on a C^*-algebra B such that B has no proper G-invariant ideals, then (B, G, β) is always induced from a system (A, H, α) with A a simple C^*-algebra; this follows from [3, Theorem], since compactness of G guarantees, by [19, Lemma 2.1], that $\operatorname{Prim} A$ is equivariantly homeomorphic to a homogeneous space G/H. For a *discrete* group G, however, our characterizations of induced coactions allow us to show that there exist numerous examples of G-simple coactions which are not induced (even in the weak sense) from simple coactions! This drawback of the theory is mainly due to the fact that the theory of coactions (at least so far) only allows us to look at quotients by normal subgroups, while for actions we can work with any closed subgroup of G. If we want to have a theory for coactions which works similarly to the full Mackey-Green theory for actions, we have to introduce something like "coactions of homogeneous spaces", *i.e.*, coactions of quotients G/H by not-necessarily-normal subgroups H of G. Up to now, there is no such theory.

Finally, in Section 6 we investigate under which conditions the pull-back bundles $q^*\mathcal{D}$ are amenable. This question is of particular interest to us, since, as mentioned above, only if $q^*\mathcal{D}$ is amenable do we have a unique choice for our induced algebra $\operatorname{Ind} D$. In [7] Exel introduced a certain approximation property (which we call property (EP)), which guarantees amenability of a given Fell bundle $(\mathcal{A}, G)$. For instance he showed that all Cuntz-Krieger bundles, which arise from the natural coactions of $\mathbb{F}_n$ on the Cuntz-Krieger algebras $\mathcal{O}_A$ as found in [25], satisfy property (EP), although the free group $\mathbb{F}_n$ with n generators is certainly not amenable if $n > 1$. If $\mathcal{D}$ is a bundle over G/N, then we will show that $q^*\mathcal{D}$ satisfies (EP) if $\mathcal{D}$ satisfies (EP) and N is amenable; the amenability of N is also necessary for $q^*\mathcal{D}$ to satisfy (EP). Note that as an immediate consequence of this we see that the Cuntz-Krieger algebras are not induced from any nontrivial quotient of $\mathbb{F}_n$, since $\mathbb{F}_n$ does not contain any nontrivial amenable normal subgroup.

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2 Preliminaries and Basic Definitions

Throughout this paper, G will be (except in certain remarks comparing with other research) a *discrete* group. We are primarily concerned with coactions of G on C^*-algebras, and for these we adopt the conventions of [23] and [22]. We can derive a few benefits from G being discrete: a *coaction* of G on A is an injective, nondegenerate homomorphism $\delta\colon A \to A \otimes C^*(G)$ (where here nondegeneracy means $\overline{\operatorname{span}}\,\delta(A)\bigl(A \otimes C^*(G)\bigr) = A \otimes C^*(G)$) such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$, where $\delta_G\colon C^*(G) \to C^*(G) \otimes C^*(G)$ is the homomorphism defined by $\delta_G(s) = s \otimes s$ for $s \in G$. The *spectral subspace* of A associated with $s \in G$ is $A_s := \{a \in A : \delta(a) = a \otimes s\}$. Since G is discrete, A is the closed span of the A_s. A *covariant representation* of (A, G, δ) in a multiplier algebra $M(B)$ is a pair (π, μ) of nondegenerate homomorphisms of A and $c_0(G)$ into $M(B)$ (where, for example, nondegeneracy of π means $\overline{\operatorname{span}}\,\pi(A)B = B$) such that

$$\pi(a_s)\mu(\chi_t) = \mu(\chi_{st})\pi(a_s) \quad \text{for } a_s \in A_s,\ t \in G,$$

where χ_t denotes the characteristic function of the singleton $\{t\}$. The closed span $C^*(\pi, \mu) := \overline{\operatorname{span}}\,\pi(A)\mu\bigl(c_0(G)\bigr)$ is a C^*-algebra, and is called a *crossed product* for (A, G, δ) if every covariant representation (ρ, ν) factors through $C^*(\pi, \mu)$ in the sense that there is a homomorphism $\rho \times \nu$ of $C^*(\pi, \mu)$ to $C^*(\rho, \nu)$ such that $(\rho \times \nu) \circ \pi = \rho$ and $(\rho \times \nu) \circ \mu = \nu$. All crossed products are isomorphic, and a generic one is denoted by $A \times_\delta G$, and moreover the covariant homomorphism generating $A \times_\delta G$ is written (j_A, j_G). The distinction among the various crossed products is frequently blurred, and any one of them is referred to as *the* crossed product. The *dual action* of G on the crossed product $A \times_\delta G$ is determined by $\hat{\delta}_s\bigl(j_A(a) j_G(\chi_t)\bigr) = j_A(a) j_G(\chi_{ts^{-1}})$ for $a \in A$ and $s, t \in G$.

The coaction δ is called *normal* if j_A is faithful. In any case, there is always a unique ideal I of A such that, with q denoting the quotient map from A to A/I, the composition $(q \otimes \mathrm{id}) \circ \delta$ factors through a normal coaction δ^n, called the *normalization* of δ, on A/I with the same crossed product as δ, that is, if (j_A, j_G^A) and $(j_{A/I}, j_G^{A/I})$ are the canonical covariant homomorphisms of (A, G, δ) into $M(A \times_\delta G)$ and $M(A/I \times_{\delta^n} G)$, respectively, then $(j_{A/I} \circ q) \times j_G^{A/I}$ is an isomorphism of $A \times_\delta G$ onto $A/I \times_{\delta^n} G$. The ideal I coincides with $\ker j_A$, as well as with $\ker(\mathrm{id} \otimes \lambda) \circ \delta$, where λ denotes the left regular representation of G.

As shown in [23], for discrete groups coactions are strongly related to Fell bundles and cross sectional algebras of Fell bundles, for which we adopt the conventions of [9] and [7]. More precisely: if (A, G, δ) is a coaction, then the spectral subspaces $\{A_s : s \in G\}$ (or, more properly, the disjoint union of these subspaces) form a Fell bundle over G, which we call the *Fell bundle associated to* δ.

Conversely, if $(\mathcal{A}, G)$ is a Fell bundle, there is a canonical coaction $\delta_{\mathcal{A}}$, which we will call the *dual coaction*, of G on the *full* cross sectional algebra $C^*(\mathcal{A})$, determined by $\delta_{\mathcal{A}}(a_s) = a_s \otimes s$ for a_s in the fiber A_s of $\mathcal{A}$. Throughout this paper, when we write something like a_s for an element of a Fell bundle, we always mean this element is to be understood to belong to the fiber over $s \in G$.

If $(\mathcal{A}, G)$ is a Fell bundle over G, then a *cross sectional algebra* A of $(\mathcal{A}, G)$ is simply a completion of $\Gamma_c(\mathcal{A})$ with respect to any given C^*-norm. A cross sectional algebra A is called *topologically graded* (see [7, Definition 3.4]) if there exists a contractive conditional expectation $F\colon A \to A_e$ which vanishes on each fiber A_s for $s \neq e$ (where we always view the fibers A_s of the bundle as subspaces of A in the canonical way). Exel showed that the full and reduced cross sectional algebras $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$ are maximal and minimal, respectively, among all topologically graded cross sectional algebras of a given Fell bundle $(\mathcal{A}, G)$. To be more precise: if A is any topologically graded cross sectional algebra of $\mathcal{A}$, then it follows from [7, Theorem 3.3] and the universal property of $C^*(\mathcal{A})$ (see [9, VIII.16.11]) that the identity map on $\mathcal{A}$ determines surjective *-homomorphisms

$$\phi\colon C^*(\mathcal{A}) \to A, \quad \lambda\colon A \to C^*_r(\mathcal{A}), \quad \text{and} \quad \Lambda\colon C^*(\mathcal{A}) \to C^*_r(A)$$

such that $\lambda\circ\phi = \Lambda$ (the map Λ is called the *regular representation* of $C^*(\mathcal{A})$). If $\|\cdot\|_{\max}$, $\|\cdot\|_\nu$ and $\|\cdot\|_{\min}$ denote the norms on $\Gamma_c(\mathcal{A})$ coming from viewing $\Gamma_c(\mathcal{A})$ as a dense subalgebra of $C^*(\mathcal{A})$, A, and $C^*_r(\mathcal{A})$, respectively, then the above result is of course equivalent to saying that $\|\cdot\|_{\max} \geq \|\cdot\|_\nu \geq \|\cdot\|_{\min}$. Thus the topologically graded cross sectional algebras are exactly the completions of $\Gamma_c(\mathcal{A})$ with respect to the C^*-norms which lie between $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$. Exel calls a Fell bundle $\mathcal{A}$ *amenable* if $C^*(\mathcal{A}) = C^*_r(\mathcal{A})$ in the sense that the regular representation of $\mathcal{A}$ is faithful on $C^*(\mathcal{A})$. In this case all topologically graded cross sectional algebras of $\mathcal{A}$ are identical.

If (A, G, δ) is any coaction, then for each $s \in G$ the map $\delta_s := (\mathrm{id}\otimes\chi_s)\circ\delta\colon A \to A$ (where here χ_s, the characteristic function of $\{s\}$, is regarded as belonging to the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, and $\mathrm{id}\otimes\chi_s$ is then the slice map of $A \otimes C^*(G)$ into A) is idempotent, with range A_s and kernel containing every A_t for $t \neq s$. In particular, $\delta_e\colon A \to A_e$ is a contractive conditional expectation which vanishes on A_s, for all $s \neq e$. Hence A is a topologically graded cross sectional algebra of the associated Fell bundle $(\mathcal{A}, G)$.

Now, [23, Comment immediately following Definition 3.5] states that the dual coaction $\delta^n_{\mathcal{A}}\colon a_s \mapsto a_s\otimes s$ on the *reduced* cross sectional algebra $C^*_r(\mathcal{A})$ is (isomorphic to) the normalization of the dual coaction $\delta_{\mathcal{A}}$ on $C^*(\mathcal{A})$. However, there is a subtlety: the constructions of $C^*_r(\mathcal{A})$ in [23] and [7] are not quite the same. So, before we can use the results from both sources, we need to check that their notions of the reduced C^*-algebra of a Fell bundle are compatible. Namely, we need to know that the kernels in $C^*(\mathcal{A})$ of the regular representations of [23] and [7] coincide. The conditional expectation $E := (\delta_{\mathcal{A}})_e$ of $C^*(\mathcal{A})$ onto the fixed-point algebra $C^*(\mathcal{A})^{\delta_{\mathcal{A}}} = A_e$ makes $C^*(\mathcal{A})$ into a Hilbert A_e-module, as in [28, Example 6.7]. Then left multiplication gives a representation of the C^*-algebra $C^*(\mathcal{A})$ on the Hilbert A_e-module $C^*(\mathcal{A})$, and this in turn gives a Rieffel inducing map from ideals of A_e to ideals of $C^*(\mathcal{A})$. In [23] the kernel of the regular representation is the ideal of $C^*(\mathcal{A})$ induced from the zero ideal of the fixed-point algebra $C^*(\mathcal{A})^{\delta_{\mathcal{A}}}$. But this coincides with the kernel of the above representation of $C^*(\mathcal{A})$ on the Hilbert A_e-module $C^*(\mathcal{A})$, which is the kernel of the regular representation of [7] (by the proof of [7, Theorem 3.3]). Hence, the definitions of $C^*_r(\mathcal{A})$ in [23] and [7] are indeed compatible.

The maps ϕ, λ and Λ considered above are clearly equivariant with respect to the coactions $\delta_{\mathcal{A}}$, δ and $\delta^n_{\mathcal{A}}$ (recall that if (A, G, δ) and (B, G, ϵ) are coactions, then a homomorphism $\phi\colon A \to B$ is called *equivariant* if $\epsilon\circ\phi = (\phi\otimes\mathrm{id})\circ\delta$). Thus we can say that any coaction $\delta\colon A \to A\otimes C^*(G)$ "lies between" the dual coaction $\delta_{\mathcal{A}}$ on $C^*(\mathcal{A})$ and its normalization $\delta^n_{\mathcal{A}}$ on $C^*_r(\mathcal{A})$, if $\mathcal{A}$ is the Fell bundle associated to δ. For reference it is useful to state the following lemma.

Lemma 2.1 *Let $\delta\colon A \to A\otimes C^*(G)$ be a coaction of the discrete group G and let $(\mathcal{A}, G)$*

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be the associated Fell bundle. Let $\delta_{\mathcal{A}}$ and $\delta^n_{\mathcal{A}}$ denote the dual coaction and its normalization on $C^*(\mathcal{A})$ and $C^*_r(\mathcal{A})$, respectively, and let ϕ, λ and Λ be as above. Then there are canonical isomorphisms

$$\operatorname{Ind}\phi\colon C^*(\mathcal{A})\times_{\delta_{\mathcal{A}}} G\to A\times_\delta G,\quad \operatorname{Ind}\lambda\colon A\times_\delta G\to C^*_r(\mathcal{A})\times_{\delta^n_{\mathcal{A}}} G,\quad \textit{and}$$
$$\operatorname{Ind}\Lambda\colon C^*(\mathcal{A})\times_{\delta_{\mathcal{A}}} G\to C^*_r(\mathcal{A})\times_{\delta^n_{\mathcal{A}}} G$$

defined by

$$\operatorname{Ind}\phi=(j_A\circ\phi)\times j_G^A,\quad \operatorname{Ind}\lambda=(j_{C^*_r(\mathcal{A})}\circ\lambda)\times j_G^{C^*_r(\mathcal{A})},\quad \textit{and}$$
$$\operatorname{Ind}\Lambda=(j_{C^*_r(\mathcal{A})}\circ\Lambda)\times j_G^{C^*_r(\mathcal{A})},$$

respectively. In particular, $\operatorname{Ind}\Lambda=\operatorname{Ind}\lambda\circ\operatorname{Ind}\phi$ *and $\delta^n_{\mathcal{A}}$ coincides with the normalization δ^n of δ.*

Proof It follows directly from the equivariance and surjectivity of the maps ϕ,λ and Λ that the maps $\operatorname{Ind}\phi$, $\operatorname{Ind}\lambda$ and $\operatorname{Ind}\Lambda$ are well defined surjections. Since $\Lambda=\lambda\circ\phi$ we also have $\operatorname{Ind}\Lambda=\operatorname{Ind}\lambda\circ\operatorname{Ind}\phi$. Since $\delta^n_{\mathcal{A}}$ is the normalization of $\delta_{\mathcal{A}}$, it follows from [22, Corollary 2.7] that $\operatorname{Ind}\Lambda$ is an isomorphism, which then implies that $\operatorname{Ind}\lambda$ and $\operatorname{Ind}\phi$ are also isomorphisms. In particular, it follows that $\ker j_A=\ker j_{C^*_r(\mathcal{A})}\circ\lambda=\ker\lambda$ (since $j_{C^*_r(\mathcal{A})}$ is injective by the normality of $\delta^n_{\mathcal{A}}$). Thus $\delta^n_{\mathcal{A}}$ coincides with the normalization of δ. ∎

Remark 2.2 In view of the above discussion one could guess that any topologically graded cross sectional algebra A of a given Fell bundle $(\mathcal{A},G)$ over the discrete group G carries a dual coaction δ which satisfies $\delta(a_s)=a_s\otimes s$. This is *not* the case.

To see a counter example let G be any non-amenable discrete group such that the direct sum $V=1_G\oplus\lambda_G$ of the trivial representation 1_G and the regular representation λ_G of G is not faithful. Then $V\big(C^*(G)\big)$ is a topologically graded cross sectional algebra of the Fell bundle $(\mathcal{A},G)$ corresponding to G (*i.e.*, $A_s=\mathbb{C}$ for all $s\in G$), since the kernel of V is contained in the kernel of λ_G. Let $U\colon C^*(G)\to\mathcal{L}(\mathcal{H})$ be any faithful representation of G. If there were a coaction δ on $V\big(C^*(G)\big)$ satisfying $\delta(a_s)=a_s\otimes s$, this would imply that the unitary representation $V\otimes U$ of G factors through a faithful representation of $V\big(C^*(G)\big)$, *i.e.*, $\ker(V\otimes U)=\ker V$ in $C^*(G)$. But $V\otimes U=(1_G\oplus\lambda_G)\otimes U=U\oplus(\lambda_G\otimes U)$ is faithful on $C^*(G)$, while V is not faithful by assumption.

To see that there are numerous examples of groups satisfying the above property on $1_G\oplus\lambda_G$, let us first note that any non-amenable group with $1_G\oplus\lambda_G$ faithful satisfies Kazhdan's property (T) (*i.e.*, the trivial representation is an isolated point in $\hat{G}$). Since the nonabelian free groups $\mathbb{F}_n$ in n generators do not satisfy Kazhdan's property (T) (which follows from the simple fact that $(\mathbb{F}_n/[\mathbb{F}_n,\mathbb{F}_n])^\wedge=\widehat{\mathbb{Z}^n}=\mathbb{T}^n$), they all serve as specific examples for our counter example. Moreover, by a theorem of Fell [8, Proposition 5.2] it is known that for any subgroup H of a discrete group G and any representation V of H, V is a direct summand of $(\operatorname{Ind}_H^G V)|_H$, which implies that any faithful representation of $C^*(G)$ restricts to a faithful representation of $C^*(H)$. Thus, if G is a non-amenable group with

$1_G \oplus \lambda_G$ faithful, it follows that $(1_G \oplus \lambda_G)|_H = 1_H \oplus \lambda_G|_H$, and hence $1_H \oplus \lambda_H$ is faithful on $C^*(H)$ for any subgroup H of G, since it follows from [12, Addendum of Theorem 1] that $\ker \lambda_G|_H = \ker \lambda_H$. In particular, $1_G \oplus \lambda_G$ is not faithful for any discrete group which contains the free group $\mathbb{F}_2$ as a subgroup. This shows that any non-amenable group with $1_G \oplus \lambda_G$ faithful must indeed be very exotic, and it is certainly an interesting question whether there exist such groups. We are grateful to Alain Valette for some useful comments on this.

We want to define a notion of induced coactions, dual to the concept of induced actions. If N is a subgroup of G, we can induce an action of N to an action of G, so dually we should expect to induce a coaction from a quotient group to the big group. For this we require N to be a normal subgroup. We don't know yet how to induce coactions in general; for dual coactions of Fell bundles the way seems fairly clear now (given the techniques of the present paper!), but for arbitrary coactions it seems much more difficult. We will develop the theory for dual coactions of Fell bundles over discrete groups, where the computations are so much cleaner than for continuous groups. It will be fairly obvious to the reader that some of what we will do in this paper can be done for Fell bundles over continuous groups, and indeed we plan to pursue this.

However, we feel it is valuable to have the machinery laid out for the case of discrete groups, since the discrete theory has a flavor all its own. We first define the Fell bundle which will be associated to the induced coaction. This will just be the "Banach *-algebraic bundle retraction" by the quotient map $G \to G/N$, as in [9, VIII.3.17], but we use different notation and terminology:

Definition 2.3 Suppose $(\mathcal{D}, G/N)$ is a Fell bundle over G/N, where N is a normal subgroup of the discrete group G, and let $q\colon G \to G/N$ be the quotient map. We define the *pull-back Fell bundle over G* as

$$q^*\mathcal{D} = \{(D_{sN}, s) : s \in G\}.$$

The bundle projection is $(d_{sN}, s) \mapsto s$, and we denote the fiber over s by $q^*D_s = (D_{sN}, s)$. Each fiber q^*D_s is given the Banach space structure of D_{sN}. The multiplication and involution are defined by

$$(d_{sN}, s)(d_{tN}, t) = (d_{sN}d_{tN}, st)$$
$$(d_{sN}, s)^* = (d_{sN}^*, s^{-1}).$$

It is completely routine to verify that the above operations indeed make $q^*\mathcal{D}$ into a Fell bundle over G.

Definition 2.4 Let $(D, G/N, \delta)$ be a coaction, $\mathcal{D}$ the associated Fell bundle over G/N, and $q^*\mathcal{D}$ the pull-back Fell bundle over G. We call the full cross sectional algebra $C^*(q^*\mathcal{D})$ the *algebra induced from D* and denote it by $\operatorname{Ind} D$, and we call the dual coaction on $C^*(q^*\mathcal{D})$ the *coaction induced from* δ and denote it by $\operatorname{Ind} \delta$.

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Remark 2.5 Note that the induced algebra and the induced coaction depend only upon the Fell bundle $\mathcal{D}$; in general D will be some intermediate algebra between $C^*(\mathcal{D})$ and $C^*_r(\mathcal{D})$. In a sense, our definition of the induced C^*-algebra $\operatorname{Ind} D$ above is somehow artificial: we could have equally well defined $\operatorname{Ind} D$ as the reduced cross sectional algebra $C^*_r(q^*\mathcal{D})$, or any algebra which lies "between" the full and the reduced cross sectional algebras and carries a coaction ϵ which satisfies $\epsilon(d_{sN}, s) = (d_{sN}, s) \otimes s$ for all $(d_{sN}, s) \in q^*D_s$. The only case where there is really a canonical choice is when $q^*\mathcal{D}$ is amenable in the sense of Exel [7], since then all cross sectional algebras are the same. We are going to study this problem in Section 6. Anyway, it follows from Lemma 2.1 that the crossed product $\operatorname{Ind} D \times_{\operatorname{Ind}\delta} G$ is *always* independent from the choice of the cross sectional algebra for $q^*\mathcal{D}$!

In view of the above remark it makes sense to give also the following

Definition 2.6 Let (A, G, δ) be a coaction of the discrete group G and let N be a normal subgroup of G. We say that (A, G, δ) is *weakly induced* from G/N if there exists a Fell bundle $(\mathcal{D}, G/N)$ such that $(q^*\mathcal{D}, G)$ is isomorphic to the Fell bundle associated to (A, G, δ).

Hence a weakly induced coaction is actually induced if and only if A is equal to the full cross sectional $C^*(\mathcal{A})$, where $(\mathcal{A}, G)$ is the Fell bundle associated to (A, G, δ).

Remark 2.7 If G is abelian, then our notion of induced coactions is the same as the notion of an induced action of the dual group $\hat{G}$ of G under the one-to-one correspondence between coactions of G and actions of $\hat{G}$. In order to explain this recall first that if $\alpha\colon \hat{G} \to \operatorname{Aut}(A)$ is an action, then the corresponding coaction δ_α of G on A is given by

$$\delta_\alpha\colon A \to C(\hat{G}, A); \quad \big(\delta_\alpha(a)\big)(\chi) := \alpha_\chi(a),$$

$\chi \in \hat{G}$. Here we made the identifications $C^*(G) \cong C(\hat{G})$ (via Fourier transform) and $A \otimes C(\hat{G}) \cong C(\hat{G}, A)$. Since the Fourier transform of $s \in C^*(G)$ is given by the function $\chi \mapsto \chi(s)$ on $\hat{G}$, we see that for $s \in G$ the spectral subspace A_s for δ_α is given by

$$A_s = \{a \in A : \alpha_\chi(a) = \chi(s)a \text{ for all } \chi \in \hat{G}\}.$$

Suppose now that N is a subgroup of G and let $\beta\colon \widehat{G/N} \to \operatorname{Aut}(B)$ be an action of the subgroup $\widehat{G/N} = N^\perp$ of $\hat{G}$. The induced C^*-algebra $\operatorname{Ind}(B, \beta)$ is then defined as

$$\operatorname{Ind}(B, \beta) := \{F \in C(\hat{G}, B) : F(\mu\chi) = \beta_{\bar{\chi}}\big(F(\mu)\big) \text{ for all } \chi \in \widehat{G/N}, \mu \in \hat{G}\},$$

with induced action $\operatorname{Ind}\beta\colon \hat{G} \to \operatorname{Aut}\big(\operatorname{Ind}(B, \beta)\big)$ given by

$$\big(\operatorname{Ind}\beta_\mu(F)\big)(\nu) := F(\bar{\mu}\nu).$$

We claim that the coactions $\big(\operatorname{Ind}(B, \beta), G, \delta_{\operatorname{Ind}\beta}\big)$ and $\big(\operatorname{Ind}(B, \delta_\beta), G, \operatorname{Ind}\delta_\beta\big)$ are isomorphic. For this it suffices to show that the Fell bundle associated to $\delta_{\operatorname{Ind}\beta}$ is isomorphic to the pull back of the bundle $(\mathcal{B}, G/N)$ associated to δ_β, since by the amenability of G there is only one topologically graded cross sectional algebra for this bundle [7, Theorem 4.7]. Indeed, we claim that the family of maps

$$\Phi_s\colon \operatorname{Ind}(B,\beta)_s \to (B_{sN}, s); \quad \Phi_s(F_s) = \big(F_s(1_G), s\big)$$

is well defined and gives the desired isomorphism of bundles. To see that it is well defined, it is enough to show that $F_s(1_G) \in B_{sN}$ for all $s \in G$. But $F_s \in \operatorname{Ind}(B,\beta)_s$ if and only if $\operatorname{Ind}\beta_\mu(F) = \mu(s)F$ for all $\mu \in \hat{G}$, so that

$$\beta_\chi\big(F_s(1_G)\big) = F_s(\bar{\chi}) = \big(\operatorname{Ind}\beta_\chi(F_s)\big)(1_G) = \chi(s)F_s(1_G),$$

for all $\chi \in \widehat{G/N}$. Thus $F_s(1_G) \in B_{sN}$. Each map Φ_s is norm preserving since

$$\|F_s(\mu)\| = \|\operatorname{Ind}\beta_{\bar{\mu}}F_s(1_G)\| = \|\bar{\mu}(s)F_s(1_G)\| = \|F_s(1_G)\|.$$

If $b_{sN} \in B_{sN}$ we can define an element $F_s \in \operatorname{Ind}(B,\beta)_s$ by $F_s(\mu) := \bar{\mu}(s)b_{sN}$, and then $\Phi_s(F_s) = (b_{sN}, s)$, thus each Φ_s is surjective. Finally, since all operations in $\operatorname{Ind}(B,\beta)$ are pointwise, it follows that the family of maps $(\Phi_s)_{s\in G}$ respects the bundle operations; for instance

$$\Phi_{st}(F_sF_t) = \big(F_s(1_G)F_t(1_G), st\big) = \big(F_s(1_G), s\big)\big(F_t(1_G), t\big) = \Phi_s(F_s)\Phi_t(F_t).$$

3 Enveloping Algebras and Discrete Coactions

In this section we show that for discrete groups all the C^*-calculations we must do are basically "pure algebra". The following terminology shows what we mean by "pure algebra".

Definition 3.1 Let B_0 be a *-algebra. By a *representation* of B_0 we always mean a *-homomorphism of B_0 into the bounded operators on a Hilbert space. We say B_0 *has an enveloping C^*-algebra* if the supremum of the C^*-seminorms on B_0 is finite, and in this case we call the Hausdorff completion of B_0 relative to this largest C^*-seminorm the *enveloping C^*-algebra* of B_0.

Thus, if B_0 is a *-subalgebra of a C^*-algebra B, and if every representation of B_0 is bounded in the norm inherited from B, then the closure of B_0 in B is the enveloping C^*-algebra of B_0. For example, we have the well known

Lemma 3.2 *If $(\mathcal{A}, G)$ is a Fell bundle, then $C^*(\mathcal{A})$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{A})$.*

Proof This follows immediately from the observation that a representation of $\Gamma_c(\mathcal{A})$ will be pointwise continuous into the operator norm topology, hence continuous for the inductive limit topology. ∎

The notion of enveloping algebras will be very convenient for crossed products by coactions: let (A, G, δ) be a coaction, and let $\mathcal{A}$ be the associated Fell bundle. Let $\mathcal{A} \times G$ be the

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product *Banach* bundle over $G \times G$. Then $\mathcal{A} \times G$ embeds in the crossed product $A \times_\delta G$ by identifying (a_s, t) with $j_A(a_s) j_G(\chi_t)$, and the algebraic operations become

$$(a_s, t)(a_u, v) = (a_s a_u, v) \quad \text{if } t = uv \text{ (and 0 else)}$$
$$(a_s, t)^* = (a_s^*, st). \tag{3.1}$$

Thus, $\mathcal{A} \times G$ acquires the structure of a Fell bundle *over the groupoid* $G \times G$ in the sense of [15], where $G \times G$ is given the transformation group groupoid structure associated with the action of G on itself by left translation:

$$(s, tr)(t, r) = (s, r) \quad \text{and} \quad (s, t)^{-1} = (s^{-1}, st).$$

Although the theory of Fell bundles over groupoids is still in its infancy, it seems reasonable to expect that many of the properties of Fell bundles over groups will carry over. In particular, the *-algebra of finitely supported sections should have an enveloping C^*-algebra. We only need this for the above special case (see Corollary 3.4 below).

We will need to know that

$$\|(a_s, t)\| := \|a_s\| = \| j_A(a_s) j_G(\chi_t)\| \tag{3.2}$$

for all $(a_s, t) \in \mathcal{A} \times G$, *i.e.*, that the above described embedding of $\mathcal{A} \times G$ into $A \times_\delta G$ is isometric. First of all, if E is any subset of G, then $\chi_E = \sum_{t \in E} \chi_t$ strictly in $\ell^\infty(G) = M\big(c_0(G)\big)$, so

$$j_G(\chi_E) = \sum_{t \in E} j_G(\chi_t),$$

strictly in $M(A \times_\delta G)$. Consequently, for $a_s \in A_s$ we have

$$j_A(a_s) = j_A(a_s)1 = j_A(a_s) \sum_{t \in G} j_G(\chi_t) = \sum_{t \in G} j_A(a_s) j_G(\chi_t).$$

Moreover, since the $j_G(\chi_t)$ are orthogonal projections summing strictly to 1 in $M(A \times_\delta G)$, and since j_A is faithful on the unit fiber algebra A_e, for $a_e \in A_e$ we have

$$\|a_e\| = \| j_A(a_e)\| = \sup_{t \in G} \| j_A(a_e) j_G(\chi_t)\|.$$

But $\| j_A(a_e) j_G(\chi_t)\|$ is independent of $t \in G$, since $j_A(a_e) j_G(\chi_s) = \hat{\delta}_{s^{-1}t}\big(j_A(a_e) j_G(\chi_t)\big)$ and $\hat{\delta}$ is isometric, being an automorphism of a C^*-algebra. Hence, we get

$$\|a_e\| = \| j_A(a_e) j_G(\chi_t)\| \quad \text{for all } a_e \in A_e,\ t \in G.$$

Since $a_s^* a_s \in A_e$ for all $a_s \in A_s$, Equation (3.2) now follows from

$$\begin{aligned} \|(a_s, t)\|^2 = \|a_s\|^2 = \|a_s^* a_s\| &= \| j_A(a_s^* a_s) j_G(\chi_t)\| \\ &= \big\|\big(j_A(a_s) j_G(\chi_t)\big)^* \big(j_A(a_s) j_G(\chi_t)\big)\big\| \\ &= \| j_A(a_s) j_G(\chi_t)\|^2. \end{aligned}$$

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We are now going to show that $\Gamma_c(\mathcal{A} \times G)$ does have an enveloping C^*-algebra, namely given by the crossed product $A \times_\delta G$, if $(\mathcal{A}, G)$ is the bundle associated to a given coaction δ. We do it in somewhat more generality, namely for (appropriate) dense subspaces of the fibers. Although we do not need the last whistle in the present paper, we include it since we feel it will be useful elsewhere.

To prepare for the statement, suppose we have a Fell bundle $(\mathcal{A}, G)$, and for each $s \in G$ we have a *linear* subspace B_s of A_s such that

$$B_sB_t \subset B_{st} \quad \text{and} \quad B_s^* = B_{s^{-1}}. \tag{3.3}$$

Then $\mathcal{B} := \bigcup_{s\in G} B_s$ is a subbundle of $\mathcal{A}$ (although not a Fell bundle, since its fibers may be incomplete). Let $\Gamma_c(\mathcal{B})$ denote the linear span of $\mathcal{B}$ in $\Gamma_c(\mathcal{A})$. Then $\Gamma_c(\mathcal{B})$ is a *-subalgebra, and conversely any *-subalgebra B of $\Gamma_c(\mathcal{A})$ which is the linear span of the intersections $B_s := B \cap A_s$ arises in this way.

Theorem 3.3 *Let $\mathcal{A}$ be a Fell bundle over the discrete group G. Suppose we have a subbundle $\mathcal{B}$ of $\mathcal{A}$ (with possibly incomplete fibers), satisfying* (3.3)*, such that A_e is the enveloping C^*-algebra of B_e and $C^*(\mathcal{A})$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{B})$. Then, regarding $\Gamma_c(\mathcal{B} \times G)$ as a *-subalgebra of $\Gamma_c(\mathcal{A} \times G) \subseteq C^*(\mathcal{A}) \times_{\delta_\mathcal{A}} G$ via the inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$, the inductive limit topology on $\Gamma_c(\mathcal{B} \times G)$ is stronger than the norm inherited from $C^*(\mathcal{A}) \times_{\delta_\mathcal{A}} G$, and $C^*(\mathcal{A}) \times_{\delta_\mathcal{A}} G$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{B} \times G)$.*

Proof Note first that by Equation (3.2) each fiber (B_s, t) embeds isometrically into $C^*(\mathcal{A}) \times_{\delta_\mathcal{A}} G$. In particular, this implies that for any finite subset E of $G \times G$ and for any element b of $\Gamma_c(\mathcal{B} \times G)$ supported in E, the norm $\|b\|$ of b as an element of $C^*(\mathcal{A}) \times_\delta G$ is bounded by $\max_{(s,t)\in E} \|b_{s,t}\|\,|E|$, where $|E|$ denotes the cardinality of E. This proves the first part of the theorem.

For the second part, let Π be a representation of $\Gamma_c(\mathcal{B} \times G)$. We must find a covariant representation (π, μ) of $\big(C^*(\mathcal{A}), G, \delta_\mathcal{A}\big)$ whose integrated form $\pi \times \mu$ is an extension of Π. For each $t \in G$ define a representation σ_t of the unit fiber algebra B_e of $\mathcal{B}$ by

$$\sigma_t(b_e) = \Pi(b_e, t).$$

Since A_e is the enveloping C^*-algebra of B_e, σ_t extends uniquely to a representation, which we continue to denote by σ_t, of A_e.

We will need to know that Π is automatically bounded on each fiber (B_s, t):

$$\begin{aligned}\|\Pi(b_s,t)\|^2 &= \|\Pi(b_s,t)^*\Pi(b_s,t)\| = \|\Pi(b_s^*,st)\Pi(b_s,t)\| \\ &= \|\Pi\big((b_s^*,st)(b_s,t)\big)\| = \|\Pi(b_s^*b_s,t)\| \\ &= \|\sigma_t(b_s^*b_s)\| \le \|b_s^*b_s\| = \|b_s\|^2 = \|(b_s,t)\|^2.\end{aligned}$$

Fix a bounded approximate identity $\{d_i\}$ for A_e, and put

$$p_t = \lim \sigma_t(d_i),$$

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the limit taken in the weak operator topology. The p_t's are orthogonal projections, and so determine a representation μ of $c_0(G)$ such that $\mu(\chi_t) = p_t$. Note that

$$\begin{aligned}\Pi(b_s, r)p_t &= \Pi(b_s, r)\lim \Pi(d_i, t)\\ &= \lim \Pi(b_s, r)\Pi(d_i, t)\\ &= \lim \Pi(b_s d_i, t) \quad \text{if } r = t \text{ (and 0 else)}\\ &= \Pi(b_s, r) \quad \text{if } t = r \text{ (and 0 else)},\end{aligned}$$

since Π is bounded on each fiber (B_s, r). Also,

$$\begin{aligned}p_{st}\Pi(b_s, t) &= \lim \Pi(d_i, st)\Pi(b_s, t)\\ &= \lim \Pi(d_i b_s, t)\\ &= \Pi(b_s, t),\end{aligned}$$

again by boundedness on the fibers. Furthermore, one of the above computations implies $\|\Pi(b_s, t)\|^2 \leq \|b_s\|^2$. Consequently, the sum

$$\pi(b_s) := \sum_{t\in G} \Pi(b_s, t)\Big(= \sum_{t\in G} \Pi(b_s, t)p_t\Big)$$

converges in the weak operator topology, and $\|\pi(b_s)\| \leq \|b_s\|$ for all $b_s \in B_s$. Standard properties of the weak operator topology allow one to show

$$\pi(b_s b_t) = \pi(b_s)\pi(b_t) \text{ and } \pi(b_s^*) = \pi(b_s)^* \quad \text{for } b_s \in B_s,\ b_t \in B_t,$$

so the map π extends uniquely to a representation of the *-algebra $\Gamma_c(\mathcal{B})$, hence to a representation, which we still call π, of $C^*(\mathcal{A})$, since $C^*(\mathcal{A})$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{B})$. The reader can check that the pair (π, μ) is covariant. Then we have

$$(\pi \times \mu)(b_s, t) = \pi(b_s)\mu(\chi_t) = \Big(\sum_{r\in G} \Pi(b_s, r)\Big)p_t = \Pi(b_s, t),$$

as desired. ∎

As a direct consequence of Theorem 3.3 together with Lemma 2.1 we obtain

Corollary 3.4 *Let (A, G, δ) be a coaction of the discrete group G and $(\mathcal{A}, G)$ the corresponding Fell bundle. Then $A \times_\delta G$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{A} \times G)$.*

4 Imprimitivity Theorem

In this section we obtain the anticipated dual mirror of Green's imprimitivity theorem [11, Theorem 17]. We remind the reader that G denotes a *discrete* group and N a normal subgroup. Starting with a Fell bundle $\mathcal{D}$ over G/N, we will construct a $C^*(q^*\mathcal{D}) \times_{\operatorname{Ind}\delta_{\mathcal{D}}} G -$

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$C^*(\mathcal{D}) \times_{\delta_{\mathcal{D}}} G/N$ imprimitivity bimodule X, where $\delta_{\mathcal{D}}$ is the dual coaction. Moreover, we will make X equivariant for the actions $\widehat{\operatorname{Ind}\delta_{\mathcal{D}}}$ and $\operatorname{Inf}\widehat{\delta_{\mathcal{D}}}$ of G, where the latter action is the inflation to G of the dual action $\widehat{\delta_{\mathcal{D}}}$ of G/N on $C^*(\mathcal{D}) \times_{\delta_{\mathcal{D}}} G/N$. This is a dual analogue of a result due to Raeburn and the first author [4, Theorem 3.3], which shows the symmetric imprimitivity theorem of [27, Theorem 1.1] (which includes Green's imprimitivity theorem) is equivariant for suitable coactions.

As usual, we will work with dense subspaces. For $C^*(\mathcal{D}) \times_{\delta_{\mathcal{D}}} G/N$ we take the dense *-subalgebra $C_0 := \Gamma_c(\mathcal{D} \times G/N)$, where we remind the reader to regard $\mathcal{D} \times G/N$ as a Fell bundle over the groupoid $G/N \times G/N$, with operations given by (3.1) (with G replaced by G/N). For $C^*(q^*\mathcal{D}) \times_{\operatorname{Ind}\delta_{\mathcal{D}}} G$ we form the corresponding dense *-subalgebra $B_0 := \Gamma_c(q^*\mathcal{D} \times G)$, except that for $(d_{sN}, s) \in q^*D_s$ and $t \in G$ we write the pair $\big((d_{sN}, s), t\big)$ simply as a triple (d_{sN}, s, t). For reference, the operations are

$$(d_{sN}, s, uv)(d_{uN}, u, v) = (d_{sN} d_{uN}, su, v)$$
$$(d_{sN}, s, t)^* = (d_{sN}^*, s^{-1}, st).$$

Also, the actions $\widehat{\operatorname{Ind}\delta_{\mathcal{D}}}$ and $\operatorname{Inf}\widehat{\delta_{\mathcal{D}}}$ on the subalgebras B_0 and C_0 are given on the generators by

$$\widehat{\operatorname{Ind}\delta_{\mathcal{D}}}(r)(d_{sN}, s, t) = (d_{sN}, s, tr^{-1})$$
$$\operatorname{Inf}\widehat{\delta_{\mathcal{D}}}(r)(d_{sN}, tN) = (d_{sN}, tr^{-1}N).$$

Our $B_0 - C_0$ pre-imprimitivity bimodule will be $X_0 := \Gamma_c(\mathcal{D} \times G)$, where $\mathcal{D} \times G$ is the product Banach bundle over $G/N \times G$. The pre-Hilbert bimodule operations are defined on the generators by

$$(d_{sN}, t) \cdot (d_{uN}, vN) = (d_{sN} d_{uN}, t) \quad \text{if } s^{-1}tN = uvN \text{ (and 0 else)}$$
$$\langle (d_{sN}, t), (d_{uN}, v) \rangle_{C_0} = (d_{sN}^* d_{uN}, u^{-1}vN) \quad \text{if } t = v \text{ (and 0 else)}$$
$$(d_{qN}, q, r) \cdot (d_{sN}, t) = (d_{qN} d_{sN}, qt) \quad \text{if } r = t \text{ (and 0 else)}$$
$${}_{B_0}\langle (d_{sN}, t), (d_{uN}, v) \rangle = (d_{sN} d_{uN}^*, tv^{-1}, v) \quad \text{if } su^{-1}N = tv^{-1}N \text{ (and 0 else)},$$

and then extended bilinearly (or sesquilinearly, as the case may be). Our action γ of G on X_0 is defined on the generators by

$$\gamma_s(d, t) = (d, ts^{-1}).$$

It is easy to check that all operations are continuous in the inductive limit topologies.

Theorem 4.1 *If $(\mathcal{D}, G/N)$ is a Fell bundle, where N is a normal subgroup of the discrete group G, the above operations make X_0 into a $B_0 - C_0$ pre-imprimitivity bimodule. Consequently, the completion is a $C^*(q^*\mathcal{D}) \times_{\operatorname{Ind}\delta_{\mathcal{D}}} G - C^*(\mathcal{D}) \times_{\delta_{\mathcal{D}}} G/N$ imprimitivity bimodule X.*

Moreover, the above formula for γ determines an action of G on X which implements a Morita equivalence between the actions $\widehat{\operatorname{Ind}\delta_{\mathcal{D}}}$ and $\operatorname{Inf}\widehat{\delta_{\mathcal{D}}}$.

Proof For the first statement, we must check:

(i) X_0 is a $B_0 - C_0$ bimodule;
(ii) ${}_{B_0}\langle b\cdot x, y\rangle = b\,{}_{B_0}\langle x, y\rangle$ and $\langle x, y\cdot c\rangle_{C_0} = \langle x, y\rangle_{C_0} c$;
(iii) ${}_{B_0}\langle x, y\rangle^* = {}_{B_0}\langle y, x\rangle$ and $\langle x, y\rangle_{C_0}^* = \langle y, x\rangle_{C_0}$;
(iv) ${}_{B_0}\langle x, y\rangle$ is linear in x and $\langle x, y\rangle_{C_0}$ is linear in y;
(v) $x\cdot\langle y, z\rangle_{C_0} = {}_{B_0}\langle x, y\rangle\cdot z$;
(vi) span ${}_{B_0}\langle X_0, X_0\rangle$ is dense in B_0 and span $\langle X_0, X_0\rangle_{C_0}$ is dense in C_0;
(vii) ${}_{B_0}\langle x, x\rangle \geq 0$ and $\langle x, x\rangle_{C_0} \geq 0$;
(viii) $\langle b\cdot x, b\cdot x\rangle_{C_0} \leq \|b\|^2\langle x, x\rangle_{C_0}$ and ${}_{B_0}\langle x\cdot c, x\cdot c\rangle \leq \|c\|^2\langle x, x\rangle_{B_0}$.

The verifications of (i)–(v) are routine, and we will just give one sample of the computations, leaving the rest to the conscientious reader. We will prove (vi)–(vii) in one whack using Rieffel's trick: it suffices to produce nets in both B_0 and C_0, each term of which is a finite sum of the form $\sum\langle x_i, x_i\rangle$, which are approximate identities for both the algebras and the module multiplications, in the inductive limit topologies (for example, see the discussion following [11, Lemma 2]). We prove (viii) by showing we have homomorphisms of B_0 and C_0 into the adjointable operators on the respective Hilbert modules, which suffices since by Corollary 3.4 the crossed products mentioned in the statement of the theorem are the enveloping C^*-algebras of B_0 and C_0. (i) follows readily from the definition. For example, the formula for the right action of C_0 on X_0 is obviously bilinear on each product $(D_{sN}, t)\times(D_{uN}, vN)$, and then the extension to $X_0\times C_0$ is bilinear by definition; similarly for the action of B_0. We check associativity of the B_0-action, leaving to the reader the similar verification for the C_0-action:

$$\begin{aligned}\big((d_{qN}, q, rt)(d_{rN}, r, t)\big)\cdot(d_{sN}, t) &= (d_{qN}d_{rN}, qr, t)\cdot(d_{sN}, t)\\ &= (d_{qN}d_{rN}d_{sN}, qrt)\\ &= (d_{qN}, q, rt)\cdot(d_{rN}d_{sN}, rt)\\ &= (d_{qN}, q, rt)\cdot\big((d_{rN}, r, t)\cdot(d_{sN}, t)\big),\end{aligned}$$

which shows associativity on the generators, and associativity follows in general by bilinearity. Similarly, commutativity of the left and right module multiplications is readily verified on the generators, hence follows in general by bilinearity. The verifications of (ii)–(v) are quite similar, and we leave them to the reader.

In order to apply the Rieffel trick, we construct an appropriate approximate identity for C_0, leaving the (easier) construction for B_0 to the reader. Let $\{d_i\}_{i\in I}$ be a bounded approximate identity for the unit fiber D_N, and let $\mathcal{F}$ denote the family of finite subsets of G/N, directed by inclusion. For each $F\in\mathcal{F}$ choose $S_F\subset G$ comprising exactly one element from each coset in F. Claim:

$$\Big\{\sum_{t\in S_F}\langle(d_i^{1/2}, t), (d_i^{1/2}, t)\rangle_{C_0}\Big\}_{(i,F)\in I\times\mathcal{F}}$$

is an approximate identity for both the algebra C_0 and the right module multiplication of C_0 on X_0, in the inductive limit topologies. First of all, note that

$$\langle(d_i^{1/2}, t), (d_i^{1/2}, t)\rangle_{C_0} = (d_i, tN).$$

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For each generator (d_{uN}, vN) we have

$$\Big(\sum_{t\in S_F}(d_i, tN)\Big)(d_{uN}, vN) = \sum_{tN\in F}(d_i d_{uN}, vN) \quad \text{if } tN = uvN \text{ (and 0 else)}$$
$$= (d_i d_{uN}, vN) \quad \text{whenever } F \geq \{uvN\},$$

which tends in the norm of the Banach space D_{uN} to (d_{uN}, vN) since $D_N D_{uN} = D_{uN}$. It follows immediately that for all $c \in C_0$ the products $\sum(d_i, tN)c$ converge to c in the inductive limit topology, and similarly for multiplication on the other side. For the C_0-module multiplication, we similarly have

$$(d_{sN}, r) \cdot \sum(d_i, tN) = (d_{sN} d_i, r) \quad \text{whenever } F \geq \{s^{-1}rN\},$$

which converges in norm to (d_{sN}, r), hence if we replace (d_{sN}, r) by any $x \in X_0$ the convergence will be in the inductive limit topology, as desired.

We show (viii) for B_0, leaving the easier verification for C_0 to the reader. Take any state ω on $C^*(\mathcal{D}) \times_{\delta_{\mathcal{D}}} G$. We get a semi-inner product $\omega(\langle\cdot,\cdot\rangle_{C_0})$ on X_0. Let H be the corresponding inner product space and $\Theta\colon X_0 \to H$ the quotient map. Then the left module action of B_0 on X_0 defines a *-homomorphism π from B_0 to the *-algebra of adjointable linear operators on H via $\pi(b)\Theta(x) = \Theta(b \cdot x)$. Claim: for all $b \in B_0$, the operator $\pi(b)$ is bounded, and $\|\pi(b)\| \leq \|b\|$. Since B_0 has the greatest C^*-norm, it suffices to show $\pi(b)$ is bounded. For this, without loss of generality let $b = (d_{sN}, s, t)$ for some $s, t \in G$. Then $b^*b = (d_{sN}^* d_{sN}, e, t) \in (D_N, e, t)$, which is a C^*-algebra. Hence, $\pi(b^*b)$ is bounded. Since $\pi(b)^*\pi(b) = \pi(b^*b)$, $\pi(b)$ must be bounded as well, and the claim is verified. Hence, for all $b \in B_0$ and $x \in X_0$,

$$\omega(\langle b\cdot x, b\cdot x\rangle_{C_0}) = \big(\pi(b)\Theta(x), \pi(b)\Theta(x)\big) \leq \|\pi(b)\|^2\big(\Theta(x), \Theta(x)\big) \leq \|b\|^2\omega(\langle x, x\rangle_{C_0}).$$

Since the state ω was arbitrary,

$$\langle b\cdot x, b\cdot x\rangle_{C_0} \leq \|b\|^2\langle x, x\rangle_{C_0},$$

as required.

Finally, γ is clearly an algebraic representation of G on the vector space X_0, and straightforward computations (first on the generators and then extending by bi-additivity) show that

$${}_{B_0}\langle\gamma_r(x), \gamma_r(y)\rangle = \widehat{\operatorname{Ind}\delta_{\mathcal{D}}}(r)({}_{B_0}\langle x, y\rangle) \quad \text{and} \quad \gamma_r(x\cdot c) = \gamma_r(x)\cdot \operatorname{Inf}\widehat{\delta_{\mathcal{D}}}(r)(c)$$

for all $r \in G$, $x, y \in X_0$, and $c \in C$. Hence, each linear map γ_r on X_0 extends to X, and we get an algebraic representation of G on X, where the above identities continue to hold. This is enough to show γ gives a Morita equivalence between the actions $\widehat{\operatorname{Ind}\delta_{\mathcal{D}}}$ and $\operatorname{Inf}\widehat{\delta_{\mathcal{D}}}$. ∎

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5 Characterizations of Induced Coactions, the Olesen-Pedersen Theorem, and the Mackey Machine

In this section we want to give a number of useful characterizations of induced coactions, which are analogues of similar characterizations in the theory of induced actions. We will then use our results for a discussion about the possible (or impossible) development of a Mackey machine for coactions.

In [3] the first author gave a characterization of induced actions: there must be an equivariant homomorphism of $C_0(G/N)$ into the central multipliers of the algebra carrying the action. Here we give a corresponding characterization of induced coactions, involving an appropriate equivariant homomorphism of N. Recall from [9] that if $(\mathcal{A}, G)$ is a Fell bundle, a *multiplier* of a fiber A_s (called a "multiplier of order s" in [9]) is a pair $m = (L, R)$ of maps from $\mathcal{A}$ to itself such that $L(A_t) \subset A_{st}$ and $R(A_t) \subset A_{ts}$, and the associativity property

$$R(a)b = aL(b) \quad \text{for } a, b \in \mathcal{A}$$

holds, and one writes $L(a) = ma$ and $R(a) = am$. We write $M(A_s)$ for the set of all multipliers of A_s. The adjoint of m is defined by $m^*a = (a^*m)^*$ and $am^* = (ma^*)^*$, and m is called *unitary* if $m^*m = mm^* = 1$, the identity element of $M(A_e)$ (and the latter fortunately agrees with the usual notion of the multipliers of the C^*-algebra A_e). We write $UM(A_s)$ for the set of all unitary multipliers of A_s, and $UM(\mathcal{A})$ for the set of all unitary multipliers of the bundle $\mathcal{A}$.

Theorem 5.1 *If N is a normal subgroup of the discrete group G, a Fell bundle $\mathcal{A}$ over G is isomorphic to a pull-back bundle $q^*\mathcal{D}$ for some Fell bundle $\mathcal{D}$ over G/N if and only if there is a homomorphism u of N into $UM(\mathcal{A})$ such that*

(i) $u_n \in M(A_n)$ for all $n \in N$;
(ii) $a_s u_n = u_{sns^{-1}} a_s$ for all $a_s \in A_s$, $n \in N$.

Consequently, a coaction (A, G, δ) is weakly induced from a coaction of G/N if and only if there is a homomorphism u of N into $UM(A)$ such that $\delta(u_n) = u_n \otimes n$ for $n \in N$ and (ii) above holds.

Proof Starting with a Fell bundle $\mathcal{D}$ over G/N, it is easy to check that $u_n := (1, n) \in M(q^*\mathcal{D})$ (where the 1 denotes the identity element of $M(D_N)$) has the required properties.

Conversely, assume we have a Fell bundle $\mathcal{A}$ and a map u satisfying (i)–(ii). Let N act on the right of $\mathcal{A}$ by $a \cdot n = au_n$. Our bundle $\mathcal{D}$ will be the orbit space of this action. Write $[a]$ for the N-orbit of $a \in \mathcal{A}$, and for $s \in G$ define

$$D_{sN} = \{[a] : a \in A_{sn} \text{ for some } n \in N\}.$$

For each s the orbit map $a \mapsto [a]$ takes A_s bijectively onto D_{sN}, and the resulting Banach space structure on D_{sN} depends only upon the coset sN. More precisely, for $a_s, b_s \in A_s$, $\lambda \in \mathbb{C}$, and $n \in N$ we have

$$[a_s + b_s] = [a_s u_n + b_s u_n]$$
$$[\lambda a_s] = [\lambda a_s u_n].$$

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Thus, each fiber D_{sN} has a well defined Banach space structure, and we get a Banach bundle $\mathcal{D}$ over G/N. Define multiplication and involution in $\mathcal{D}$ by

$$[a][b] = [ab]$$
$$[a]^* = [a^*].$$

The reader can easily check that these operations are well defined: for example, if $a_s \in A_s$, $a_t \in A_t$, and $n, k \in N$, then

$$[(a_s u_n)(a_t u_k)] = [a_s a_t u_{t^{-1}ntk}] = [a_s a_t],$$

and that they give $\mathcal{D}$ a Fell bundle structure: for example, if $a_s \in A_s$ and $a_t \in A_t$, then $[a_s a_t] \in D_{stN}$, so $D_{sN} D_{tN} \subset D_{sNtN}$, and for the C^*-norm property we have

$$\begin{aligned}\|[a_s]^*[a_s]\| &= \|[a_s^*][a_s]\| = \|[a_s^* a_s]\| = \|a_s^* a_s\| \\ &= \|a_s\|^2 = \|[a_s]\|^2.\end{aligned}$$

To finish, just check that the map $\phi\colon \mathcal{A} \to q^*\mathcal{D}$ defined by

$$\phi(a_s) = ([a_s], s) \quad \text{for } a_s \in A_s$$

is a Fell bundle isomorphism: for example,

$$\begin{aligned}\phi(a_s)\phi(a_t) &= ([a_s], s)([a_t], t) = ([a_s][a_t], st) \\ &= ([a_s a_t], st) = \phi(a_s a_t).\end{aligned}$$

The other part follows immediately since a coaction is weakly induced if and only if the associated Fell bundle is a pull-back. ■

Remark 5.2 Of course, if $A = C^*(\mathcal{A})$, where $\mathcal{A}$ is the bundle associated to (A, G, δ) (which is for instance always true if $\mathcal{A}$ is amenable, which in turn is always true if G is amenable by [7, Theorem 4.7]), then we can replace the term "weakly induced" by the term "induced" in the above theorem.

Note that when G is abelian the above characterization reduces to that of [3], since we then have an equivariant nondegenerate homomorphism of $C_0(\hat{G}/N^\perp)$ into the central multipliers of $C^*(\mathcal{A})$ (since $u_{sns^{-1}} = u_n$ when G is abelian), and then the Dauns-Hoffman theorem shows that [3] applies.

We now give several applications of the above result in connection with crossed products by twisted actions in the sense of Green [11]. Let (B, G, α) be an action, and let $(B \times G, G)$ be the associated semidirect product Fell bundle as in [9], so that $C^*(B \times G)$ is naturally isomorphic to the crossed product $B \times_\alpha G$ and $C_r^*(B \times G)$ to $B \times_{\alpha,r} G$. For reference, the operations on the bundle are

$$(b, s)(c, t) = \big(b\alpha_s(c), st\big) \quad \text{and} \quad (b, s)^* = \big(\alpha_{s^{-1}}(b)^*, s^{-1}\big).$$

Recall that a Green-twisted system (B, G, N, α, τ) consists of an action α of G on B together with a (strictly continuous) homomorphism $\tau\colon N \to UM(B)$ satisfying the equations $\alpha_s(\tau_n) = \tau_{sns^{-1}}$ and $\alpha_n(b) = \tau_n b \tau_{n^{-1}}$ for all $s \in G, n \in N, b \in B$.

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Definition 5.3 Let (B, G, N, α, τ) be a twisted action, and let N act from the right on the (untwisted) semidirect product bundle $B \times G$ via

$$(b, s) \cdot n = (b\tau_n, n^{-1}s).$$

The *twisted semidirect product bundle* over G/N is the orbit space, which we denote by $B \times_N G$. Let $[b, s]$ denote the N-orbit of $(b, s) \in B \times G$. Then the bundle projection is defined to be $[b, s] \mapsto sN$, and (since $[b, ns] = [b\tau_n, s]$ for $n \in N$) the fibers are given by

$$(B \times_N G)_{sN} = \{[b, s] : b \in B\}.$$

The full and reduced twisted crossed products are naturally isomorphic to the respective full and reduced cross sectional C^*-algebras of the Fell bundle $B \times_N G$. Landstad [16] gave a characterization of reduced crossed products by actions, in terms of (what we now call) the dual coaction, and it would be useful to have a version of this "Landstad duality" for twisted crossed products. We prove such a result here when the group is discrete, in which case it is equivalent to characterize twisted semidirect product Fell bundles.

Theorem 5.4 *If N is a normal subgroup of the discrete group G, a Fell bundle $\mathcal{D}$ over G/N is isomorphic to a twisted semidirect product bundle if and only if there is a homomorphism u of G into $UM(\mathcal{D})$ with $u_s \in D_{sN}$ for all $s \in G$.*

Consequently, a C^*-algebra D is isomorphic to a reduced twisted crossed product $B \times_{\alpha,\tau,r} G$ by an action of G which is twisted over N if and only if there are a normal coaction δ of G/N on D and a homomorphism u of G into $UM(D)$ such that $\delta(u_s) = u_s \otimes sN$ for $s \in G$.

Proof Given a twisted action (B, G, N, α, τ), it is easy to check that taking $u = j_G$, where $j_G \colon G \to UM(B \times_{\alpha,\tau,r} G)$ is the canonical homomorphism, gives a map with the required property.

Conversely, assume we have a Fell bundle $\mathcal{D}$ over G/N and a map u as in the statement of the theorem. Put $B = D_N$. Then $u_s B u_s^* = B$ for all $s \in G$, so $\alpha_s = \operatorname{Ad} u_s$ gives an action α of G on B. Moreover, it is easy to check that the map $\tau := u|_N$ is a twist for α. So, we have a twisted action (B, G, N, α, τ). It is readily verified that the map $\phi \colon B \times_N G \to \mathcal{D}$ defined by

$$\phi([b, s]) = bu_s$$

is well defined (since $\tau = u|_N$) and gives a Fell bundle isomorphism of the twisted semidirect product bundle $B \times_N G$ onto $\mathcal{D}$, using the equality $D_{sN} = D_N u_s$ for $s \in G$.

The other part follows immediately, since the coaction δ is normal if and only if D is isomorphic to the reduced cross sectional algebra of the associated Fell bundle. ∎

Remark 5.5 Alternatively, we could prove the above theorem by pulling back to a Fell bundle over G, suitably carrying along the map u, then using Landstad's original theorem, and finally appealing to a uniqueness clause in the characterization of $q^*\mathcal{D}$. This would be closer to the strategy of [24], but the above proof is much shorter and more direct.

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Olesen and Pedersen [20] proved that if (B, G, N, α, τ) is a twisted action of an abelian (not necessarily discrete) group G, then the dual action of $\hat{G}$ on the untwisted crossed product $B \times_\alpha G$ is induced from the dual action of $N^\perp$ on the twisted crossed product $B \times_{\alpha,\tau} G$. It often happens that results about actions of abelian groups can be "naturally" viewed as results about coactions, and in this case Raeburn and the second author proved in [24] an extension of this result for nonabelian G: if $(B, G, G/N, \delta, \kappa)$ is a twisted coaction (in the sense of Phillips and Raeburn [21]), then the dual action of G on the untwisted crossed product $B \times_\delta G$ is induced from the dual action of N on the twisted crossed product $B \times_{\delta,\kappa} G$. Here we prove a different sort of (partial, since our groups are discrete) extension:

Theorem 5.6 *If (B, G, N, α, τ) is a twisted action of the discrete group G, then the semidirect product bundle $B \times G$ is isomorphic to the pull-back of the twisted semidirect product bundle $B \times_N G$. Consequently, the dual coaction of G on the crossed product $B \times_\alpha G$ is induced from the dual coaction of G/N on the twisted crossed product $B \times_{\alpha,\tau} G$.*

Conversely, if (B, G, α) is an action, and if the semidirect product bundle $B \times G$ is pulled back from a Fell bundle over G/N, or equivalently the dual coaction $\hat{\alpha}$ is induced from a coaction of G/N, then α is twisted over N.

Proof If (B, G, N, α, τ) is a twisted action, the reader can easily check that the assignment

$$(b, s) \mapsto ([b, s], s)$$

gives a Fell bundle isomorphism of $B \times G$ onto $q^*(B \times_N G)$. The statement concerning induced coactions follows immediately from the fact that $B \times_\alpha G = C^*(B \times G)$.

Conversely, suppose $B \times G$ is isomorphic to a pull-back. Then by Theorem 5.1 there is a homomorphism $u \colon N \to UM(B \times G)$ satisfying (i)–(ii) of that theorem. The reader can easily check that the map

$$\tau_n := (1, n)u_{n^{-1}}$$

gives a twist for the action α. ∎

A very important step in the modern version of the Mackey machine for actions is the fact that under favourable circumstances a G-simple action of a group G on a C^*-algebra A is automatically induced from a simple system, *i.e.*, from a system (D, H, β) with D simple. This works especially well for actions of compact groups. To be more precise: suppose that a compact group G acts on a C^*-algebra A with Hausdorff primitive ideal space $\mathrm{Prim}(A)$ such that there exists no nontrivial G-invariant ideal of A. Then it follows directly from the compactness of G that if we pick any $P \in \mathrm{Prim}(A)$ then $\mathrm{Prim}(A)$ is homeomorphic to G/H as a G-space, where $H = \{s \in G : \alpha_s(P) = P\}$ denotes the stabilizer of P. Applying [3, Theorem], it then follows that (A, G, α) is isomorphic to the induced system $(\mathrm{Ind}_H^G D, G, \mathrm{Ind}\,\beta)$, where $D = A/P$ and β is the associated action (determined by $\alpha|_H$) of H on the quotient A/P (note that the Hausdorff assumption of $\mathrm{Prim}(A)$ could even be omitted, by [19, Lemma 2.1]). The resulting Morita equivalence between $D \times_\beta H$ and $A \times_\alpha G$ constitutes one of the main steps for the Mackey machine for actions (compare with [11, Theorem 18]).

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Since coactions of discrete groups behave somehow similarly to actions of compact groups (and for abelian G they actually correspond directly to the actions of the compact group $\hat{G}$), it would be an easy guess that a statement similar to the above should be true for coactions of discrete groups. But trying to do this we run into trouble very soon: although one can make sense of a definition of a G-simple coaction (see below), it is certainly not clear at all what quotient of G should serve as a "stabilizer" of a primitive ideal of A. Actually, in what follows next we are going to show that a result similar to the above can not hold in general for coactions. We do this by using the previous characterizations of induced coactions.

***Definition 5.7* (*cf.* [18, Section 2], [17, Definition 4.1])** Let $\delta\colon A \to A \otimes C^*(G)$ be a coaction of a group G on a C^*-algebra A. A closed ideal I of A is called *δ-invariant* if $\delta(I)\big(1 \otimes C^*(G)\big) = I \otimes C^*(G)$. (A, G, δ) is called *G-simple* if A has no δ-invariant ideals.

Remark 5.8 Of course, the above definition makes sense also for coactions of non-discrete groups. Notice that there are weaker notions of G-invariant ideals (and hence, stronger notions of G-simple cosystems). For instance one could define an ideal of A to be G-invariant if $I = \ker(\phi \otimes \mathrm{id}) \circ \delta$, where $\phi\colon A \to A/I$ is the quotient map (*e.g.*, see [5, Definition 2.4]). However, if G is amenable then it is an easy consequence of [17, Proposition 4.3] that both definitions coincide.

The following lemma is a special case of [10, Corollary 3.5] if G is amenable.

Lemma 5.9 *Let (B, G, β) be a G-simple system (i.e., B contains no G-invariant closed ideals). Then $(B \times_\beta G, G, \hat{\beta})$ is a G-simple cosystem.*

Proof First note that β is G-simple if and only if its double dual action $\hat{\hat{\beta}}$ is G-simple. This follows easily from the generalized duality theorem of Imai and Takai for full crossed products [26, Theorem 7]. Suppose that there exists a nontrivial closed $\hat{\beta}$-invariant ideal I of $A = B \times_\beta G$. By [18, Propositions 2.1 and 2.2] there are well defined "restrictions" $\hat{\beta}_I$ and $\hat{\beta}_{A/I}$ of $\hat{\beta}$ to I and A/I, and by [18, Theorem 2.3] we get an exact sequence

$$0 \to I \times_{\hat{\beta}_I} G \to A \times_{\hat{\beta}} G \to A/I \times_{\hat{\beta}_{A/I}} G \to 0$$

with respect to the canonical maps, which are all equivariant with respect to the double dual action $\hat{\hat{\beta}}$ (the last assertion following directly from the definition of the maps as given in [18, Theorem 2.3]). But this shows that $I \times_{\hat{\beta}_I} G$ is a nontrivial $\hat{\hat{\beta}}$-invariant ideal, which shows that $\hat{\hat{\beta}}$ and hence β is not G-simple. ∎

Remark 5.10 For amenable G Gootman and Lazar proved in [10, Corollary 3.5] that an action β is G-simple if and only if the dual coaction $\hat{\beta}$ is G-simple, and it follows from [10, Theorem 3.7] that the dual version of this result is also true: if G is amenable, then a cosystem (A, G, δ) is G-simple if and only if the dual system $(A \times_\delta G, G, \hat{\delta})$ is G-simple.

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As mentioned above, a G-simple action of a *compact* group is always induced from an action on a simple C^*-algebra. We are now going to show that, unfortunately, a similar result does *not* hold for coactions of discrete groups. In fact, we can create a multitude of counterexamples by using the following easy corollary of Theorem 5.6

Corollary 5.11 *Let (B, G, β) be an action with G discrete, and for each $P \in \operatorname{Prim}(B)$ denote by $S_P \subseteq G$ the stabilizer of P under the corresponding action of G on $\operatorname{Prim}(B)$. Suppose further that $\bigcap\{S_P : P \in \operatorname{Prim}(B)\} = \{e\}$. Then the dual coaction $\hat{\beta}$ is not (weakly) induced from any nontrivial quotient of G.*

Proof Assume that there is a nontrivial subgroup N of G such that $\hat{\beta}$ is (weakly) induced from G/N. Then β is twisted over N, by Theorem 5.6, which in particular implies that the restriction of β to N is implemented by a homomorphism $\tau\colon N \to UM(B)$. But this implies that N stabilizes all primitive ideals of B, a contradiction to $\bigcap\{S_P : P \in \operatorname{Prim}(B)\} = \{e\}$. ∎

Example 5.12 Let G be a discrete group which has a nontrivial subgroup H such that $\bigcap_{s\in G} sHs^{-1} = \{e\}$. There are many examples of such groups: *e.g.*, take G any finite simple group and H any nontrivial subgroup, or take $G = \mathbb{F}_n$ and H a cyclic subgroup. Put $B = C_0(G/H)$ and let $\beta\colon G \to \operatorname{Aut}(B)$ denote the action given by left translation. Then β is a G-simple action, and hence $\hat{\beta}$ is G-simple by Lemma 5.9. It follows from Corollary 5.11 that $\hat{\beta}$ is not induced from any nontrivial quotient G/N, since the stabilizers of β are just the groups sHs^{-1}, $s \in G$. However, it follows from Rieffel's version of the imprimitivity theorem for groups [28] that $B \times_\beta G$ is Morita equivalent to $C^*(H)$, which in both specific examples mentioned above is not simple and has Hausdorff primitive ideal space.

6 Amenability For Pull-Back Bundles

Recall that a Fell bundle $(\mathcal{A}, G)$ is called *amenable* if the regular representation $\Lambda\colon C^*(\mathcal{A}) \to C_r^*(\mathcal{A})$ is faithful. In this section we want to investigate under what conditions the pull-back bundle $(q^*\mathcal{D}, G)$ of a bundle $(\mathcal{D}, G/N)$ is amenable. This question is particularly interesting for us, since only for amenable bundles is there a unique choice for the induced algebra $\operatorname{Ind} D$, where D is any cross sectional algebra of $\mathcal{D}$ which carries a coaction ϵ of G/N given by $\epsilon(d_{sN}) = d_{sN} \otimes sN$. Inspired by earlier work of C. Anantharaman-Delaroche, Exel introduced in [7] a certain approximation property for Fell bundles, and he proved that this gives a sufficient condition for $\mathcal{A}$ to be amenable. Let us recall his condition:

Definition 6.1 **(*cf.* [7, Definition 4.4])** Let $\mathcal{A}$ be a Fell bundle over the discrete group G. We say that $\mathcal{A}$ has property (EP), if there exists a net of functions $f_i\colon G \to A_e$ with finite supports and satisfying:

(i) $\sup_{i\in I} \|\sum_{s\in G} f_i(s)^* f_i(s)\| < \infty$;
(ii) $\lim_{i\to\infty} \sum_{s\in G} f_i(ts)^* a_t f_i(s) = a_t$ for all $a_t \in A_t$.

Note that Exel did not require finite supports of the functions f_i in his definition of the approximation property. But by [7, Proposition 4.5] his definition is equivalent to the

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above. In [7, Theorem 4.6], Exel showed that property (EP) implies amenability of $\mathcal{A}$. As the main result of this section we will prove that a pull-back bundle $(q^*\mathcal{D}, G)$ satisfies (EP) at least if $(\mathcal{D}, G/N)$ satisfies (EP) and N is amenable. Moreover, amenability of N is also a necessary condition. In what follows, if $(\mathcal{A}, G)$ is a Fell bundle over G and H is a subgroup of G, then $(\mathcal{A}_H, H)$ denotes the restriction of $\mathcal{A}$ to H. We start with a lemma.

Lemma 6.2 *Let $\mathcal{A}$ be a Fell bundle over the discrete group G and let H be a subgroup of G. Suppose that $C^*(\mathcal{A}_H)$ is represented faithfully as a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and let $\delta\colon C^*(\mathcal{A}_H) \to C^*(\mathcal{A}_H) \otimes C^*(H)$ denote the coaction given by $\delta(h) = a_h \otimes h$. Further, let λ^G denote the left regular representation of G on $l^2(G)$. Then $(\mathrm{id} \otimes \lambda^G|_H) \circ \delta$ factors through a faithful representation of $C^*_r(\mathcal{A}_H)$ on $\mathcal{H} \otimes l^2(G)$.*

Proof It follows from [12, Addendum of Theorem 1] that the restriction $\lambda^G|_H$ of λ^G to H is weakly equivalent to the left regular representation λ^H of H. Hence it follows that the kernels of $\mathrm{id} \otimes \lambda^G|_H$ and $\mathrm{id} \otimes \lambda^H$ coincide in $C^*(\mathcal{A}_H) \otimes C^*(H)$. Thus we get $\ker(\mathrm{id} \otimes \lambda^G|_H) \circ \delta = \ker(\mathrm{id} \otimes \lambda^H) \circ \delta$. But $\ker(\mathrm{id} \otimes \lambda^H) \circ \delta$ coincides with $\ker j_{C^*(\mathcal{A}_H)}$. Hence, since the normalization of δ is the "dual" coaction on $C^*_r(\mathcal{A}_H)$, it follows that $(\mathrm{id} \otimes \lambda^G|_H) \circ \delta$ factors through a faithful map on $C^*_r(\mathcal{A}_H)$ (see Section 2). ∎

In the next proposition we are going to refine Exel's arguments in order to show that property (EP) for a Fell bundle $(\mathcal{A}, G)$ even implies amenability of all restrictions $(\mathcal{A}_H, H)$ of $\mathcal{A}$ to subgroups H of G. It is not clear to us whether amenability, as defined above, is always inherited by restrictions of Fell bundles to subgroups—only if $\mathcal{A}$ is saturated were we able to show that this is true for normal subgroups (see Remark 6.4 below). We were also not able (so far) to show that property (EP) is inherited by restrictions $\mathcal{A}_H$ of $\mathcal{A}$.

Proposition 6.3 (cf. [7, Theorem 4.6]) *Suppose a Fell bundle $\mathcal{A}$ over the discrete group G satisfies property* (EP)*. Then the restricted bundle $\mathcal{A}_H$ of $\mathcal{A}$ is amenable for any subgroup H of G, i.e., the regular representation $\Lambda^H\colon C^*(\mathcal{A}_H) \to C^*_r(\mathcal{A}_H)$ is an isomorphism.*

Proof Let $(f_i)_{i\in I}$ be a net of finitely supported functions on G satisfying the conditions of Definition 6.1. For each $i \in I$ we define a map $\Psi_i\colon \mathcal{A}_H \to \mathcal{A}_H$ by

$$\Psi_i(a_h) = \sum_{s\in G} f_i(hs)^* a_h f_i(s).$$

We claim that each map Ψ_i extends to a bounded linear map $\Psi_i\colon C^*_r(\mathcal{A}_H) \to C^*(\mathcal{A}_H)$ satisfying $\|\Psi_i\| \le \|\sum_{s\in G} f_i(s)^* f_i(s)\|$. If this is shown, then $(\Psi_i \circ \Lambda^H)(a)$ will converge to a for any section $a \in \Gamma_c(\mathcal{A}_H)$ by property (2) of Definition 6.1, and hence for any $a \in C^*(\mathcal{A}_H)$, since the Ψ_i are uniformly bounded by property (1) of Definition 6.1. Hence, if $a \in \ker \Lambda^H$, then $a = \lim_{i\to\infty} (\Psi_i \circ \Lambda^H)(a) = 0$.

In order to prove the claim assume that $C^*(\mathcal{A}_H)$ is faithfully represented on a Hilbert space $\mathcal{H}$ as in Lemma 6.2, and let $f\colon G \to A_e$ be any finitely supported map. Recall also from the lemma that $C^*_r(\mathcal{A}_H)$ is represented faithfully on $\mathcal{H} \otimes l^2(G)$ via $(\mathrm{id} \otimes \lambda^G|_H) \circ \delta$,

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which maps an element a_h in the fibre A_h of $\mathcal{A}_H$ to the operator $a_h \otimes \lambda^G(h)$. As in the proof of [7, Lemma 4.2], we define an operator $V: \mathcal{H} \to \mathcal{H} \otimes l^2(G)$ by

$$V\xi = \sum_{s\in G} f(s)\xi \otimes \chi_s,$$

where χ_s denotes the characteristic function of $\{s\}$. Since

$$\|V\xi\|^2 = \sum_{s\in G} \langle f(s)^* f(s)\xi, \xi\rangle \leq \Big\| \sum_{s\in G} f(s)^* f(s) \Big\| \, \|\xi\|^2,$$

it follows that $\|V\|^2 \leq \| \sum_{s\in G} f(s)^* f(s)\|$. Using the equation $V^*(\xi \otimes \chi_s) = f(s)^*\xi$, we easily compute

$$\begin{aligned} V^*\big(a_h \otimes \lambda^G(h)\big)V\xi &= V^*\big(a_h \otimes \lambda^G(h)\big)\Big(\sum_{s\in G} f(s)\xi \otimes \chi_s\Big) \\ &= V^*\Big(\sum_{s\in G} a_h f(s) \otimes \chi_{hs}\Big) = \sum_{s\in G} f(hs)^* a_h f(s)\xi. \end{aligned}$$

Hence, if we define $\Psi: C_r^*(\mathcal{A}_H) \to C^*(\mathcal{A}_H)$ by

$$\Psi\big(\Lambda^H(a)\big) = V^*\big((\mathrm{id} \otimes \lambda^G|_H) \circ \delta(a)\big)V,$$

we see that Ψ is a linear map whose norm is bounded by $\| \sum_{s\in G} f(s)^* f(s)\|$, and which maps $a_h \in A_h$ to $\sum_{s\in G} f(hs)^* a_h f(s)$. Replacing f by the f_i gives the desired result. ∎

Remark 6.4 Note that if $\mathcal{A}$ is saturated (*i.e.*, $A_s A_t$ is dense in A_{st} for all $s, t \in G$), and N is a normal subgroup of G, then it is not to hard to see that amenability of $\mathcal{A}$ implies amenability of $\mathcal{A}_N$. The proof involves showing that the restriction of the regular representation of $\mathcal{A}$ to $\mathcal{A}_N$ has the same kernel as the regular representation of $\mathcal{A}_N$, and that $C^*(\mathcal{A}_N)$ imbeds faithfully as a subalgebra of $C^*(\mathcal{A})$. The first can be shown by using Exel's description of the kernel of the regular representation via the canonical conditional expectation [7, Proposition 3.6]. The second fact follows from the saturatedness, which allows one to apply [9, Corollary XI.12.8] in order to show that any faithful representation of $C^*(\mathcal{A})$ restricts to a faithful representation of $C^*(\mathcal{A}_N)$. We do not know whether this result still holds without the saturatedness condition. Since we don't need these results later, we omit further details.

As mentioned earlier, we also do not know whether property (EP) is inherited to (normal) subgroups. As for pull backs: we have the strong feeling that the pull back $(q^*\mathcal{D}, G)$ of a bundle $(\mathcal{D}, G/N)$ is amenable if and only $(\mathcal{D}, D/N)$ and N are, but all we could prove (so far) is

Theorem 6.5 *Suppose that G is a discrete group with normal subgroup N, and that $\mathcal{D}$ is a Fell bundle over G/N. If $(\mathcal{D}, G/N)$ satisfies Exel's property* (EP) *and N is amenable, then $(q^*\mathcal{D}, G)$ satisfies* (EP). *Conversely, if $(q^*\mathcal{D}, G)$ satisfies* (EP)*, then N has to be amenable.*

Proof Suppose first that $q^*\mathcal{D}$ satisfies (EP). In order to see that N is amenable we consider the restriction $(q^*\mathcal{D}_N, N)$ of $q^*\mathcal{D}$ to N. It follows straight from the definitions that $q^*\mathcal{D}_N$ is the trivial bundle $D_N \times N$. Hence, the full cross sectional algebra is isomorphic to $D_N \otimes_{\max} C^*(N)$ and the reduced cross sectional algebra is isomorphic to $D_N \otimes C_r^*(N)$. Further, the regular representation can be identified with the quotient map $\mathrm{id} \otimes \lambda^N \colon D_N \otimes_{\max} C^*(N) \to D_N \otimes C_r^*(N)$. Proposition 6.3 implies that $D_N \times N$ is amenable, from which it follows that $\mathrm{id} \otimes \lambda^N$ is an isomorphism. But this implies that λ^N is an isomorphism, too. Hence N is amenable.

Suppose now that N is amenable and $(\mathcal{D}, G/N)$ satisfies (EP). Let $(f_i)_{i\in I}$ be a net of finitely supported maps $f_i \colon G/N \to D_N$ which satisfies the conditions of Definition 6.1 for $\mathcal{D}$. Since N is amenable, we can also choose a net $(g_j)_{j\in J}$ of finitely supported complex-valued functions on N satisfying $\sum_{n\in N} |g_j(n)|^2 \leq 1$ such that the corresponding matrix coefficients

$$n \mapsto \langle \lambda^N(n) g_j, g_j \rangle = \sum_{m\in N} \overline{g_j(nm)} g_j(m)$$

converge pointwise to the trivial function 1_N on N. Let $c \colon G/N \to G$ be any cross section satisfying $c(eN) = e$ (in what follows we will simply write $c(s)$ instead of $c(sN)$). Every $s \in G$ has a unique decomposition $s = c(s)n_s$ with $n_s = c(s)^{-1}s \in N$. For each $(i, j) \in I \times J$ we define $f_i \times g_j \colon G \to q^*D_e = (D_N, e)$ by $f_i \times g_j(s) = (f_i(sN) g_j(n_s), e)$.

Thus, if $(d_{tN}, t) \in (D_{tN}, t)$ we compute

$$\begin{aligned}
&\sum_{s\in G} f_i \times g_j(ts)^* (d_{tN}, t) f_i \times g_j(s) \\
&\quad = \sum_{sN\in G/N} \sum_{m\in N} f_i \times g_j\big(tc(s)m\big)^* (d_{tN}, t) f_i \times g_j\big(c(s)m\big) \\
&\quad = \sum_{sN\in G/N} \sum_{m\in N} f_i \times g_j\big(c(ts) n_{tc(s)} m\big)^* (d_{tN}, t) f_i \times g_j\big(c(s)m\big) \\
&\quad = \Big(\sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) \sum_{m\in N} \overline{g_j(n_{tc(s)}m)} g_j(m), t \Big).
\end{aligned}$$

Now, if F is any finite subset of G and $\epsilon > 0$ is given, we may choose $j := j_{(i,\epsilon,F)} \in J$ such that $|1 - \sum_{m\in N} \overline{g_j(n_{tc(s)}m)} g_j(m)| < \frac{\epsilon}{c_i}$, for any $sN \in \operatorname{supp} f_i$ and $t \in F$, where we put $c_i := \sup_{t\in F} \big(\sum_{sN\in G/N} \| f_i(tsN) \| \| f_i(sN) \| \big)$ (assuming f_i to be non-zero). Hence, for any $t \in F$ and $(d_{tN}, t) \in (D_{tN}, t)$ we get

$$\begin{aligned}
&\Big\| \sum_{s\in G} f_i \times g_j(ts)^* (d_{tN}, t) f_i \times g_j(s) - (d_{tN}, t) \Big\| \\
&\quad = \Big\| \Big(\sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) \sum_{m\in N} \overline{g_j(n_{tc(s)}m)} g_j(m) - d_{tN}, t \Big) \Big\| \\
&\quad \leq \Big\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) \sum_{m\in N} \overline{g_j(n_{tc(s)}m)} g_j(m) - d_{tN} \Big\|
\end{aligned}$$

$$\leq \Big\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) - d_{tN} \Big\| + \Big\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) \Big(1 - \sum_{m\in N} \overline{g_j(n_{tc(s)}m)} g_j(m)\Big) \Big\|$$

$$\leq \Big\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) - d_{tN} \Big\| + \|d_{tN}\| \sum_{sN\in G/N} \|f_i(tsN)\| \, \|f_i(sN)\| \frac{\epsilon}{c_i}$$

$$\leq \Big\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) - d_{tN} \Big\| + \|d_{tN}\|\epsilon.$$

Now we are ready to produce a net $(h_k)_{k\in K}$ which will enforce (EP) on $q^*\mathcal{D}$. For this let $K = I \times (0,\infty) \times \mathcal{F}$, where $\mathcal{F}$ denotes the set of all finite subsets of G, equipped with the ordering

$$(i,\epsilon,F) \geq (i',\epsilon',F') \Leftrightarrow i \geq i', \quad \epsilon \leq \epsilon', \quad \text{and} \quad F \supseteq F'.$$

If $k = (i,\epsilon,F) \in K$ is given, we define $h_k = f_i \times g_j$, where $j = j_{(i,\epsilon,F)}$ is chosen as above. Then it is a direct consequence of the above computations that $\sum_{s\in G} h_k(ts)^*(d_{tN},t)h_k(s)$ converges to (d_{tN},t) for all $(d_{tN},t) \in q^*\mathcal{D}$; in fact, if (d_{tN},t) and $\delta > 0$ are given, choose $k_0 = (i_0,\epsilon_0,F_0)$ such that $t \in F_0$, $\epsilon_0 < \frac{\delta}{2\|d_{tN}\|}$ and $\| \sum_{sN\in G/N} f_i(tsN)^* d_{tN} f_i(sN) - d_{tN} \| < \frac{\delta}{2}$ for all $i \geq i_0$. Then the above computations show that

$$\Big\| \sum_{s\in G} h_k(ts)^*(d_{tN},t)h_k(s) - (d_{tN},t) \Big\| < \delta$$

for all $k \geq k_0$. Hence the $(h_k)_{k\in K}$ satisfy condition (i) of Definition 6.1. Finally, it is trivial to check that, if $h_k = f_i \times g_j$, then

$$\Big\| \sum_{s\in G} h_k(s)^* h_k(s) \Big\| \leq \Big\| \sum_{sN\in G/N} f_i(sN)^* f_i(sN) \Big\| \cdot \Big(\sum_{n\in N} |g_j(n)|^2 \Big) \leq \Big\| \sum_{sN\in G/N} f_i(sN)^* f_i(sN) \Big\|,$$

which proves condition (ii) of Definition 6.1. ∎

Remark 6.6 In [25] Raeburn and the second author discovered that the Cuntz-Krieger algebras carry a natural coaction of $\mathbb{F}_n$, $n \geq 2$, and are therefore isomorphic to cross sectional algebras of the associated Fell bundles $(\mathcal{A}, \mathbb{F}_n)$. Later [7] Exel showed that all these Fell bundles satisfy property (EP), and hence are amenable, which shows that the Cuntz-Krieger algebras are completely determined by the corresponding Fell bundles (even if they do not satisfy condition (I) as studied by Cuntz and Krieger in [2]). Since any nontrivial normal subgroup of $\mathbb{F}_n$ is nonamenable, it follows from Theorem 6.5 that the Cuntz-Krieger algebras are not induced from any coaction on any proper quotient G/N of G.

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